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$\mathcal{R}(p,q)$-calculus: differentiation and integration

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Abstract. We build a framework for $\mathcal{R}(p,q)$-deformed calculus, which provides a method of computation for deformed $\mathcal{R}(p,q)$-derivative and integration, generalizing known deformed derivatives and integrations of analytic functions defined on a complex disc as particular cases corresponding to conveniently chosen meromorphic functions. Under prescribed conditions, we define the $\mathcal{R}(p,q)$-derivative and integration. Relevant examples are also given.

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§1. Introduction

The twin basic factors or (p,q)-factors (also called (p,q)-factors in physics literature) were introduced in order to generalize or unify several forms of q-oscillator algebras well known in the earlier physics literature related to the representation theory of single parameter quantum algebras [3, 13]. In Ref. [5], the authors examined the relation between the representation theory of a two-parameter deformation of the oscillator algebra and certain bibasic Laguerre functions and polynomials. Burban and Klimyk [1] investigated in detail the (p,q)-differentiation, (p,q)-integration, and the (p,q)-hypergeometric series. Gelfand *et al.*, in Refs.[7, 8], generalized the two-parameter deformed derivative and developed a very general theory of deformation of classical hypergeometric functions.

Generalizing the definition of deformed hypergeometric functions by Burban and Klimyk [1], Jagannathan and Srinivasa Rao[12] gave a method to embed the q-series by Gasper and Rahman[6], in a (p,q)-series and derived the corresponding (p,q)-extensions of q-identities. In the same way, Hounkonnou and Ngompe[10] and Burban[2] defined the (p,q,μ,ν)-derivative and derived the related deformed hypergeometric series and associated Hopf algebra structures.

In our previous paper [9], based on the K-derivative developed by Odzijewicz in a nice, mathematically based work published in 1998 [15], but unfortunately hushed up in the recent literature on the topic, we introduced the $\mathcal{R}(p,q)$-derivative relative to a meromorphic function defined on a bidisc and derived generalized deformed factors, deformed factorials and deformed exponential functions.

The aim of this paper is to develop a deformed calculus for $\mathcal{R}(p,q)$-deformations, especially the differential and integration calculi.

The paper is organized as follows. We first recall in Section 2 the definition of the $\mathcal{R}(p,q)$-factors and their associated quantum algebra. We introduce a new algebra generated by four quantities provided some conditions are satisfied. The Section 3 is devoted to the definition of the $\mathcal{R}(p,q)$-differential calculus yielding the $\mathcal{R}(p,q)$-integration. In Section 4 we show that some particular cases can be deduced from the constructed general formalism. Then follow some concluding remarks in Section 5.

§2. $\mathcal{R}(p,q)$-factors and their associated quantum algebras

In our previous paper [9] we have built the $\mathcal{R}(p,q)$-factors which are a generalization of Heine q-factors (also called Heine q-number in physics literature)

$$[n]_q = \frac{1-q^n}{1-q}, \qquad n = 0,1,2,\cdots \tag{2.1}$$

and Jagannathan-Srinivasa (p,q)-factors [12]

$$[n]_{p,q} = \frac{p^n-q^n}{p-q}, \qquad n = 0,1,2,\cdots \tag{2.2}$$

as follows. Let p and q be two positive real numbers such that $0 < q < p \leq 1$. Consider a meromorphic function $\mathcal{R}$, defined on $\mathbb{C}\times\mathbb{C}$ by

$$\mathcal{R}(x,y) = \sum_{k,l=-L}^{\infty} r_{kl}x^k y^l \tag{2.3}$$

with an eventual isolated singularity at the zero, where r_{kl} are complex numbers, $L \in \mathbb{N}\cup\{0\}$, $\mathcal{R}(p^n,q^n) > 0\ \forall n \in \mathbb{N}$, and $\mathcal{R}(1,1) = 0$ by definition. Then, the $\mathcal{R}(p,q)$-factors denoted by $\mathcal{R}(p^n,q^n)$, $n = 0,1,2,\cdots$ are used to deduce the $\mathcal{R}(p,q)$-factorial

$$\mathcal{R}!(p^n,q^n) = \begin{cases} 1 & \text{for} \quad n=0 \\ \mathcal{R}(p,q)\cdots\mathcal{R}(p^n,q^n) & \text{for} \quad n\geq 1, \end{cases} \tag{2.4}$$

the $\mathcal{R}(p,q)$-binomial coefficient

$$\left[\begin{array}{c} m \\ n \end{array}\right]_{\mathcal{R}(p,q)} = \frac{\mathcal{R}!(p^m,q^m)}{\mathcal{R}!(p^n,q^n)\mathcal{R}!(p^{m-n},q^{m-n})}, \quad m,n=0,1,2,\cdots, \quad m\geq n, \tag{2.5}$$

and the $\mathcal{R}(p,q)$-exponential function

$$\mathrm{Exp}_{\mathcal{R}(p,q)}(z) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{R}!(p^n,q^n)} z^n. \tag{2.6}$$

Denote by $\mathbb{D}_R = \{z\in\mathbb{C} : |z| < R\}$ a complex disc and by $\mathcal{O}(\mathbb{D}_R)$ the set of holomorphic functions defined on $\mathbb{D}_R$, where R is the radius of convergence of the series (2.6).

We then define the following linear operators on $\mathcal{O}(\mathbb{D}_R)$ by (see [9] and references therein):

$$\begin{aligned} Q &: \varphi \longmapsto Q\varphi(z) = \varphi(qz), \\ P &: \varphi \longmapsto P\varphi(z) = \varphi(pz), \\ \partial_{p,q} &: \varphi \longmapsto \partial_{p,q}\varphi(z) = \frac{\varphi(pz)-\varphi(qz)}{z(p-q)}, \end{aligned} \tag{2.7}$$

$\varphi\in\mathcal{O}(\mathbb{D}_R)$, $0<q<p\leq 1$, and the $\mathcal{R}(p,q)$-derivative by

$$\partial_{\mathcal{R}(p,q)} := \partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q) = \frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\partial_{p,q}. \tag{2.8}$$

Note that the $\mathcal{R}(p,q)$-exponential function is invariant under the action of the $\mathcal{R}(p,q)$-derivative since

$$\partial_{\mathcal{R}(p,q)} z^n = \begin{cases} 0 & \text{for} \quad n=0 \\ \mathcal{R}(p^n,q^n)z^{n-1} & \text{for} \quad n\geq 1. \end{cases} \tag{2.9}$$

In [9], we also studied the $\mathcal{R}(p,q)$-deformed quantum algebra generated by the set of operators $\{1, A, A^\dagger, N\}$ and the commutation relations

$$[N,A] = -A \quad \text{and} \quad \left[N, A^\dagger\right] = A^\dagger \tag{2.10}$$

with

$$AA^\dagger = \mathcal{R}(p^{N+1},q^{N+1}), \quad \text{and} \quad A^\dagger A = \mathcal{R}(p^N,q^N). \tag{2.11}$$

This algebra is defined on $\mathcal{O}(\mathbb{D}_R)$ as:

$$A^\dagger := z, \qquad A := \partial_{\mathcal{R}(p,q)}, \qquad N := z\partial_z, \tag{2.12}$$

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$. Therefore, the following holds:

Proposition 1.

$$P = p^{z\partial z}, \qquad Q = q^{z\partial z} \tag{2.13}$$

and the algebra $\mathcal{A}_{\mathcal{R}(p,q)}$ *generated by* $\{1,\ z,\ z\partial_z,\ \partial_{\mathcal{R}(p,q)}\}$ *satisfies the relations:*

$$\begin{array}{ll} z\,\partial_{\mathcal{R}(p,q)} = \mathcal{R}(P,Q), & \partial_{\mathcal{R}(p,q)}\ z = \mathcal{R}(pP,qQ), \\ [z\partial_z,\ z] = z, & [z\partial_z,\ \partial_{\mathcal{R}(p,q)}] = -\partial_{\mathcal{R}(p,q)}. \end{array} \tag{2.14}$$

Proposition 2. *If there exist two functions* Ψ_1 *and* $\Psi_2 : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ *such that*

$$\Psi_i(p,q) > 0 \quad \textit{for} \ \ i = 1,\ 2 \tag{2.15}$$

(2.16)
$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{\mathcal{R}(p,q)} = \Psi_1^k(p,q)\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)} + \Psi_2^{n+1-k}(p,q)\left[\begin{array}{c} n \\ k-1 \end{array}\right]_{\mathcal{R}(p,q)},$$

(2.17)
$$ba = \Psi_1(p,q)ab,\ xy = \Psi_2(p,q)yx,\ \textit{and}\ [i,\ j] = 0\ \textit{for}\ i\in\{a,b\}, j\in\{x,y\}$$

for some algebra elements a, b, x, y, *then*

$$(ax+by)^n = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{n-k}b^k y^k x^{n-k}. \tag{2.18}$$

Proof. By induction over n. Indeed, the equality (2.18) holds for $n = 1$ since

$$\begin{aligned}(ax+by)^1 = ax+by &= \left[\begin{array}{c} 1 \\ 0 \end{array}\right]_{\mathcal{R}(p,q)} a^1b^0y^0x^2 + \left[\begin{array}{c} 1 \\ 1 \end{array}\right]_{\mathcal{R}(p,q)} a^0b^1y^1x^0 \\ &= \sum_{k=0}^{1}\left[\begin{array}{c} 1 \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{1-k}b^ky^kx^{1-k}.\end{aligned}$$

Suppose that the equality (2.18) holds for $n \le m$, this means in particular for $n = m$,

$$(ax+by)^m = \sum_{k=0}^{m}\left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m-k}b^ky^kx^{m-k}, \tag{2.19}$$

and let us prove that it remains valid for $n = m+1$. Indeed,

$$\begin{aligned}
(ax+by)^{m+1} &= (ax+by)^m(ax+by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m-k}b^k y^k x^{m-k}(ax+by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m-k}b^k y^k x^{m-k}ax + \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m-k}b^k y^k x^{m-k}by \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} \Psi_1^k(p,q)a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} \Psi_2^{m-k}(p,q)a^{m-k}b^{k+1} y^{k+1} x^{m-k} \\
&= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} \Psi_1^k(p,q)a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m-1} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} \Psi_2^{m-k}(p,q)a^{m-k}b^{k+1} y^{k+1} x^{m-k} + b^{m+1}y^{m+1} \\
&= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \Psi_1^k(p,q) \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=1}^{m} \Psi_2^{m+1-k}(p,q) \left[\begin{array}{c} m \\ k-1 \end{array}\right]_{\mathcal{R}(p,q)} a^{m+1-k}b^k y^k x^{m+1-k} + b^{m+1}y^{m+1} \\
&= a^{m+1}x^{m+1} + b^{m+1}y^{m+1} + \sum_{k=1}^{m} \Bigg(\Psi_1^k(p,q) \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} \\
&\quad + \Psi_2^{m+1-k}(p,q) \left[\begin{array}{c} m \\ k-1 \end{array}\right]_{\mathcal{R}(p,q)} \Bigg) a^{m+1-k}b^k y^k x^{m+1-k} \\
&= a^{m+1}x^{m+1} + b^{m+1}y^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{\mathcal{R}(p,q)} a^{m+1-k}b^k y^k x^{m+1-k}.
\end{aligned}$$

□

§3. $\mathcal{R}(p,q)$-differential and integration calculi

3.1. Differential calculus

We define a linear operator $d_{\mathcal{R}(p,q)}$ on $\mathcal{A}_{\mathcal{R}(p,q)}$ by

$$d_{\mathcal{R}(p,q)} = (dz)\partial_{\mathcal{R}(p,q)}. \tag{3.1}$$

It follows that

$$\begin{aligned} &d_{\mathcal{R}(p,q)}1 = 0, \quad d_{\mathcal{R}(p,q)}z = (dz)\mathcal{R}(p,q), \quad d_{\mathcal{R}(p,q)}\partial_{\mathcal{R}(p,q)} = (dz)\partial^2_{\mathcal{R}(p,q)} \\ &d_{\mathcal{R}(p,q)}(z\partial_z) = (dz)(z\partial_z + 1)\partial_{\mathcal{R}(p,q)} \qquad \text{and} \quad d^2_{\mathcal{R}(p,q)} = 0. \end{aligned} \tag{3.2}$$

Hence, the set of 0-forms $\Omega^0(\mathcal{A}_{\mathcal{R}(p,q)})$ is naturally $\mathcal{A}_{\mathcal{R}(p,q)}$, while a 1-form ω, element of $\Omega^1(\mathcal{A}_{\mathcal{R}(p,q)})$, is given by

$$\omega = (dz)\omega_0(z, z\partial_z, \partial_{\mathcal{R}(p,q)}), \tag{3.3}$$

where $\omega_0(z, z\partial_z, \partial_{\mathcal{R}(p,q)}) = \sum_{i,j,k=0}^{\infty} \alpha_{ijk}(z)^i(z\partial_z)^j(\partial_{\mathcal{R}(p,q)})^k$ with α_{ijk} belonging to $\mathbb{C}$. Therefore, $d\omega = 0$ for $\omega \in \Omega^1(\mathcal{A}_{\mathcal{R}(p,q)})$.

Proposition 3. *For a nonnegative integer n, the following equalities hold:*

$$\begin{aligned} &d_{\mathcal{R}(p,q)}(z^n) = (dz)\mathcal{R}(p^n, q^n)z^{n-1}, \\ &d_{\mathcal{R}(p,q)}(z\partial_z)^n = (dz)(z\partial_z + 1)^n\partial_{\mathcal{R}(p,q)}, \\ &d_{\mathcal{R}(p,q)}(\partial^n_{\mathcal{R}(p,q)}) = (dz)\partial^{n+1}_{\mathcal{R}(p,q)}. \end{aligned} \tag{3.4}$$

Moreover if $f \in \mathcal{O}(\mathbb{D}_R)$ then

$$d_{\mathcal{R}(p,q)}f(z) = (dz)\partial_{\mathcal{R}(p,q)}f(z). \tag{3.5}$$

Proof. The equalities in (3.4) follow from the definition of the $\mathcal{R}(p,q)$-derivative (2.8), the commutation relations (2.14) and the definition of the differential (3.1). Then, (3.5) follows by definition (2.8). □

Proposition 4. *The differential $d_{\mathcal{R}(p,q)}$ obeys the two following equivalent Leibniz rules*

$$d_{\mathcal{R}(p,q)}(fg) = (dz)\frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\left\{\partial_{p,q}(f))(Pg) + (Qf)(\partial_{p,q}(g))\right\}, \tag{3.6}$$

$$d_{\mathcal{R}(p,q)}(fg) = (dz)\frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\left\{(\partial_{p,q}(f))(Qg) + (Pf)(\partial_{p,q}(g))\right\} \tag{3.7}$$

for $f, g \in \mathcal{O}(\mathbb{D}_R)$.

Proof. This follows from

$$\partial_{p,q}(fg) = (\partial_{p,q}(f))(Qg) + (Pf)(\partial_{p,q}(f) = (\partial_{p,q}(f)(Pg) + (Qf)(\partial_{p,q}(g)).$$

□

3.2. $\mathcal{R}(p,q)$-integration

We define the operator $\mathcal{I}_{\mathcal{R}(p,q)}$ over $\mathcal{O}(\mathbb{D}_R)$ as the inverse image of the $\mathcal{R}(p,q)$-derivative. For elements z^n of the basis of $\mathcal{O}(\mathbb{D}_R)$, $\mathcal{I}_{\mathcal{R}(p,q)}$ acts as follows:

$$\mathcal{I}_{\mathcal{R}(p,q)} z^n := \left(\partial_{\mathcal{R}(p,q)}\right)^{-1} z^n = \frac{1}{\mathcal{R}(p^{n+1},q^{n+1})} z^{n+1} + c, \tag{3.8}$$

where $n \geq 0$ and c is an integration constant. Hence, if $f \in \mathcal{O}(\mathbb{D}_R)$ then

$$\mathcal{I}_{\mathcal{R}(p,q)}\, \partial_{\mathcal{R}(p,q)} f(z) = f(z) + c \quad \text{and} \quad \partial_{\mathcal{R}(p,q)}\, \mathcal{I}_{\mathcal{R}(p,q)} f(z) = f(z) + c', \tag{3.9}$$

where c and c' are integration constants.

Provided that $\mathcal{R}(P,Q)$ is invertible, one can define the $\mathcal{R}(p,q)$-integration by the following formula

$$\mathcal{I}_{\mathcal{R}(p,q)} = \mathcal{R}^{-1}(P,Q)\, z, \tag{3.10}$$

with $c = c' = 0$.

One can also derive the definite integrals:

$$\int_\alpha^\beta f(z) d_{\mathcal{R}(p,q)} z = \mathcal{I}_{\mathcal{R}(p,q)} f(\beta) - \mathcal{I}_{\mathcal{R}(p,q)} f(\alpha), \quad \alpha, \beta \in \mathbb{D}_R; \tag{3.11}$$

$$\int_\alpha^{+\infty} f(z) d_{\mathcal{R}(p,q)} z = \lim_{n\to\infty} \int_\alpha^{p^n/q^n} f(z) d_{\mathcal{R}(p,q)} z; \tag{3.12}$$

$$\int_{-\infty}^{+\infty} f(z) d_{\mathcal{R}(p,q)} z = \lim_{n\to\infty} \int_{-p^n/q^n}^{p^n/q^n} f(z) d_{\mathcal{R}(p,q)} z. \tag{3.13}$$

Moreover, the Eqs. (3.6) and (3.7) lead to the following formulae:

$$\begin{aligned} \mathcal{I}_{\mathcal{R}(p,q)}\, & \partial_{\mathcal{R}(p,q)} (f(z)g(z)) \\ &= f(z)g(z) + c \\ &= \mathcal{I}_{\mathcal{R}(p,q)} \left\{ \frac{p-q}{pP-qQ} \mathcal{R}(pP,qQ)(\partial_{p,q}(f))(Pg) \right\} \\ &\quad + \mathcal{I}_{\mathcal{R}(p,q)}\, \frac{p-q}{pP-qQ} \mathcal{R}(pP,qQ) \left\{ (Qf)(\partial_{p,q}(g)) \right\} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \mathcal{I}_{\mathcal{R}(p,q)}\, & \partial_{\mathcal{R}(p,q)} (f(z)g(z)) \\ &= f(z)g(z) + c \\ &= \mathcal{I}_{\mathcal{R}(p,q)} \left\{ \frac{p-q}{pP-qQ} \mathcal{R}(pP,qQ)(\partial_{p,q}(f))(Qg) \right\} \\ &\quad + \mathcal{I}_{\mathcal{R}(p,q)}\, \frac{p-q}{pP-qQ} \mathcal{R}(pP,qQ) \left\{ (Pf)(\partial_{p,q}(g)) \right\}, \end{aligned} \tag{3.15}$$

respectively. These relation can be viewed as formulae of integration by parts.

§4. Relevant particular cases

Let us now apply the above general formalism to particular deformed algebras, well spred in the literature.

4.1. Jagannathan-Srinivasa deformation

A. Taking $\mathcal{R}(x,y)=\frac{x-y}{p-q}$, we obtain the Jagannathan-Srinivasa (p,q)- factors and (p,q)-factorials

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},$$

and

$$[n]!_{p,q}=\begin{cases}1 \quad \text{for} \quad n=0\\ \frac{((p,q);(p,q))_n}{(p-q)^n} \quad \text{for} \quad n\geq 1,\end{cases} \tag{4.1}$$

respectively.

Referring the readers to [12] for details on (p,q)-calculus, let us restrict the present description to some new relevant properties.

Proposition 5. *If n and m are nonnegative integers, then*

$$\begin{aligned}
[n]_{p,q} &= \sum_{k=0}^{n-1} p^{n-1-k}q^k,\\
[n+m]_{p,q} &= q^m[n]_{p,q}+p^n[m]_{p,q}\\
&= p^m[n]_{p,q}+q^n[m]_{p,q},\\
[-m]_{p,q} &= -q^{-m}p^{-m}[m]_{p,q},\\
[n-m]_{p,q} &= q^{-m}[n]_{p,q}-q^{-m}p^{n-m}[m]_{p,q}\\
&= p^{-m}[n]_{p,q}-q^{n-m}p^{-m}[m]_{p,q},\\
[n]_{p,q} &= [2]_{p,q}[n-1]_{p,q}-pq[n-2]_{p,q}.
\end{aligned} \tag{4.2}$$

Proposition 6. *The (p,q)-binomial coefficients*

$$\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q}=\frac{((p,q);(p,q))_n}{((p,q);(p,q))_k((p,q);(p,q))_{n-k}}, \quad 0\leq k\leq n;\ \ n\in\mathbb{N}, \tag{4.3}$$

where $((p,q);(p,q))_m=(p-q)(p^2-q^2)\cdots(p^m-q^m)$, $m\in\mathbb{N}$ satisfy the following identities

$$\begin{aligned}
\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q} &= \left[\begin{array}{c} n\\ n-k\end{array}\right]_{p,q}\\
&= p^{k(n-k)}\left[\begin{array}{c} n\\ k\end{array}\right]_{q/p}=p^{k(n-k)}\left[\begin{array}{c} n\\ n-k\end{array}\right]_{q/p},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}(4.5)\quad \left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q} &= p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} + q^{n+1-k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q}, \\ (4.6)\quad \left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q} &= p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} + p^{n+1-k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q} \\ &\quad -(p^n - q^n) \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q}\end{aligned}$$

with

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{q/p} = \frac{(q/p;q/p)_n}{(q/p;q/p)_k (q/p;q/p)_{n-k}},$$

where $(q/p;q/p)_n = (1-q/p)(1-q^2/p^2)\cdots(1-q^n/p^n)$ *and the* (p,q)*-shifted factorial*

$$\begin{aligned}((a,b);(p,q))_n &\equiv (a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1}) \\ &= \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} (-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k.\end{aligned}$$

Proposition 7. *If the quantities* x*,* y*,* a *and* b *are such that* $xy = qyx$*,* $ba = pab$*,* $[i,\ j] = 0$ *for* $i \in \{a,b\}$ *and* $j \in \{x,y\}$*, and, moreover,* p *and* q *commute with each element of the set* $\{a,b,x,y\}$*, then*

$$(4.7)\qquad (ax+by)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k} b^k y^k x^{n-k}.$$

The latter result is a generalization of noncommutative form of the q-binomial theorem [6], which can be obtained setting a, b and p equal to 1, i.e.

$$(4.8)\qquad (x+y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q} y^k x^{n-k},$$

where

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{q} = (q;q)_n/(q;q)_k (q;q)_{n-k},$$

with $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

Proof. An alternative proof has been also proposed in [12]. Here we provide another one by induction on n. Indeed, the result is true for

$n = 1$. Suppose it remains valid for all $n \leq m$ and prove that this is also true for $n = m + 1$:

$$\begin{aligned}
&(ax+by)^{m+1} \\
&\quad = (ax+by)^m(ax+by) \\
&\quad = \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} a^{m-k}b^k y^k x^{m-k}(ax+by) \\
&\quad = \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} p^k a^{m+1-k}b^k y^k x^{m+1-k} \\
&\qquad + \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} q^{m-k} a^{m-k}b^{k+1} y^{k+1} x^{m-k} \\
&\quad = a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} p^k a^{m+1-k}b^k y^k x^{m+1-k} \\
&\qquad + \sum_{k=0}^{m-1} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} q^{m-k} a^{m-k}b^{k+1} y^{k+1} x^{m-k} + b^{m+1}y^{m+1} \\
&\quad = a^{m+1}x^{m+1} + b^{m+1}y^{m+1} \\
&\qquad + \sum_{k=1}^{m} \left(p^k \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q} q^{m+1-k} \left[\begin{array}{c} m \\ k-1 \end{array}\right]_{p,q} \right) a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad = a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m+1 \\ k \end{array}\right]_{p,q} a^{m+1-k}b^k y^k x^{m+1-k} + b^{m+1}y^{m+1} \\
&\quad = \sum_{k=0}^{m+1} \left[\begin{array}{c} m+1 \\ k \end{array}\right]_{p,q} a^{m+1-k}b^k y^k x^{m+1-k},
\end{aligned}$$

where the use of (4.5) has been made. Hence the result is true for all $n \in \mathbb{N}$. □

The $\mathcal{R}(p,q)$-derivative is thus reduced to the (p,q)-derivative [12]

$$\partial_{p,q} = \frac{1}{(p-q)z}(P-Q), \tag{4.9}$$

namely, for $f \in \mathcal{O}(\mathbb{D}_R)$,

$$\partial_{p,q} f(z) = \frac{f(pz) - f(qz)}{z(p-q)}. \tag{4.10}$$

The associated algebra $\mathcal{A}_{p,q}$, generated by $\{1,\ A,\ A^\dagger,\ N\}$, satisfies the relations:

$$A\,A^\dagger - pA^\dagger A = q^N, \qquad A\,A^\dagger - qA^\dagger A = p^N, \tag{4.11}$$

$$[N,\ A^\dagger] = A^\dagger, \qquad [N,\ A] = -A, \tag{4.12}$$

and its realization on $\mathcal{O}(\mathbb{D}_R)$, engendered by $\{1,\ z,\ z\partial_z,\ \partial_{p,q}\}$, satisfies the relations

$$\begin{aligned} &z\,\partial_{p,q} - p\,\partial_{p,q}\,z = q^{z\partial_z}, \quad z\,\partial_{p,q} - q\,\partial_{p,q}\,z = p^{z\partial_z}, \\ &[z\partial_z,\ z] = z, \qquad [z\partial_z,\ \partial_{p,q}] = -\partial_{p,q}. \end{aligned} \tag{4.13}$$

Therefore, the differential operator $d_{p,q}$ is then given by

$$d_{p,q} = (dz)\frac{1}{(p-q)z}(P-Q) \tag{4.14}$$

with the following properties:

$$\begin{aligned} &d_{p,q}1 = 0, \qquad d_{p,q}z = (dz), \qquad d_{p,q}\partial_{p,q} = (dz)\partial^2_{p,q}, \\ &d_{p,q}(z\partial_z) = (dz)(z\partial_z + 1)\partial_{p,q} \qquad \text{and} \quad d^2_{p,q} = 0. \end{aligned} \tag{4.15}$$

The differential of $f \in \mathcal{O}(\mathbb{D}_R)$ is then

$$d_{p,q}f(z) = (dz)\frac{f(pz) - f(qz)}{(p-q)z} \tag{4.16}$$

affording the Leibniz rule

$$\begin{aligned} d_{p,q}(fg)(z) &= (dz)\frac{f(pz) - f(qz)}{(p-q)z}g(qz) \\ &\quad + (dz)f(pz)\frac{g(pz) - g(qz)}{(p-q)z} \\ &= \{d_{p,q}f(z)\} \cdot g(qz) + f(pz) \cdot d_{p,q}g(z) \end{aligned} \tag{4.17}$$

or, equivalently,

$$\begin{aligned} d_{p,q}(fg)(z) &= (dz)\frac{f(pz) - f(qz)}{(p-q)z}g(pz) \\ &\quad + (dz)f(qz)\frac{g(pz) - g(qz)}{(p-q)z} \\ &= \{d_{p,q}f(z)\} \cdot g(pz) + f(qz) \cdot d_{p,q}g(z). \end{aligned} \tag{4.18}$$

The (p,q)-integration is obtained from (3.10) as follows:

$$\mathcal{I}_{p,q}f(z) = \frac{p-q}{P-Q}zf(z) = (p-q)\sum_{\nu=0}^{\infty}\frac{Q^\nu}{P^{\nu+1}}zf(z) \tag{4.19}$$
$$= (p-q)z\sum_{\nu=0}^{\infty} f(zq^\nu/p^{\nu+1})q^\nu/p^{\nu+1}.$$

Setting $p=1$, one recovers the q-derivative and q-integral of Jackson [14].

B. Taking $\mathcal{R}(x,y) = \dfrac{x-y}{\frac{a}{p}x - \frac{b}{q}y}$, where $a,b\in\mathbb{C}$ with $a\neq b$, the $\mathcal{R}(p,q)$-factors are given by

$$\mathcal{R}(p^n,q^n) = [n]^{a,b}_{p,q} = \frac{p^n-q^n}{ap^{n-1}-bq^{n-1}}, \quad n = 0,1,\cdots. \tag{4.20}$$

The $\mathcal{R}(p,q)$-factorials become

$$[n]!^{a,b}_{p,q} = \begin{cases} 1 \quad \text{for} \quad n=0 \\ \dfrac{((p,q);(p,q))_n}{((a,b);(p,q))_n} \quad \text{for} \quad n\geq 1. \end{cases} \tag{4.21}$$

The derivative is now given by

$$\partial_{\mathcal{R}(p,q)} = \partial_{p,q}\frac{p-q}{P-Q}\frac{P-Q}{\frac{a}{p}P-\frac{b}{q}Q} = \frac{1}{z}\frac{P-Q}{\frac{a}{p}P-\frac{b}{q}Q}, \tag{4.22}$$

so that for $f\in\mathcal{O}(\mathbb{D}_R)$ we have

$$\partial_{\mathcal{R}(p,q)}f(z) = \frac{1}{z}\frac{P-Q}{\frac{a}{p}P-\frac{b}{q}Q}f(z) \tag{4.23}$$
$$= \frac{1}{z}(P-Q)\frac{p}{aP}\sum_{\nu=0}^{\infty}(bp/aq)^\nu(Q/P)^\nu f(z)$$
$$= \frac{p}{az}\sum_{\nu=0}^{\infty}(bp/aq)^\nu\left[(Q/P)^\nu - (Q/P)^{\nu+1}\right]f(z)$$
$$= \frac{p}{az}\sum_{\nu=0}^{\infty}(bp/aq)^\nu\left[f\left((q/p)^\nu z\right) - f\left((q/p)^{\nu+1}z\right)\right].$$

Moreover,

$$\mathcal{I}_{\mathcal{R}(p,q)} = \frac{\frac{a}{p}P-\frac{b}{q}Q}{P-Q}z. \tag{4.24}$$

Applying this to $f \in \mathcal{O}(\mathbb{D}_R)$ we obtain

$$
\begin{aligned}
(4.25)\quad \mathcal{I}_{\mathcal{R}(p,q)} f(z) &= \frac{\frac{a}{p}P - \frac{b}{q}Q}{P-Q} z f(z) \\
&= \left(\frac{a}{p}P - \frac{b}{q}Q\right)\frac{1}{P}\sum_{\nu=0}^{\infty}(Q/P)^{\nu} z f(z) \\
&= \left(\frac{a}{p} - \frac{b}{q}(Q/P)\right)\sum_{\nu=0}^{\infty}(Q/P)^{\nu} z f(z) \\
&= \sum_{\nu=0}^{\infty}\Big[(a/p)\,(Q/P)^{\nu} - (b/q)\,(Q/P)^{\nu+1}\Big] z f(z) \\
&= \sum_{\nu=0}^{\infty}[(a/p)(q/p)^{\nu} z f\left((q/q)^{\nu} z\right) \\
&\qquad -(b/q)(q/p)^{\nu+1} z f\left((q/p)^{\nu+1}\right)] \\
&= (z/p)\sum_{\nu=0}^{\infty}(q/p)^{\nu}\left[a f\left((q/q)^{\nu} z\right) - b f\left((q/p)^{\nu+1}\right)\right].
\end{aligned}
$$

4.2. Chakrabarty and Jagannathan deformation

The algebra of Chakrabarty and Jagannathan [3] can be obtained from our general formalism by taking $\mathcal{R}(x,y) = \frac{1-xy}{(p^{-1}-q)x}$. Indeed, the $\mathcal{R}(p,q)$-factors and $\mathcal{R}(p,q)$-factorials are reduced to (p^{-1},q)-factors and (p^{-1},q)-factorials, namely

$$[n]_{p^{-1},q} = \frac{p^{-n} - q^{n}}{p^{-1} - q},$$

and

$$
(4.26)\qquad [n]!_{p^{-1},q} = \begin{cases} 1 \quad \text{for} \quad n = 0 \\ \dfrac{((p^{-1},q);(p^{-1},q))_n}{(p^{-1}-q)^n} \quad \text{for} \quad n \geq 1, \end{cases}
$$

respectively. The properties of this deformation can be readily recovered from the previous section 4.1 by replacing the parameter p by p^{-1}.

The $\mathcal{R}(p,q)$-derivative is also reduced to (p^{-1},q)-derivative. Indeed,

$$
\begin{aligned}
(4.27)\qquad \partial_{\mathcal{R}(p,q)} &= \partial_{p,q}\frac{p-q}{P-Q}\frac{1-PQ}{(p^{-1}-q)P} \\
&= \frac{1}{(p^{-1}-q)z}(P^{-1}-Q) =: \partial_{p^{-1},q}.
\end{aligned}
$$

Therefore, for $f \in \mathcal{O}(\mathbb{D}_R)$

$$\partial_{p^{-1},q} f(z) = \frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} \tag{4.28}$$

and the differential of $f \in \mathcal{O}(\mathbb{D}_R)$ is given by

$$d_{p^{-1},q} f(z) = (dz)\frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)}. \tag{4.29}$$

Computing the Leibniz rule we get

$$\begin{aligned} d_{p^{-1},q}(fg)(z) &= (dz)\frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} g(qz) \\ &\quad + (dz) f(p^{-1}z)\frac{g(p^{-1}z) - g(qz)}{z(p^{-1} - q)} \\ &= \{d_{p^{-1},q} f(z)\} \cdot g(qz) + f(p^{-1}z) \cdot d_{p^{-1},q} g(z) \end{aligned} \tag{4.30}$$

or, equivalently,

$$\begin{aligned} d_{p^{-1},q}(fg)(z) &= (dz)\frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} g(p^{-1}z) \\ &\quad + (dz) f(qz)\frac{g(p^{-1}z) - g(qz)}{z(p^{-1} - q)} \\ &= \{d_{p^{-1},q} f(z)\} \cdot g(p^{-1}z) + f(qz) \cdot d_{p^{-1},q} g(z). \end{aligned} \tag{4.31}$$

We obtain from (3.10) the action of the (p^{-1}, q)-integration on $f \in \mathcal{O}(\mathbb{D}_R)$ as follows:

$$\begin{aligned} \mathcal{I}_{p^{-1},q} f(z) &= \frac{p^{-1} - q}{P^{-1} - Q} z f(z) = (p^{-1} - q)\sum_{\nu=0}^{\infty} Q^{\nu} P^{\nu+1} z f(z) \\ &= (1 - pq) z \sum_{\nu=0}^{\infty} f(zq^{\nu} p^{\nu+1})(pq)^{\nu}. \end{aligned} \tag{4.32}$$

4.3. Generalized q-Quesne deformation

The generalized Quesne algebra [10, 16] can be found by taking $\mathcal{R}(x, y) = \frac{xy-1}{(q-p^{-1})y}$. Indeed, the (p, q)-Quesne factors and factorials are given by

$$[n]^{Q}_{p,q} = \frac{p^n - q^{-n}}{q - p^{-1}},$$

and

$$(4.33)\qquad [n]^Q_{p,q}! = \begin{cases} 1 & \text{for} \quad n = 0 \\ \frac{((p,q^{-1});(p,q^{-1}))_n}{(q-p^{-1})^n} & \text{for} \quad n \geq 1, \end{cases}$$

respectively. There follow some relevant new properties:

Proposition 8. *If n and m are nonnegative integers, then*

$$(4.34)\qquad [-m]^Q_{p,q} = -p^{-m}q^m[m]^Q_{p,q},$$

$$(4.35)\qquad [n+m]^Q_{p,q} = q^{-m}[n]^Q_{p,q} + p^n[m]^Q_{p,q} = p^m[n]^Q_{p,q} + q^{-n}[m]^Q_{p,q},$$

$$(4.36)\qquad [n-m]^Q_{p,q} = q^m[n]^Q_{p,q} - p^{n-m}q^m[m]^Q_{p,q} = p^{-m}[n]^Q_{p,q} + p^{-m}q^{m-n}[m]^Q_{p,q},$$

$$(4.37)\qquad [n]^Q_{p,q} = \frac{q-p^{-1}}{p-q^{-1}}[2]^Q_{p,q}[n-1]^Q_{p,q} - pq^{-1}[n-2]^Q_{p,q}.$$

Proof. We obtain Eqs.(4.34) and (4.35) applying the relations

$$p^{-m} - q^m = -p^{-m}q^m(p^m - q^{-m})$$

and

$$\begin{aligned} p^{n+m} - q^{-n-m} &= q^{-m}(p^n - q^{-n}) + p^n(p^m - q^{-m}) \\ &= p^m(p^n - q^{-n}) + q^{-n}(p^m - q^{-m}), \end{aligned}$$

respectively. Eq.(4.36) follows combining Eqs.(4.34) and (4.35). Note that

$$(4.38)\qquad [n]_{p,q^{-1}} = \frac{p^n - q^{-n}}{p - q^{-1}} = \frac{q-p^{-1}}{p-q^{-1}}\frac{p^n - q^{-n}}{q - p^{-1}} = [n]^Q_{p,q}, \quad n = 1, 2, \cdots$$

which, combined with the following identity

$$[n]_{p,q-1} = [2]_{p,q-1}[n-1]_{p,q-1} - pq^{-1}[n-2]_{p,q^{-1}}$$

gives Eq.(4.37). □

Proposition 9. *The (p,q)-Quesne binomial coefficients*

$$(4.39)\qquad \left[\begin{array}{c} n \\ k \end{array}\right]^Q_{p,q} = \frac{((p,q^{-1});(p,q^{-1}))_n}{((p,q^{-1});(p,q^{-1}))_k((p,q^{-1});(p,q^{-1}))_{n-k}},$$

where $0 \leq k \leq n,\ n \in \mathbb{N}$, *satisfy the following properties:*

$$(4.40)\qquad \left[\begin{array}{c} n \\ k \end{array}\right]^Q_{p,q} = \left[\begin{array}{c} n \\ n-k \end{array}\right]^Q_{p,q} = p^{k(n-k)}\left[\begin{array}{c} n \\ k \end{array}\right]_{1/qp} = p^{k(n-k)}\left[\begin{array}{c} n \\ n-k \end{array}\right]_{1/qp},$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q}^{Q} = p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}^{Q} + q^{-n-1+k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q}^{Q}, \tag{4.41}$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]_{p,q}^{Q} = p^k \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}^{Q} + p^{n+1-k} \left[\begin{array}{c} n \\ k-1 \end{array}\right]_{p,q}^{Q} - (p^n - q^{-n}) \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q}^{Q}.$$

Proof. This is direct using the Proposition 5 and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}^{Q} = \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q^{-1}}. \tag{4.42}$$

□

Proposition 10. *If the quantities x, y, a and b are such that $xy = q^{-1}yx$, $ba = pab$, $[i,\ j] = 0$ for $i \in \{a, b\}$ and $j \in \{x, y\}$, and, moreover, p and q commute with each element of the set $\{a, b, x, y\}$, then*

$$(ax + by)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}^{Q} a^{n-k} b^k y^k x^{n-k}. \tag{4.43}$$

Proof. By induction on n. Indeed, the result is true for $n = 1$. Suppose it remains valid for $n \leq m$ and prove that this is also true for $n = m+1$:

$$\begin{aligned}
(ax + by)^{m+1} &= (ax + by)^m (ax + by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}^{Q} a^{m-k} b^k y^k x^{m-k} (ax + by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}^{Q} p^k a^{m+1-k} b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}^{Q} q^{-m+k} a^{m-k} b^{k+1} y^{k+1} x^{m-k} \\
&= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}^{Q} p^k a^{m+1-k} b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m-1} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q}^{Q} q^{-m+k} a^{m-k} b^{k+1} y^{k+1} x^{m-k} + b^{m+1} y^{m+1}
\end{aligned}$$

$$= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left(p^k \left[\begin{array}{c} m \\ k \end{array} \right]_{p,q}^{Q} \right.$$
$$\left. + q^{-m-1+k} \left[\begin{array}{c} m \\ k-1 \end{array} \right]_{p,q}^{Q} \right) a^{m+1-k} b^k y^k x^{m+1-k} + b^{m+1} y^{m+1}$$
$$= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m+1 \\ k \end{array} \right]_{p,q}^{Q} a^{m+1-k} b^k y^k x^{m+1-k}$$
$$+ b^{m+1} y^{m+1}$$
$$= \sum_{k=0}^{m+1} \left[\begin{array}{c} m+1 \\ k \end{array} \right]_{p,q}^{Q} a^{m+1-k} b^k y^k x^{m+1-k},$$

where the use of (4.41) has been made. Therefore the result is true for all $n \in \mathbb{N}$. □

The (p,q)-Quesne derivative is then given by

$$\partial_{p,q}^{Q} = \partial_{p,q} \frac{p-q}{P-Q} \frac{PQ-1}{(q-p^{-1})Q} = \frac{1}{(q-p^{-1})z}(P-Q^{-1}). \tag{4.44}$$

Therefore, for $f \in \mathcal{O}(\mathbb{D}_R)$

$$\partial_{p,q}^{Q} f(z) = \frac{f(pz) - f(q^{-1}z)}{z(q-p^{-1})} \tag{4.45}$$

and the differential is given by

$$d_{p,q}^{Q} f(z) = (dz) \frac{f(pz) - f(q^{-1}z)}{z(q-p^{-1})} \tag{4.46}$$

leading to the Leibniz rule

$$\begin{aligned} d_{p,q}^{Q}(fg)(z) &= (dz) \frac{f(pz) - f(q^{-1}z)}{z(q-p^{-1})} g(q^{-1}z) \\ &\quad + (dz) f(pz) \frac{g(pz) - g(q^{-1}z)}{z(q-p^{-1})} \\ &= \{d_{p,q}^{Q} f(z)\} \cdot g(q^{-1}z) + f(pz) \cdot d_{p,q}^{Q} g(z) \end{aligned} \tag{4.47}$$

or, equivalently,

$$\begin{aligned} d_{p,q}^{Q}(fg)(z) &= (dz) \frac{f(pz) - f(q^{-1}z)}{z(q-p^{-1})} g(pz) \\ &\quad + (dz) f(q^{-1}z) \frac{g(pz) - g(q^{-1}z)}{z(q-p^{-1})} \\ &= \{d_{p,q}^{Q} f(z)\} \cdot g(pz) + f(q^{-1}z) \cdot d_{p,q}^{Q} g(z). \end{aligned} \tag{4.48}$$

The action of the (p,q)-Quesne integration on $f \in \mathcal{O}(\mathbb{D}_R)$ is obtained from (3.10) as follows:

$$\mathcal{I}^{Q}_{p,q} f(z) = \frac{q-p^{-1}}{P-Q^{-1}} z f(z) = (p^{-1}-q)\sum_{\nu=0}^{\infty} P^{\nu} Q^{\nu+1} z f(z) \tag{4.49}$$
$$= (p^{-1}-q) z \sum_{\nu=0}^{\infty} f(zp^{\nu}q^{\nu+1}) p^{\nu} q^{\nu+1}.$$

4.4. $(p,q;\mu,\nu,h)$-deformation

The deformed Hounkonnou-Ngompe generalized algebra [11] can be obtained by taking

$$\mathcal{R}(x,y) = h(p,q)\frac{y^{\nu}}{x^{\mu}}\frac{xy-1}{(q-p^{-1})y},$$

such that $0 < pq < 1$, $p^{\mu} < q^{\nu-1}$, $p > 1$, and $h(p,q)$ is a well behaved real and non-negative function of deformation parameters p and q such that $h(p,q) \to 1$ as $(p,q) \to (1,1)$. Here the $\mathcal{R}(p,q)$-factors become $(p,q;\mu,\nu,h)$-factors, namely

$$[n]^{\mu,\nu}_{p,q,h} = h(p,q)\frac{q^{\nu n}}{p^{\mu n}}\frac{p^n - q^{-n}}{q-p^{-1}}. \tag{4.50}$$

Proposition 11. *The $(p,q;\mu,\nu,h)$-factors verify the following properties, for $m,n \in \mathbb{N}$:*

$$[-m]^{\mu,\nu}_{p,q,h} = -\frac{q^{-2\nu m+m}}{p^{-2\mu m+m}}[m]^{\mu,\nu}_{p,q,h}, \tag{4.51}$$

$$[n+m]^{\mu,\nu}_{p,q,h} = \frac{q^{\nu m-m}}{p^{\mu m}}[n]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu n}}{p^{\mu n-n}}[m]^{\mu,\nu}_{p,q,h} \tag{4.52}$$
$$= \frac{q^{\nu m}}{p^{\mu m-m}}[n]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu n-n}}{p^{\mu n}}[m]^{\mu,\nu}_{p,q,h}\ ,$$

$$[n-m]^{\mu,\nu}_{p,q,h} = \frac{q^{-\nu m+m}}{p^{-\mu m}}[n]^{\mu,\nu}_{p,q,h} - \frac{q^{\nu(n-2m)+m}}{p^{\mu(n-2m)-n+m}}[m]^{\mu,\nu}_{p,q,h} \tag{4.53}$$
$$= \frac{q^{-\nu m}}{p^{-\mu m+m}}[n]^{\mu,\nu}_{p,q,h} - \frac{q^{\nu(n-2m)-n+m}}{p^{\mu(n-2m)+m}}[m]^{\mu,\nu}_{p,q,h},$$

$$[n]^{\mu,\nu}_{p,q,h} = \frac{q-p^{-1}}{p-q^{-1}}\frac{q^{-\nu}}{p^{-\mu}}\frac{1}{h(p,q)}[2]^{\mu,\nu}_{p,q,h}[n-1]^{\mu,\nu}_{p,q,h} - \frac{q^{2\nu-1}}{p^{2\nu-1}}[n-2]^{\mu,\nu}_{p,q,h}. \tag{4.54}$$

Proof. This is direct using the Proposition 8 and the fact that

$$[n]^{\mu,\nu}_{p,q,h} = h(p,q)\frac{q^{\nu n}}{p^{\mu n}}[n]^{Q}_{p,q,h}. \tag{4.55}$$

□

Proposition 12. *he (p,q,μ,ν,h)-binomial coefficients*

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} := \frac{[n]^{\mu,\nu}_{p,q,h}!}{[k]^{\mu,\nu}_{p,q,h}![n-k]^{\mu,\nu}_{p,q,h}!} = \frac{q^{\nu k(n-k)}}{p^{\mu k(n-k)}}\left[\begin{array}{c} n \\ k \end{array}\right]^{Q}_{p,q}, \tag{4.56}$$

where $0 \le k \le n, \;\; n \in \mathbb{N}$, satisfy the following properties:

$$\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \left[\begin{array}{c} n \\ n-k \end{array}\right]^{\mu,\nu}_{p,q,h}, \tag{4.57}$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \frac{q^{\nu k}}{p^{(\mu-1)k}}\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} + \frac{q^{(\nu-1)(n+1-k)}}{p^{\mu(n+1-k)}}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h}, \tag{4.58}$$

$$\left[\begin{array}{c} n+1 \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} = \frac{q^{\nu k}}{p^{(\mu-1)k}}\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} + \frac{q^{\nu(n+1-k)}}{p^{(\mu-1)(n+1-k)}}\left[\begin{array}{c} n \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h} - (p^n - q^{-n})\frac{q^{\nu n}}{p^{\mu n}}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]^{\mu,\nu}_{p,q,h}. \tag{4.59}$$

Proof. This is direct using the Proposition 9 and the fact that

$$[n]^{\mu,\nu}_{p,q,h}! = h^n(p,q)\frac{q^{n(n+1)/2}}{p^{n(n+1)/2}}[n]^{Q}_{p,q}!, \tag{4.60}$$

where the use of Eq.(4.55) has been made. □

Proposition 13. *If the quantities x, y, a and b are such that $xy = \frac{q^{\nu-1}}{p^{\mu}}yx$, $ba = \frac{q^{\nu}}{p^{\mu-1}}ab$, $[i,\ j] = 0$ for $i \in \{a,b\}$ and $j \in \{x,y\}$, and, moreover, p and q commute with each element of the set $\{a,b,x,y\}$, then*

$$(ax+by)^n = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]^{\mu,\nu}_{p,q,h} a^{n-k}b^k y^k x^{n-k}. \tag{4.61}$$

Proof. By induction on n. Indeed, the result is true for $n = 1$. Suppose it remains valid for $n \leq m$ and prove that this is also true for $n = m + 1$:

$$\begin{aligned}
(ax+by)^{m+1} &= (ax+by)^m(ax+by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} a^{m-k}b^k y^k x^{m-k}(ax+by) \\
&= \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} \frac{q^{\nu k}}{q^{(\mu-1)k}} a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} \frac{q^{(\nu-1)(m-k)}}{p^{\mu(m-k)}} a^{m-k}b^{k+1} y^{k+1} x^{m-k} \\
&= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} \frac{q^{\nu k}}{q^{(\mu-1)k}} a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + \sum_{k=0}^{m-1} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} \frac{q^{(\nu-1)(m-k)}}{p^{\mu(m-k)}} a^{m-k}b^{k+1} y^{k+1} x^{m-k} \\
&\quad + b^{m+1}y^{m+1} \\
&= a^{m+1}x^{m+1} + b^{m+1}y^{m+1} + \sum_{k=1}^{m} \left(\frac{q^{\nu k}}{q^{(\mu-1)k}} \left[\begin{array}{c} m \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} \right. \\
&\quad \left. + \frac{q^{(\nu-1)(m+1-k)}}{p^{\mu(m+1-k)}} \left[\begin{array}{c} m \\ k-1 \end{array}\right]_{p,q,h}^{\mu,\nu} \right) a^{m+1-k}b^k y^k x^{m+1-k} \\
&= a^{m+1}x^{m+1} + \sum_{k=1}^{m} \left[\begin{array}{c} m+1 \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} a^{m+1-k}b^k y^k x^{m+1-k} \\
&\quad + b^{m+1}y^{m+1} \\
&= \sum_{k=0}^{m+1} \left[\begin{array}{c} m+1 \\ k \end{array}\right]_{p,q,h}^{\mu,\nu} a^{m+1-k}b^k y^k x^{m+1-k},
\end{aligned}$$

where the use of (4.58) has been made. Therefore, the result is true for all $n \in \mathbb{N}$. □

The $\mathcal{R}(p,q)$-derivative is then reduced to the $(p,q;\mu,\nu,h)$-derivative, given by

$$\begin{aligned}
\partial_{\mathcal{R}(p,q)} &= \partial_{p,q} \frac{p-q}{P-Q} h(p,q) \frac{Q^\nu}{P^\mu} \frac{PQ-1}{(q-p^{-1})Q} \\
&= \frac{h(p,q)}{(q-p^{-1})z} \frac{Q^\nu}{P^\mu}(P-Q^{-1}) \equiv \partial_{p,q,h}^{\mu,\nu}.
\end{aligned} \tag{4.62}$$

Therefore the $(p,q;\mu,\nu,h)$-derivative and the $(p,q;\mu,\nu,h)$-differential of $f\in\mathcal{O}(\mathbb{D}_R)$ are given by

$$\partial^{\mu,\nu}_{p,q,h}=h(p,q)\frac{f(zq^{\nu}/p^{\mu-1})-f(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}\tag{4.63}$$

and

$$d^{\mu,\nu}_{p,q,h}f(z)=(dz)h(p,q)\frac{f(zq^{\nu}/p^{\mu-1})-f(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}\tag{4.64}$$

respectively, with the Leibniz rule

$$\begin{aligned}d^{\mu,\nu}_{p,q,h}(fg)(z)&=(dz)h(p,q)\frac{f(zq^{\nu}/p^{\mu-1})-f(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}g(zq^{\nu-1}/p^{\mu})\\&\quad+(dz)f(zq^{\nu}/p^{\mu-1})h(p,q)\frac{g(zq^{\nu}/p^{\mu-1})-g(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}\\&=\{d^{\mu,\nu}_{p,q,h}f(z)\}\cdot g(zq^{\nu-1}/p^{\mu})+f(zq^{\nu}/p^{\mu-1})\cdot d^{\mu,\nu}_{p,q,h}g(z)\end{aligned}$$

which is equivalent to

$$\begin{aligned}d^{\mu,\nu}_{p,q,h}(fg)(z)&=(dz)h(p,q)\frac{f(zq^{\nu}/p^{\mu-1})-f(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}g(zq^{\nu}/p^{\mu-1})\\&\quad+(dz)f(zq^{\nu-1}/p^{\mu})h(p,q)\frac{g(zq^{\nu}/p^{\mu-1})-g(zq^{\nu-1}/p^{\mu})}{z(q-p^{-1})}\\&=\{d^{\mu,\nu}_{p,q,h}f(z)\}\cdot g(zq^{\nu}/p^{\mu-1})+f(zq^{\nu-1}/p^{\mu})\cdot d^{\mu,\nu}_{p,q,h}g(z).\end{aligned}$$

From (3.10) we obtain the action of the (p,q,μ,ν,h)-integration on $f\in\mathcal{O}(\mathbb{D}_R)$ as follows:

$$\begin{aligned}\mathcal{I}^{\mu,\nu}_{p,q,h}f(z)&=\frac{q-p^{-1}}{h(p,q)}\frac{P^{\mu}/Q^{\nu}}{P-Q^{-1}}zf(z)=\frac{p^{-1}-q}{h(p,q)}\frac{P^{\mu}/Q^{\nu-1}}{1-PQ}zf(z)\\&=\frac{p^{-1}-q}{h(p,q)}\frac{P^{\mu}}{Q^{\nu-1}}\sum_{j=0}^{\infty}P^jQ^jzf(z)\\&=\frac{p^{-1}-q}{h(p,q)}\sum_{j=0}^{\infty}P^{j+\mu}Q^{j+1-\nu}zf(z)\\&=\frac{z(p^{-1}-q)}{h(p,q)}\frac{p^{\mu}}{q^{\nu-1}}\sum_{j=0}^{\infty}f(zp^{j+\mu}q^{j+1-\nu}).\end{aligned}\tag{4.65}$$

§5. Concluding remarks

In this paper we have provided a new noncommutative algebra related to the $\mathcal{R}(p,q)$-deformation and shown that the notions of differentiation and integration can be extended to it, thus generalizing well known q or/and (p,q)-differential and integration calculi [1, 4, 14]. The whole formalism has been illustrated by relevant examples.

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