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LP-Sasakian manifolds with quasi-conformal curvature tensor

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Abstract. The object of the present paper is to study LP-Sasakian manifolds with quasi-conformal curvature tensor.

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§1. Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [10] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], De and Shaikh [3], Ozgur [12] and many others.

The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki [16]. According to them a quasi-conformal curvature tensor is defined by

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z \\ &\quad +b[S(Y,Z)X - S(X,Z)Y \\ &\quad +g(Y,Z)QX - g(X,Z)QY] \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1}+2b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

where a and b are non-zero constants, R is the curvature tensor, S is the Ricci tensor, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the Riemannian manifold $(M^n, g)(n \geq 3)$. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the form

$$\begin{aligned}
\tilde{C}(X,Y)Z &= R(X,Y)Z \\
&\quad -\frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\
&\quad +g(Y,Z)QX - g(X,Z)QY] \\
&\quad +\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\
&= C(X,Y)Z,
\end{aligned} \tag{1.2}$$

where C is the conformal curvature tensor [15]. Thus the conformal curvature tensor C is a particular case of the tensor $\widetilde{C}$. For this reason $\widetilde{C}$ is called the quasi-conformal curvature tensor. A Riemannian manifold (M^n, g) $(n > 3)$ shall be called quasi-conformally flat if the quasi-conformal curvature tensor $\widetilde{C} = 0$. It is known [1] that the quasi-conformally flat Riemannian manifold is either conformally flat if $a \neq 0$ or, Einstein if $a = 0$ and $b \neq 0$. Since they give no restrictions if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

In [5], De and Matsuyama studied quasi-conformally flat Riemannian manifolds satisfying a certain condition on the Ricci tensor. From Theorem 5 of [5], it can be proved that a 4-dimensional quasiconformally flat semi-Riemannian manifold is the Robertson-Walker space time. Robertson-Walker spacetime is the warped product $I \times_f M^*$, where M^* is a space of constant curvature and I is an open interval [11]. From (1.1), we obtain

$$\begin{aligned}
(\nabla_W \widetilde{C})(X,Y)Z &= a(\nabla_W R)(X,Y)Z + b[(\nabla_W S)(Y,Z)X \\
&\quad -(\nabla_W S)(X,Z)Y \\
&\quad +g(Y,Z)(\nabla_W Q)(X) - g(X,Z)(\nabla_W Q)(Y)] \\
&\quad -\frac{dr(W)}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)X - g(X,Z)Y],
\end{aligned} \tag{1.3}$$

where ∇ denotes the Levi-Civita connection. If the condition

$$\nabla R = 0$$

holds on M^n, then M^n is called locally symmetric. An LP-Sasakian manifold (M^n, g) is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_X R)(Y,Z)W) = 0 \tag{1.4}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [14]. Later in [2], Blair, Koufogiorgos and Sharma studied locally ϕ-symmetric contact metric manifolds.

In (1.4), if X, Y, Z and W are not horizontal vectors then we call the manifold globally ϕ-symmetric.

In this paper, we study locally ϕ-quasiconformally symmetric and globally ϕ-quasiconformally symmetric LP-Sasakian manifolds. An LP-Sasakian manifold is called locally ϕ-quasiconformally symmetric if the condition

$$\phi^2\left(\left(\nabla_X \widetilde{C}\right)(Y,Z)W\right) = 0 \tag{1.5}$$

holds on M^n, where X, Y, Z and W are horizontal vectors. If X, Y, Z and W are arbitrary vectors then the manifold is called globally ϕ-quasiconformally symmetric.

A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([13], [6]) if $R(X,Y)\cdot R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y)\cdot\widetilde{C} = 0$, then the manifold is said to be quasi-conformally semi-symmetric manifold.

The paper is organized as follows.
After introduction in Section 2, we give a brief account of LP-Sasakian manifolds. In the next two sections, we prove that in a complete simply connected LP-Sasakian manifold if M is quasi-conformally flat, then M is isometric to the Lorentz sphere $\mathbb{S}^n_1(1)$, and if M is a quasi-conformally semi-symmetric and $a+(n-2)b \neq 0$, then M is isometric to the Lorentz sphere $\mathbb{S}^n_1(1)$. In Section 5, we study globally ϕ-quasiconformally symmetric LP-Sasakian manifolds. We prove that a globally ϕ-quasiconformally symmetric LP-Sasakian manifold is globally ϕ-symmetric if $a \neq 0$. In the next Section, we study 3-dimensional locally ϕ-quasiconformally symmetric LP-Sasakian manifolds. We prove that a 3-dimensional LP-Sasakian manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant if $a+b \neq 0$ and $r \neq 6$. Finally, we construct an example of a 3-dimensional quasi-conformally flat LP-Sasakian manifold.

§2. Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p\,\varepsilon M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,\ldots,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi,\ \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [7] :

$$\phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(X, \phi Y)$.
Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1$, $\epsilon_2 = \cdots = \epsilon_n = 1$.
A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called an LP-Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP-Sasakian manifold the 1-form η is closed. Also in [7], it is proved that if an n-dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an LP-Sasakian structure. Also since the 1-form η is closed in an LP-Sasakian manifold, we have ([7], [8])

$$(\nabla_X \eta)Y = \Omega(X,Y), \tag{2.5}$$

$$\Omega(X,\xi) = 0, \tag{2.6}$$

$$\nabla_X \xi = \phi X \tag{2.7}$$

for any vector field X and Y.
Further, on such an LP-Sasakian manifold M^n (ϕ, ξ, η, g), the following relations hold [7]:

$$\eta(R(X,Y)Z) = [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.9}$$

$$R(X,Y)\xi = [\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.11}$$

$$(\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] \tag{2.12}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

§3. Quasi-conformally flat LP-Sasakian manifold

When the quasi-conformal curvature tensor vanishes identically on the Lorentzian manifold, then we find from (1.1)

$$\begin{aligned} a\widetilde{R}(X,Y,Z,W) &= b\{S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ &\quad + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)\} \\ &\quad + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right) \\ &\quad \{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \end{aligned} \tag{3.1}$$

which implies that

$$\{a + (n-2)b\}\left\{S(Y,Z) - \frac{r}{n}g(Y,Z)\right\} = 0. \tag{3.2}$$

Thus we obtain $a + (n-2)b = 0$ or $S(Y,Z) = \frac{r}{n}g(Y,Z)$. If $a + (n-2)b = 0$, then the conformal curvature tensor vanishes identically. It is known that a conformally flat LP-Sasakian manifold is of constant curvature [4]. When M is an Einstein LP-Sasakian manifold, we get $r = n(n-1)$. It is easy to see form (3.1) that M is of constant curvature 1. Conversely, if M is of contant curvature, then the quasi-conformal curvature tensor vanishes. Hence we have

Theorem 3.1. *Let M^n $(n > 3)$ be an LP-Sasakian manifold. Then M is quasi-conformally flat if and only if it is of constant curvature.*

From [11], we have

Theorem 3.2. *Let M^n $(n > 3)$ be a complete simply connected LP-Sasakian manifold. If M is quasi-conformally flat, then M is isometric to the Lorentz sphere $\mathbb{S}^n_1(1)$.*

§4. LP-Sasakian manifolds satisfying $R(\xi, Y) \cdot \widetilde{C} = 0$

In this section we consider an LP-Sasakian manifold M^n $(n > 3)$ satisfying the condition

$$(R(\xi, Y) \cdot \widetilde{C})(U, V)W = 0, \tag{4.1}$$

which yields from (2.11) that

$$\begin{aligned} &g(\widetilde{C}(U, V)W, Y)\xi - \eta(\widetilde{C}(U, V)W)Y - g(Y, U)\widetilde{C}(\xi, V)W \\ &+\eta(U)\widetilde{C}(Y, V)W - g(Y, V)\widetilde{C}(U, \xi)W + \eta(V)\widetilde{C}(U, Y)W \\ &-g(Y, W)\widetilde{C}(U, V)\xi + \eta(W)\widetilde{C}(U, V)Y = 0. \end{aligned}$$

Operating η to the above equation and using of (1.1), (2.8) $\sim$ (2.11) we obtain

$$\begin{aligned} &g(\widetilde{C}(U, V)W, Y) + bg(U, Y)\{S(V, W) + (n-1)\eta(V)\eta(W)\} \\ &-bg(Y, V)\{S(U, W) + (n-1)\eta(U)\eta(W)\} \\ &+b\{S(V, Y)\eta(U) - S(Y, U)\eta(V)\}\eta(W) \\ &-\left\{a + (n-1)b - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right\} \\ &\{g(V, W)g(Y, U) - g(U, W)g(V, Y)\} = 0. \end{aligned} \tag{4.2}$$

Putting $Y = U = e_i$ in the above equation and taking summation over i, we get

$$\begin{aligned} &(a-b)S(V, W) - \{(n-1)a + (n-1)^2 b - br\}g(V, W) \\ &+b\{r - n(n-1)\}\eta(V)\eta(W) = 0, \end{aligned} \tag{4.3}$$

moreover, we find $\{a + (n-2)b\}\{r - n(n-1)\} = 0$.
We can consider the two cases. At first, in the case of $r = n(n-1)$, we have form (4.3)

$$(a-b)\{S(V, W) - (n-1)g(V, W)\} = 0.$$

If $a \neq b$, then $S(V,W) = (n-1)g(V,W)$. Therefore it is clear form (4.2) that the quasi-conformal curvature tensor vanishes, namely, M is of constant curvature 1 from Theorem 3.1. Also, if $a = b(\neq 0)$, then we get form (1.1) and (4.2)

$$\begin{aligned}&g(R(U,V)W,Y) + \{2S(V,W) - ng(V,W) + (n-1)\eta(V)\eta(W)\}g(U,Y)\\&-\{2S(U,W) - ng(U,W) + (n-1)\eta(U)\eta(W)\}g(V,Y)\\&+\{g(V,W) - \eta(V)\eta(W)\}S(U,Y) - \{g(U,W) - \eta(U)\eta(W)\}S(V,Y) = 0.\end{aligned}$$

If we put $W{=}\,\xi$, then we have $\eta(V)S(U,Y) - \eta(U)S(V,Y) = 0$. Furthermore, putting $U{=}\,\xi$, we get $S(V,Y) = -(n-1)\eta(V)\eta(Y)$, that is $r = n-1$. This is the contradiction. Thus $a \neq b$ holds.
Secondly, in the case of $a+(n-2)b = 0$, equation (4.2) is rewritten as follows:

$$\begin{aligned}&(n-2)g(R(U,V)W,Y)\\&-\{2S(V,W) - g(V,W) + (n-1)\eta(V)\eta(W)\}g(U,Y)\\&+\{2S(U,W) - g(U,W) + (n-1)\eta(U)\eta(W)\}g(V,Y)\\&-\{g(V,W) - \eta(V)\eta(W)\}S(U,Y)\\&+\{g(U,W) - \eta(U)\eta(W)\}S(V,Y) = 0.\end{aligned} \tag{4.4}$$

We put $U = W = \xi$. Then we find $S(V,Y) = -(n-1)\eta(V)\eta(Y)$, which yields form (4.2) that

$$\begin{aligned}R(U,V)W &= \frac{1}{4}(c+3)\{g(V,W)U - g(U,W)V\}\\&+\frac{1}{4}(c-1)\{\eta(V)\eta(W)U - \eta(U)\eta(W)V\\&+g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi\},\end{aligned} \tag{4.5}$$

where $c = -\frac{3n-2}{n-2}$. Hence we have

Theorem 4.1. *Let $M^n (n > 3)$ be an LP-Sasakian manifold satisfying $R(\xi,Y)\cdot \widetilde{C} = 0$ for any Y.*
(1) *If $a + (n-2)b \neq 0$, then M is of constant curvature* 1.
(2) *If $a + (n-2)b = 0$, then M is a space satisfying* (4.5).

Corollary 4.1. *Let $M^n (n > 3)$ be an LP-Sasakian manifold. If M is a quasi-conformally semi-symmetric, then*
(1) *when $a + (n-2)b \neq 0$, then M is of constant curvature* 1.
(2) *when $a + (n-2)b = 0$, then M is a space satisfying* (4.5).

From [11], we have
Theorem 4.2. *Let $M^n (n > 3)$ be a complete simply connected LP-Sasakian*

manifold satisfying $R(\xi, Y) \cdot \widetilde{C} = 0$ *for any* Y*. If* $a + (n-2)b \neq 0$*, then* M *is isometric to the Lorentz sphere* $\mathbb{S}_1^n(1)$*.*

Corollary 4.2. *Let* $M^n (n > 3)$ *be a complete simply connected* LP*-Sasakian manifold. If* M *is a quasi-conformally semi-symmetric and* $a + (n-2)b \neq 0$*, then* M *is isometric to the Lorentz sphere* $\mathbb{S}_1^n(1)$*.*

§5. Globally ϕ-quasiconformally symmetric LP-Sasakian manifolds

Let us suppose that M is a globally ϕ-quasiconformally symmetric LP-Sasakian manifold. Then by definition

$$\phi^2 \left(\left(\nabla_W \widetilde{C} \right) (X, Y)Z \right) = 0.$$

Using (2.1) we have

$$\left(\nabla_W \widetilde{C} \right) (X, Y)Z + \eta \left(\left(\nabla_W \widetilde{C} \right) (X, Y)Z \right) \xi = 0.$$

From (1.2) it follows from (2.7) and (2.9) that

$$\begin{aligned}
(5.1) \quad & a\{g((\nabla_W R)(X, Y)Z, U) + \eta(U)\eta((\nabla_W R)(X, Y)Z)\} \\
& +b[\{g(X, U) + \eta(X)\eta(U)\}(\nabla_W S)(Y, Z) \\
& -\{g(Y, U) + \eta(Y)\eta(U)\}(\nabla_W S)(X, Z) \\
& +g(Y, Z)\{(\nabla_W S)(X, U) - \eta(U)S(X, \phi W) + (n-1)\eta(U)g(X, \phi W)\} \\
& -g(X, Z)\{(\nabla_W S)(Y, U) - \eta(U)S(Y, \phi W) + (n-1)\eta(U)g(Y, \phi W)\}] \\
& -\frac{dr(W)}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y, Z)\{g(X, U) + \eta(X)\eta(U)\} \\
& \qquad -g(X, Z)\{g(Y, U) + \eta(Y)\eta(U)\}] = 0.
\end{aligned}$$

Putting $Z = \xi$, in (5.1) and using of (2.7), (2.9) and (2.10), we obtain

$$\begin{aligned}
(5.2) \qquad & a\{g(Y, \phi W)g(X, U) - g(X, \phi W)g(Y, U) - g(R(X, Y)\phi W, U)\} \\
& +b[\eta(Y)(\nabla_W S)(X, U) - \eta(X)(\nabla_W S)(Y, U) \\
& -g(X, U)\{S(Y, \phi W) - (n-1)g(Y, \phi W)\} \\
& +g(Y, U)\{S(X, \phi W) - (n-1)g(X, \phi W)\}] \\
& -\frac{dr(W)}{n}\left(\frac{a}{n-1} + 2b\right)\{\eta(Y)g(X, U) \\
& \qquad -\eta(X)g(Y, U)\} = 0.
\end{aligned}$$

Moreover, putting $X = U = e_i$ in (5.2) and taking summation over i, we obtain

$$\{a+(n-2)b\}\left\{S(Y,\phi W)-(n-1)g(Y,\phi W)+\frac{dr(W)}{n}\eta(Y)\right\}=0.$$

Thus if $a+(n-2)b \neq 0$, then we find $S(Y,\phi W)=(n-1)g(Y,\phi W)-\frac{dr(W)}{n}\eta(Y)$. Setting $Y=\xi$, we have $dr(W)=0$, that is, the scalar curvature is constant. It is easy to see form (2.9) that $S(Y,W)=(n-1)g(Y,W)$, that is, M is an Einstein. Since (5.2), we find that M is of constant curvature 1. Next, when $a+(n-2)b=0$, it is clear from (5.2) that

$$\begin{aligned}
&(n-2)\{g(R(X,Y)\phi W,U)-g(Y,\phi W)g(X,U)+g(X,\phi W)g(Y,U)\}\\
&+\eta(Y)(\nabla_W S)(X,U)-g(X,U)\{S(Y,\phi W)-(n-1)g(Y,\phi W)\}\\
&-\eta(X)(\nabla_W S)(Y,U)+g(Y,U)\{S(X,\phi W)-(n-1)g(X,\phi W)\}\\
&-\frac{dr(W)}{n-1}\{\eta(Y)g(X,U)-\eta(X)g(Y,U)\}=0.
\end{aligned}$$

Setting $Y=\xi$ in the above equation, we get

$$\begin{aligned}
(\nabla_W S)(X,U) &= \eta(X)\{S(U,\phi W)-(n-1)g(U,\phi W)\}\\
&\quad+\eta(U)\{S(X,\phi W)-(n-1)g(X,\phi W)\}\\
&\quad+\frac{dr(W)}{n-1}\{g(X,U)+\eta(X)\eta(U)\}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
R(X,Y)Z &= g(Y,Z)X-g(X,Z)Y\\
&\quad+\frac{1}{n-2}\{S(Y,Z)-(n-1)g(Y,Z)\}\phi^2 X\\
&\quad-\frac{1}{n-2}\{S(X,Z)-(n-1)g(X,Z)\}\phi^2 Y.
\end{aligned}\tag{5.3}$$

Hence we have

Theorem 5.1. *Let M^n $(n>3)$ be a globally ϕ-quasiconformally symmetric LP-Sasakian manifold.*
(1) *If $a+(n-2)b\neq 0$, then M is of constant curvature* 1*.*
(2) *If $a+(n-2)b=0$, then M is a space satisfying* (5.3)*.*

From [11], we have
Theorem 5.2. *Let $M^n(n>3)$ be a complete simply connected LP-Sasakian manifold. If M is globally ϕ-quasiconformally symmetric and $a+(n-2)b\neq 0$,*

then M is isometric to the Lorentz sphere $\mathbb{S}^n_1(1)$.

Moreover, by virtue of (5.1) and Theorem 5.1, we find

$$a[(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi] = 0,$$

which implies that $\phi^2((\nabla_W R)(X,Y)Z) = 0$ if $a \neq 0$. Hence we can state:

Theorem 5.3. *A globally ϕ-quasiconformally symmetric LP-Sasakian manifold is globally ϕ-symmetric if $a \neq 0$.*

§6. 3-dimensional locally ϕ-quasiconformally symmetric LP-Sasakian manifolds

Let us consider a 3-dimensional LP-Sasakian manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6.1}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.
Putting $Z = \xi$ in (6.1) and using (2.10) we have

$$\eta(Y)QX - \eta(X)QY = \left(\frac{r}{2} - 1\right)[\eta(Y)X - \eta(X)Y]. \tag{6.2}$$

Putting $Y = \xi$ in (6.2) and using (2.1) and (2.9), we get

$$QX = \frac{1}{2}[(r-2)X + (r-6)\eta(X)\xi], \tag{6.3}$$

that is,

$$S(X,Y) = \frac{1}{2}[(r-2)g(X,Y) + (r-6)\eta(X)\eta(Y)]. \tag{6.4}$$

Using (6.3) in (6.1), we get

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r-4}{2}\right)[g(Y,Z)X - g(X,Z)Y] \\ &\quad + \left(\frac{r-6}{2}\right)[g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{6.5}$$

Putting (6.3), (6.4) and (6.5) into (1.1) we have

$$\begin{aligned}\widetilde{C}(X,Y)Z &= (a+b)(r-6)\Big[\frac{1}{3}\{g(Y,Z)X-g(X,Z)Y\}\\ &+\frac{1}{2}\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\ &+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\Big].\end{aligned}\tag{6.6}$$

Thus we have

Lemma 6.1. *Let M be a 3-dimensional LP-Sasakian manifold. If $a+b=0$ or $r=6$, then the quasi-conformal curvature tensor vanishes identically.*

Next, we assume that $a+b\neq 0$ or $r\neq 6$. Taking the covariant differentiation of (6.6), we get

$$\begin{aligned}(\nabla_W\widetilde{C})(X,Y)Z &= \frac{dr(W)}{3}(a+b)\{g(Y,Z)X-g(X,Z)Y\}\\ &+\frac{dr(W)}{2}(a+b)\{g(Y,Z)\eta(X)\xi\\ &-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\}\\ &+\frac{1}{2}(r-6)(a+b)[\{g(Y,Z)g(X,\phi W)\\ &-g(X,Z)g(Y,\phi W)\}\xi\\ &+\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\phi W\\ &+\{\eta(Z)g(Y,\phi W)+\eta(Y)g(Z,\phi W)\}X\\ &-\{\eta(Z)g(X,\phi W)+\eta(X)g(Z,\phi W)\}Y].\end{aligned}$$

Operating ϕ^2 to the above equation, then we find

$$\begin{aligned}\phi^2((\nabla_W\widetilde{C})(X,Y)Z) &= \frac{dr(W)}{3}(a+b)\{g(Y,Z)\phi^2X-g(X,Z)\phi^2Y\}\\ &+\frac{dr(W)}{2}(a+b)\eta(Z)\{\eta(Y)\phi^2X-\eta(X)\phi^2Y\}\\ &+\frac{1}{2}(r-6)(a+b)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\phi W\\ &+\{\eta(Z)g(Y,\phi W)+\eta(Y)g(Z,\phi W)\}\phi^2X\\ &-\{\eta(Z)g(X,\phi W)+\eta(X)g(Z,\phi W)\}\phi^2Y].\end{aligned}$$

If the vector fields X, Y and Z are horizontal, then the above equation is rewritten as follows:

$$\phi^2((\nabla_W\widetilde{C})(X,Y)Z)=\frac{dr(W)}{3}(a+b)\{g(Y,Z)X-g(X,Z)Y\}.$$

Hence we conclude the following theorem:

Theorem 6.1. *A 3-dimensional LP-Sasakian manifold is locally ϕ-quasiconformally symmetric if and only if the scalar curvature r is constant if $a+b \neq 0$ and $r \neq 6$.*

§7. Example

We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3_1\}$, where (x, y, z) are standard coordinates of $\mathbb{R}^3_1$.

The vector fields

$$e_1 = e^z \frac{\partial}{\partial y}, \quad e_2 = e^z \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1,$$
$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any vector field $Z \varepsilon \chi(M)$.

Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(e_1) = -e_1, \quad \phi(e_2) = -e_2, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = -1,$$
$$\phi^2 Z = Z + \eta(Z)e_3,$$
$$g(\phi Z, \phi W) = g(Z, W) + \eta(Z)\eta(W)$$

for any vector fields $Z, W \in \chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = -e_1$$

and

$$[e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\begin{array}{lll} \nabla_{e_1} e_1 = -e_3, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = -e_1, \\ \nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_3 = -e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. With the help of the above results it can be easily verified that

$$\begin{array}{lll} R(e_1,e_2)e_3 = 0, & R(e_2,e_3)e_3 = -e_2, & R(e_1,e_3)e_3 = -e_1, \\ R(e_1,e_2)e_2 = e_1, & R(e_2,e_3)e_2 = -e_3, & R(e_1,e_3)e_2 = 0, \\ R(e_1,e_2)e_1 = -e_2, & R(e_2,e_3)e_1 = 0, & R(e_1,e_3)e_1 = -e_3. \end{array}$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1,e_1) &= g(R(e_1,e_2)e_2,e_1) - g(R(e_1,e_3)e_3,e_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(e_2,e_2) = 2,\ S(e_3,e_3) = -2$$

and

$$S(e_i,e_j) = 0 (i \neq j).$$

Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) - S(e_3,e_3) = 6.$$

Thus the 3-dimensional LP-Sasakian manifold is quasi-conformally flat. Therefore Lemma 6.1. is verified.

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