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On totally magic cordial labeling

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Abstract. A graph G is said to have totally magic cordial(TMC) labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod 2$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i. In this paper, we investigate some new families of graphs that admit totally magic cordial labeling.

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§1. Introduction

All graphs considered here are finite, simple and undirected. We follow the basic notations and terminologies of graph theory as in Harary [5]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. A detailed survey of graph labeling is available in [4]. The concept of cordial labeling was introduced by Cahit [1] and he proved that every tree is cordial, K_n is cordial if $n \leq 3$, $K_{m,n}$ is cordial for all m and n, the friendship graph $C_3^{(t)}$ is cordial if and only if $t \not\equiv 2 \pmod 4$, all fans are cordial and the wheel graph W_n is cordial if and only if $n \not\equiv 3 \pmod 4$. In [2] he proved that a k-angular cactus with t cycles is cordial if and only if $kt \not\equiv 2 \pmod 4$. Further results on cordial labelings were discussed in [6, 7, 8, 9].

Based on cordial labeling Cahit [3] introduced another two well known graph labelings namely totally magic cordial labeling (TMC) and total sequential cordial labeling (TSC). In this paper, we show that the graph G is TMC if and only if G is TSC, a graph with number of vertices and number of edges differ by atmost 1 is TMC and also investigate that the TMC labelings of some families of graphs. In Theorem 10 [3], Cahit proved that the complete graph K_n is TMC if and only if $n \in \{2, 3, 5, 6\}$. This observation is not correct. We rectify this error in Theorem 2.11.

We use the following definitions in the subsequent section:

Definition 1.1. *A graph G is said to have totally magic cordial(TMC) labeling with constant C if there exists a mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that $f(a) + f(b) + f(ab) \equiv C \pmod 2$ for all $ab \in E(G)$ and $|n_f(0) - n_f(1)| \leq 1$, where $n_f(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i.*

Definition 1.2. *A graph G is said to have total sequential cordial(TSC) labeling if there is a total mapping $f : V(G) \cup E(G) \to \{0, 1\}$ such that for each edge $e = \{a, b\}$, $f(e) = |f(a) - f(b)|$ and the condition $|n_f(0) - n_f(1)| \leq 1$ holds.*

Definition 1.3. *A wheel graph W_n is obtained from a cycle C_n by adding a new vertex and joining it to all the vertices of the cycle by an edge, then the new edges are called spokes of the wheel.*

Definition 1.4. *Flower graph $Fl_n(n \geq 3)$ is constructed from a wheel W_n by attaching a pendant edge at each vertex of the n-cycle and by joining each pendant vertex to the central vertex.*

Definition 1.5. *Ladder graph $L_n(n \geq 2)$ is a product graph $P_2 \times P_n$ with $2n$ vertices and $3n - 2$ edges.*

Definition 1.6. *An (n, t)-kite graph is a cycle C_n with a t-edge path (the tail) attached to one vertex.*

Definition 1.7. *An n-sun graph is a cycle C_n with a pendant edge attached to each vertex of a cycle C_n.*

Definition 1.8. *A friendship graph $T_n(n \geq 2)$ is the one-point union of t cycles of length n.*

§2. Main Results

Theorem 2.1. *If G is a (p, q) graph with $|p - q| \leq 1$ then G is TMC.*

Proof. If we assign 0 to all the edges of G and 1 to all the vertices of G then we get $C = 0$. If we assign 1 to all the edges of G and 0 to all the vertices of G then we get $C = 1$. In either case, $|n_f(0) - n_f(1)| = |p - q| \leq 1$. Clearly, G is TMC. □

Corollary 2.2. *All trees, cycles $(n \geq 3)$, unicyclic graphs, (n, t)-kite graphs $(n \geq 3)$ and n-sun graphs $(n \geq 3)$ are TMC.*

Theorem 2.3. *A graph G is TMC if and only if G is TSC.*

Proof. A mapping $f : V(G) \cup E(G) \to \{0, 1\}$ is a TMC labeling with constant 0 if and only if f is a TSC labeling, and f is a TMC labeling with constant 1 if and only if $\bar{f}$ is a TSC labeling, where $\bar{f}$ is defined by $\bar{f}(x) = 1 - f(x)$, for all $x \in V(G) \cup E(G)$. Hence a graph G has a TMC labeling if and only if G has a TSC labeling. □

Cahit [3] proved that every cordial graph is TSC and the friendship graph T_n is TMC for all $n \geq 2$. Hence, we obtain the following results:

Corollary 2.4. *Every cordial graph is TMC.*

Corollary 2.5. *The friendship graph T_n is TMC for all $n \geq 2$.*

Lemma 2.6. *The flower graph Fl_n is TMC for $n \geq 3$.*

Proof. Let $V = \{u, u_i, v_i | 1 \leq i \leq n\}$ be the vertex set and $E = \{uu_i, u_iv_i, uv_i | 1 \leq i \leq n\} \cup \{u_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be the edge set for $n \geq 3$. Clearly, $|V| = 2n + 1$ and $|E| = 4n$. Define $f : V \cup E \to \{0, 1\}$ as follows: $f(u) = 0$, $f(u_i) = 0$, $f(v_i) = 1$, $f(uu_i) = 1$, $f(u_iv_i) = 0$ and $f(uv_i) = 0$ for $0 \leq i \leq n$ and $f(u_iu_{i+1}) = f(u_nu_1) = 1$ for $0 \leq i < n$. Clearly, $f(a) + f(b) + f(ab) \equiv 1 \pmod 2$ for all $ab \in E$. Also, $n_f(0) = n_f(1) = 3n + 1$. Thus, $|n_f(0) - n_f(1)| \leq 1$. Hence, Fl_n is TMC for $n \geq 3$. □

Lemma 2.7. *The ladder graph L_n is TMC for all $n \geq 2$.*

Proof. Let the vertex set be $V = \{u_i, v_i | 1 \leq i \leq n\}$ and the edge set be $E = \{u_iv_i | 1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1} | 1 \leq i < n\}$. Clearly, $|V| = 2n$ and $|E| = 3n - 2$. Define $f : V \cup E \to \{0, 1\}$ as follows: $f(u_i) = 0$ for $i = 1, 2, ..., n$ and $f(u_iu_{i+1}) = 1$ for $i = 1, 2, ..., n-1$. $f(v_i) = f(v_{i+1}) = 0$, $f(u_iv_i) = f(u_{i+1}v_{i+1}) = 1$ for $i \equiv 1 \pmod 4$, $f(v_i) = f(v_{i+1}) = 1$, $f(u_iv_i) = f(u_{i+1}v_{i+1}) = 0$ for $i \equiv 3 \pmod 4$ and

$$f(v_iv_{i+1}) = \begin{cases} 1 & \text{if} \quad i \text{ is odd,} \\ 0 & \text{if} \quad i \text{ is even.} \end{cases}$$

Clearly, $C = 1$ and $n_f(0) = n_f(1) + 1 = \frac{5n-1}{2}$ if n is odd and $n_f(0) = n_f(1) = \frac{5n-2}{2}$ if n is even. Hence, the ladder graph L_n is TMC for all $n \geq 2$. □

Lemma 2.8. *If G is a graph obtained by identifying a vertex of the cycle $C_m (m \geq 3)$ with each vertex of the cycle $C_n (n \geq 3)$ then G is TMC.*

Proof. Let $V(G) = \left\{u_i^j | 1 \leq i \leq m, 1 \leq j \leq n\right\}$ and $E(G) = \left\{u_i^j u_{i+1}^j | 1 \leq i \leq m-1, 1 \leq j \leq n\right\} \cup \left\{u_m^j u_1^j | 1 \leq j \leq n\right\}$

$\cup \left\{u_1^j u_1^{j+1} | 1 \le j \le n-1\right\} \cup \{u_1^n u_1^1\}$. Clearly, $|V(G)| = mn$ and $|E(G)| = mn + n$. Define $f : V(G) \cup E(G) \to \{ 0, 1\}$ as follows: For $j = 1, 2, ..., n$,

$$f(u_2^j) = \begin{cases} 0 & \text{if } \ j \text{ is odd,} \\ 1 & \text{if } \ j \text{ is even} \end{cases}$$

and $f(u_i^j) = 0$ for $i \neq 2$ and $i = 1, 3, ..., m$.

$$f(u_1^j u_2^j) = f(u_2^j u_3^j) = \begin{cases} 1 & \text{if } \ j \text{ is odd,} \\ 0 & \text{if } \ j \text{ is even.} \end{cases}$$

For $i = 3, 4, ..., m$, $f(u_i^j u_{i+1}^j) = 1$ and for $j = 1, 2, ..., n-1$, $f(u_1^j u_1^{j+1}) = f(u_1^n u_1^1) = 1$. Clearly, $C = 1$ and

$$n_f(1) = \begin{cases} n_f(0) + 1 & \text{if } \ j \text{ is odd,} \\ n_f(0) & \text{if } \ j \text{ is even.} \end{cases}$$

Hence, G is TMC. □

Theorem 2.9. *If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two disjoint TMC graphs and $p_1 = q_1$ or $p_2 = q_2$ then $G_1 \cup G_2$ is also TMC.*

Proof. Let f and g be TMC labeling of G_1 and G_2 respectively with the same constant C. Without loss of generality, we assume that $p_1 = q_1$. Then $n_f(0) = n_f(1)$. Define $h : V(G_1 \cup G_2) \cup E(G_1 \cup G_2) \to \{0, 1\}$ by $h/V(G_1) \cup E(G_1) = f$ and $h/V(G_2) \cup E(G_2) = g$. Now $n_h(0) = n_f(0) + n_g(0) = n_h(1)$ if $n_g(0) = n_g(1)$. Similarly, $n_h(0) = n_h(1) + 1$ if $n_g(0) = n_g(1) + 1$ and $n_h(1) = n_h(0) + 1$ if $n_g(1) = n_g(0) + 1$. Thus, h is a TMC labeling of $G_1 \cup G_2$ and hence, $G_1 \cup G_2$ is TMC. □

Corollary 2.10. *The disjoint union of cycle with the TMC graph G is TMC.*

Theorem 2.11. *The complete graph K_n is TMC if and only if*

$$\begin{cases} \sqrt{4k+1} \textit{ has an integer value when } n = 4k, \\ \sqrt{k+1} \textit{ or } \sqrt{k} \textit{ has an integer value when } n = 4k+1, \\ \sqrt{4k+5} \textit{ or } \sqrt{4k+1} \textit{ has an integer value when } n = 4k+2, \\ \sqrt{k+1} \textit{ has an integer value when } n = 4k+3. \end{cases}$$

Proof. Assume that f is a TMC labeling of K_n. Without loss of generality, we assume that $C = 1$. Then for any edge $e = uv \in E(K_n)$, we have either $f(e) = f(u) = f(v) = 1$ or $f(e) = f(u) = 0$ and $f(v) = 1$ or $f(e) = f(v) = 0$ and $f(u) = 1$ or $f(u) = f(v) = 0$ and $f(e) = 1$. Hence, under the labeling f, the complete graph can be decomposed as $K_n = K_p \cup K_r \cup K_{p,r}$, where K_p is the subgraph whose vertices and edges are labeled with 1, K_r is the sub

graph whose vertices labeled with 0 and its edges labeled with 1 and $K_{p,r}$ is the subgraph of K_n with the bipartition $V(K_p) \cup V(K_r)$ in which the edges are labeled with 0. Thus, we have $n_f(0) = r + pr$ and $n_f(1) = p + \frac{p(p-1)}{2} + \frac{r(r-1)}{2}$.
Also, for any TMC labeling f of K_n we must have the following:
(i) $n_f(0) = n_f(1)$ if $n \equiv 0, 3 \pmod 4$.
(ii) $n_f(1) = n_f(0) + 1$ or $n_f(0) = n_f(1) + 1$ if $n \equiv 1, 2 \pmod 4$.
Case i. $n \equiv 0, 3 \pmod 4$, $n > 2$.
Then $n_f(0) = n_f(1)$, which implies $p^2 + p(1-2r) + r^2 - 3r = 0$. Since $p = n - r$, we have $4r^2 - 4r(n+1) + n^2 + n = 0$. Hence, $r = \frac{1}{2}\left[(n+1) \pm \sqrt{n+1}\right]$. Since r is the order of subgraph K_r, it can be seen that K_{4k}, $k \geq 1$, is TMC only if $\sqrt{4k+1}$ has an integer value and K_{4k+3}, $k \geq 0$, is TMC only if $\sqrt{k+1}$ has an integer value.
Case ii. $n \equiv 1, 2 \pmod 4$, $n > 2$.
Then, $n_f(1) = n_f(0) + 1$ or $n_f(0) = n_f(1) + 1$.
If $n_f(1) = n_f(0) + 1$, $p^2 + p(1-2r) + r^2 - 3r - 2 = 0$. Since $p = n - r$, $4r^2 - 4r(n+1) + n^2 + n - 2 = 0$. Hence, $r = \frac{1}{2}\left[(n+1) \pm \sqrt{n+3}\right]$. For $k \geq 1$, K_{4k+1} is TMC only if $\sqrt{k+1}$ has an integer value and for $k \geq 1$, K_{4k+2} is TMC only if $\sqrt{4k+5}$ has an integer value.

Again, if $n_f(0) = n_f(1) + 1$, $p^2 + p(1-2r) + r^2 - 3r + 2 = 0$. Since $p = n - r$, $4r^2 - 4r(n+1) + n^2 + n + 2 = 0$. Hence, $r = \frac{1}{2}\left[(n+1) \pm \sqrt{n-1}\right]$. For $k \geq 1$, K_{4k+1} is TMC only if $\sqrt{k}$ has an integer value and for $k \geq 1$, K_{4k+2} is TMC only if $\sqrt{4k+1}$ has an integer value.
Thus, the complete graph K_n is TMC if and only if

$$\begin{cases} \sqrt{4k+1} \text{ has an integer value when } n = 4k, \\ \sqrt{k+1} \text{ or } \sqrt{k} \text{ has an integer value when } n = 4k+1, \\ \sqrt{4k+5} \text{ or } \sqrt{4k+1} \text{ has an integer value when } n = 4k+2, \\ \sqrt{k+1} \text{ has an integer value when } n = 4k+3. \end{cases}$$

□

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