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On the existence of a regular factorial subring and a p-basis of a polynomial ring in two variables in characteristic $p = 3$

Tomoaki Ono

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Abstract. When K is an algebraically closed field of characteristic $p = 3$, we shall investigate the existence of a regular factorial subring R' of $R = K[x, y]$ containing $R^p = K[x^p, y^p]$ and the existence of a p-basis of R over R'.

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§1. Introduction

Throughout this paper let K be always an algebraically closed field of odd prime characteristic p, R the polynomial ring $K[x, y]$, R^p the polynomial ring $K[x^p, y^p]$ and R' a subring of R such that $R^p \subset R' \subset R$. In the previous paper [10], we showed the following statements:

(A1) (Theorem 3.4 of [10]). *For a given integer d greater than or equal to 3, there is a polynomial $f \in R$ with $\deg f = d$ such that $\{f\}$ is a p-basis of $R^p[f]$ over R^p, $R^p[f]$ is regular non-factorial and R has a p-basis over $R^p[f]$.*

(A2) (Theorem 3.5 of [10]). *For a given integer d greater than or equal to 4, there is a polynomial $f \in R$ with $\deg f = d$ such that $\{f\}$ is a p-basis of $R^p[f]$ over R^p, $R^p[f]$ is regular non-factorial and R has no p-basis over $R^p[f]$.*

In the other paper [3], we showed the fact that R' has a p-basis over R^p if R' is regular and factorial. From (A1), (A2) and this fact, it is natural to ask whether a statement similar to (A1) or (A2) holds if 'non-factorial' is replaced with 'factorial'. In particular we take an interest in the existence of a regular factorial subring which is not a polynomial ring in two variables over K. Under the condition $p = 3$ we shall consider this question (see Theorems 3.1 and 3.4). In [10], when $f \in R$ has no monomial which belongs to R^p, we have classified $R^p[f]$ when $\deg f \leq 3$ as below.

(B) (Theorem 6.1 and Corollary 6.4 of [10]). *Let f be a non-zero polynomial of R, and let $R' = R^p[f]$. Assume that no monomial appearing in f belongs to R^p, and R' is regular. Then R has a p-basis over R'. Moreover the following hold:*

(1) *if $\deg f = 1$ or 2, then R' is a polynomial ring in two variables over K.*

(2) *if $\deg f = 3$, then R' is either a polynomial ring in two variables over K or a non-factorial ring $R^p[u + u^2v]$ for some system u and v of variables of R.*

Similarly when $\deg f = 4$, we shall classify $R^p[f]$ under the condition $p = 3$ (see Theorem 4.3). Consequently we see that there is no polynomial f with $\deg f \leq 4$ such that $R^p[f]$ is regular and factorial, but is not a polynomial ring in two variables over K (see Corollary 4.4).

§2. Preliminary facts

In this paper, for the terminology and notation of algebraic geometry resp. commutative algebra, we use those of [1] resp. [6] and [7]. Let $A^p \subseteq A' \subseteq A$ be a tower of commutative rings of prime characteristic p where $A^p = \{a^p \mid a \in A\}$. A subset $\{g_1, \ldots, g_n\}$ of A is called a *p-basis* of A over A' if the monomials $g_1^{i_1} \cdots g_n^{i_n}$ $(0 \leq i_1, \ldots, i_n \leq p-1)$ are linearly independent over A' and $A = A'[g_1, \ldots, g_n]$. Considering a tower of rings $(A')^p \subseteq A^p \subseteq A'$, a *$p$-basis* of A' over A^p is defined similarly. Under the conditions for R and R' that are specified in the previous section, we recall the results of the previous papers ([9], [10]). First note the well-known fact that $\{f\}$ $(f \in R)$ is a p-basis of $R^p[f]$ over R^p if $f \notin R^p$ (cf. Lemma 2.1 of [10]).

Lemma 2.1 (Lemma 3.1 of [9]). *Let f and g be polynomials of $R - R^p$. Then the following hold:*

(1) *$R^p[f]$ is regular if and only if $\partial f/\partial x$, $\partial f/\partial y$ generate R as an R-module.*

(2) *$\{f, g\}$ is a p-basis of R over R^p if and only if the Jacobian determinant $\partial(f,g)/\partial(x,y) \in K - \{0\}$.*

Lemma 2.2 (Lemma 2.6 of [10]). *Let f be a polynomial of R such that $\partial f/\partial x$ has a non-zero constant term, $R' = R^p[f]$ and $Q(R')$ the field of fractions of R'. Suppose that R' is regular. Then,*

$$\Big(\frac{\partial f}{\partial x}\Big)^{p-1} \in \bigoplus_{i=0}^{p-2} Q(R')y^i \text{ if and only if } R \text{ has a } p\text{-basis over } R'.$$

Lemma 2.3 (Lemma 2.7 of [10]). *Let $f = c_0 + c_1x + c_2x^2$ where $c_0, c_1, c_2 \in K[x^p, y]$. Then*

$$\Big(\frac{\partial f}{\partial x}\Big)^{p-1} = \Big\{c_1^2 + 4c_2(f - c_0)\Big\}^{(p-1)/2}.$$

Lemma 2.4 (Lemma 2.5 of [10]). *Let f be a polynomial of $R - R^p$ which has no monomial belonging to R^p. If $R^p[f]$ is regular and factorial, then $f + h^p$ is irreducible for any polynomial $h \in R$ such that $p \deg h \leq \deg f$.*

§3. Regular factorial subrings of a polynomial ring

Theorem 3.1. *Assume that $p = 3$. Then, for each $d \in \mathbb{N}$ with $d \geq 5$ and $d \not\equiv 0 \pmod 3$, there exists $f \in R$ with $\deg f = d$ which satisfies the following conditions:*

(1) *No monomial appearing in f belongs to R^p;*

(2) *$R^p[f]$ is regular and factorial, but is not a polynomial ring in two variables over K;*

(3) *R has a p-basis over $R^p[f]$.*

Proof. Let $f_1 = x - y(y + x^2)^p$ and $f_2 = x - y(y + x^2)^{2p}$. For any odd prime characteristic p, we proved in [10] that $R^p[f_1]$ is regular and factorial, but is not a polynomial ring in two variables over K, and R has a p-basis over $R^p[f_1]$. By the same argument as that in Example 4.2 of [10], we can show that $R^p[f_2]$ is not a polynomial ring in two variables over K if p is an odd prime number, as follows. Suppose that $R^p[f_2]$ is a polynomial ring in two variables over K. According to the main theorem of [2] there exists a system u and v of variables of R such that $R^p[f_2] = K[u, v^p]$. Set $x := \theta(u, v)$ and $y := \phi(u, v)$. Since u is of the form $\sum_{i=0}^{p-1} c_i f_2^i$ $(c_i \in R^p)$, we have $1 = (\sum_{i=1}^{p-1} ic_i f_2^{i-1})\partial f_2/\partial u$, and so

$$\frac{\partial f_2}{\partial u} = \frac{\partial \theta}{\partial u} - (\phi + \theta^2)^{2p}\frac{\partial \phi}{\partial u} \in K - \{0\}.$$

Since $f_2 \in K[u, v^p] = \operatorname{Ker} \partial/\partial v$, we obtain

$$\frac{\partial f_2}{\partial v} = \frac{\partial \theta}{\partial v} - (\phi + \theta^2)^{2p}\frac{\partial \phi}{\partial v} = 0.$$

Hence, we see that

$$\deg_{\{u,v\}} \frac{\partial\theta}{\partial u} > \deg_{\{u,v\}} \frac{\partial\phi}{\partial u},$$
$$\deg_{\{u,v\}} \frac{\partial\theta}{\partial v} > \deg_{\{u,v\}} \frac{\partial\phi}{\partial v},$$
$$\max\Big\{\deg_{\{u,v\}} \frac{\partial\theta}{\partial u}, \deg_{\{u,v\}} \frac{\partial\theta}{\partial v}\Big\} \geq 2p \deg_{\{u,v\}}(\phi + \theta^2),$$

where $\deg_{\{u,v\}} f$ is the degree of f for the system u and v of variables of R. We denote by d_θ the maximal degree of monomials (for the system u and v of variables of R) appearing in θ which do not belong to R^p. Similarly we use the notations d_{θ^2}, d_ϕ and $d_{\phi+\theta^2}$. The above first and second inequalities for $\deg_{\{u,v\}}$ imply $d_\theta > d_\phi$, and so $d_{\theta^2} > d_\phi$. Hence $d_{\phi+\theta^2} = d_{\theta^2}$. It follows that $\deg_{\{u,v\}}(\phi + \theta^2) \geq d_{\phi+\theta^2} > d_\theta$. On the other hand, we easily see that

$$d_\theta > \max\Big\{\deg_{\{u,v\}} \frac{\partial\theta}{\partial u}, \deg_{\{u,v\}} \frac{\partial\theta}{\partial v}\Big\}.$$

This is a contradiction. Thus R' is not a polynomial ring in two variables over K.

Next we shall prove that $R^p[f_2]$ is regular factorial if $p = 3$. First note that by Lemma 2.1 $R^p[f_2]$ is regular and $\{y\}$ is a p-basis of R over $R^p[f_2]$, since $\partial f_2/\partial x = 1$. The second fact implies $R = \bigoplus_{i=0}^2 R^p[f_2]y^i$. To show that $R^p[f_2]$ is factorial, we make use of a derivation of R over $R^p[f_2]$. Let $D = (y + x^2)^6\partial/\partial x + \partial/\partial y$. First we shall show that $\operatorname{Ker} D = R^p[f_2]$. Writing $g \in R$ as $a_0 + a_1y + a_2y^2$ ($a_0, a_1, a_2 \in R^p[f_2]$), we have $D(g) = (a_1 - a_2y)D(y)$. Since $D(y) \neq 0$, we see the following:

$$D(g) = 0 \Leftrightarrow a_1 - a_2y = 0 \Leftrightarrow a_1 = a_2 = 0 \Leftrightarrow g \in R^p[f_2].$$

This implies $\operatorname{Ker} D = R^p[f_2]$. Since $\{D, \partial/\partial x\}$ forms a basis for $\operatorname{Der}_{R^p}(R)$ and $\partial f_2/\partial x \neq 0$, $\{D\}$ forms a basis for $\operatorname{Der}_{R^p[f_2]}(R)$. Clearly $D^3 \in \operatorname{Der}_{R^p[f_2]}(R)$. Hence $D^3 = aD$ for some $a \in R$. From $Dy = 1$ we have $a = 0$, i.e.,

$$D^3 = 0.$$

Put $s := x$ and $t := y + x^2$. Then $\partial/\partial x = \partial/\partial s - s\partial/\partial t$ and $\partial/\partial y = \partial/\partial t$, so that D is expressed as $t^6\partial/\partial s + (1 - st^6)\partial/\partial t$. By a straightforward computation we obtain

$$D^2 = t^{12}\frac{\partial^2}{\partial s^2} - t^6(1 - st^6)\frac{\partial^2}{\partial s\partial t} + (1 - st^6)^2\frac{\partial^2}{\partial t^2} - t^{12}\frac{\partial}{\partial t}.$$

To prove that $R^p[f_2]$ is factorial, suppose that $R^p[f_2]$ is not factorial. By Lemma 4.1 of [10] there exist non-zero polynomials h, ξ of R such that $D(h) =$

$h\xi$, since $\operatorname{Ker} D = R^p[f_2]$. Then $D^2(h) = D(h)\xi + hD(\xi)$, and so

$$\begin{aligned} D^3(h) &= D(D(h)\xi + hD(\xi)) \\ &= D^2(h)\xi + 2D(h)D(\xi) + hD^2(\xi) \\ &= (D(h)\xi + hD(\xi))\xi + 2h\xi D(\xi) + hD^2(\xi) \\ &= h(\xi^3 + D^2(\xi)). \end{aligned}$$

Since $D^3 = 0$ and $h \neq 0$, it follows that $D^2(\xi) = -\xi^3$. Here note that $\deg_{\{s,t\}} \xi \leq 6$, because $D(h) = \xi h$. Let $[D^2(\xi)]_r$ be the homogeneous part of $D^2(\xi)$ with degree r (for the system s and t of variables of R). Hence, writing ξ as $\sum_{k=0}^{6} \xi_k$ where ξ_k is the homogeneous part of ξ with degree k, we have

$$\frac{\partial^2 \xi_2}{\partial t^2} = [D^2(\xi)]_0 = -\xi_0^3, \tag{1}$$

$$\frac{\partial^2 \xi_5}{\partial t^2} = [D^2(\xi)]_3 = -\xi_1^3, \tag{2}$$

$$\frac{\partial^2 \xi_6}{\partial t^2} = [D^2(\xi)]_4 = 0, \tag{3}$$

$$-t^6 \frac{\partial^2 \xi_2}{\partial s \partial t} = [D^2(\xi)]_6 = -\xi_2^3, \tag{4}$$

$$-t^6 \frac{\partial^2 \xi_5}{\partial s \partial t} + st^6 \frac{\partial^2 \xi_4}{\partial t^2} = [D^2(\xi)]_9 = -\xi_3^3, \tag{5}$$

$$t^{12} \frac{\partial^2 \xi_2}{\partial s^2} - t^{12} \frac{\partial \xi_1}{\partial t} = [D^2(\xi)]_{12} = -\xi_4^3, \tag{6}$$

$$t^{12} \frac{\partial^2 \xi_5}{\partial s^2} + st^{12} \frac{\partial^2 \xi_4}{\partial s \partial t} + s^2 t^{12} \frac{\partial \xi_2}{\partial t} = [D^2(\xi)]_{15} = -\xi_5^3, \tag{7}$$

$$st^{12} \frac{\partial^2 \xi_6}{\partial s \partial t} + s^2 t^{12} \frac{\partial^2 \xi_5}{\partial t^2} - t^{12} \frac{\partial \xi_6}{\partial t} = [D^2(\xi)]_{17} = 0, \tag{8}$$

$$s^2 t^{12} \frac{\partial^2 \xi_6}{\partial t^2} = [D^2(\xi)]_{18} = -\xi_6^3. \tag{9}$$

From (9) it follows that $\xi_6 = cst^5$ ($c \in K$). Hence we obtain $\xi_6 = 0$ by (3), so that $\partial^2 \xi_5 / \partial t^2 = 0$ by (8). Moreover we have $\xi_1 = 0$ by (2). From (4) it follows that $\xi_2 = 0$. Hence we get $\xi_0 = 0$ by (1), and moreover $\xi_4 = 0$ by (6). Since $\xi_2 = \xi_4 = 0$, ξ_5 is of the form $t^4(c_0 s + c_1 t)$ ($c_0, c_1 \in K$) by (7), and so

$\partial^2\xi_5/\partial s^2 = 0$. Hence $\xi_5 = 0$ by (7), so that $\xi_3 = 0$ by (5). Consequently we obtain $\xi = 0$, which is a contradiction. Thus $R^p[f_2]$ is factorial.

Now, set $u := x-(y+x^2)^\alpha$ and $v := y+x^2$ where α is a positive integer such that $\alpha \not\equiv 0 \pmod 3$. Then u and v form a system of variables of R, and we have $f_1 = u+v^\alpha - v^3(v-u^2+uv^\alpha-v^{2\alpha})$ and $f_2 = u+v^\alpha-v^6(v-u^2+uv^\alpha-v^{2\alpha})$. Hence $\deg_{\{u,v\}} f_1 = 2\alpha+3$ and $\deg_{\{u,v\}} f_2 = 2\alpha+6$. Thus f_1 and f_2 have the desired properties. □

Corollary 3.2. *Assume that $p = 3$. Then, for each $d \in \mathbb{N}$ with $d \not\equiv 0 \,(\mathrm{mod}\, 3)$, there exist a polynomial $g \in R$ with $\deg g = d$ and a subring R' of R containing R^p which satisfy the following properties:*

(1) *No monomial appearing in g belongs to R^p;*

(2) *$\{g\}$ is a p-basis of R over R';*

(3) *R' is regular and factorial, but is not a polynomial ring in two variables over K.*

Proof. Let f_2 be as in the proof of Theorem 3.1. Then, we have already seen that $\{y\}$ is a p-basis of R over $R^p[f_2]$. Put $g := y$ and $y' := y + x^d$. Then g is expressed as $y' - x^d$. So the assertion holds. □

Lemma 3.3. *Let f be a polynomial of $R - R^p$, D a derivation of R over K such that $D(x) \neq 0$ and $D(y) \neq 0$, and $K(x^p, y^p)$ the field of fractions of R^p. Suppose that $D(f) = 0$. Then $\operatorname{Ker} D = K(x^p, y^p)[f] \cap R$. Furthermore, $\operatorname{Ker} D = R^p[f]$ if and only if $R^p[f] \cap hR \subset hR^p[f]$ holds for any $h \in R^p - \{0\}$.*

Proof. Let $K(x,y)$ be the field of fractions of R, and $\bar{D}$ the extension of D to $K(x,y)$. Set $L := K(x^p,y^p)[f]$. Clearly L is a subfield of $K(x,y)$, and $[L : K(x^p,y^p)] = p$, since $f \notin K(x^p,y^p)$ and $[K(x,y) : K(x^p,y^p)] = p^2$. Hence $[K(x,y) : L] = p$, and so $K(x,y)$ is either $\bigoplus_{i=0}^{p-1} Lx^i$ or $\bigoplus_{i=0}^{p-1} Ly^i$. We consider the case where $K(x,y) = \bigoplus_{i=0}^{p-1} Lx^i$. Then any $g \in K(x,y)$ is of the form $\sum_{i=0}^{p-1} a_i x^i$ ($a_i \in L$). Therefore $\bar{D}(g) = (\sum_{i=1}^{p-1} i a_i x^{i-1}) D(x)$. Since $D(x) \neq 0$, we have the following:

$$\bar{D}(g) = 0 \Leftrightarrow \sum_{i=1}^{p-1} i a_i x^{i-1} = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_{p-1} = 0 \Leftrightarrow g \in L.$$

Hence $\operatorname{Ker} \bar{D} = L$. Similarly we can show $\operatorname{Ker} \bar{D} = L$ in the case where $K(x,y) = \bigoplus_{i=0}^{p-1} Ly^i$.

Next, we shall prove the second assertion. Suppose that $\operatorname{Ker} D = R^p[f]$. Let h be a polynomial of $R^p - \{0\}$. For each $g \in R^p[f] \cap hR$, there exists an element g' of R such that $g = hg'$. Since $0 = D(g) = hD(g')$ from the assumption, we have $D(g') = 0$ so that $g' \in R^p[f]$. This implies $g \in hR^p[f]$. Thus $R^p[f] \cap hR \subset hR^p[f]$ for each $h \in R^p - \{0\}$. Conversely suppose that $R^p[f] \cap hR \subset hR^p[f]$ for each $h \in R^p - \{0\}$. Any $g \in \operatorname{Ker} D$ is of the form $\sum_{i=0}^{p-1}(a_i/b_i)f^i$ ($a_i, b_i \in R^p$) by the first assertion. Set $h := \prod_{i=0}^{p-1} b_i$. Then $hg \in R^p[f] \cap hR$ and so $hg \in hR^p[f]$. Hence $g \in R^p[f]$. Thus $\operatorname{Ker} D = R^p[f]$. □

Theorem 3.4. *Assume that $p = 3$. Then, for each $d \in \mathbb{N}$ with $d \geq 5$ and $d \not\equiv 0 \pmod 3$, there exists a polynomial $f \in R$ with $\deg f = d$ which satisfies the following properties:*

(1) *No monomial appearing in f belongs to R^p;*

(2) *$R^p[f]$ is regular and factorial, but is not a polynomial ring in two variables over K;*

(3) *R has no p-basis over $R^p[f]$.*

Proof. Let $f_1 = x - y^2 + x^2y^3$ and $f_2 = x - y^5 + x^2y^6$. We already treated f_1 in Example 4.3 of [10]. So we only give a proof of the assertion that $R^p[f_2]$ is regular and factorial, but is not a polynomial ring in two variables over K. Set $R_2' := R^p[f_2]$. First note that R_2' is regular by Lemma 2.1 (1). From Lemma 2.3 we see that

$$\left(\frac{\partial f}{\partial x}\right)^2 = 1 + 4y^6(f_2 + y^5) \notin Q(R_2') \oplus Q(R_2')y.$$

According to Lemma 2.2 this implies that R has no p-basis over R_2', hence R_2' is not a polynomial ring in two variables over K by the result of [2] (also see [5]).

Let $D = y^4\partial/\partial x - (1 + 2xy^6)\partial/\partial y$. To show that $\operatorname{Ker} D = R_2'$, by Lemma 3.3 it is sufficient to verify the condition that $R_2' \cap hR \subset hR_2'$ holds for any $h \in R^p - \{0\}$. Suppose that $hg \in hR$ belongs to R_2'. From the assumption hg is of the form $h_0 + h_1f_2 + h_2f_2^2$ ($h_0, h_1, h_2 \in R^p$). Since $f_2 = x - y^3y^2 + y^6x^2$, we obtain

$$\begin{aligned} h_0 + h_1f_2 + h_2f_2^2 = {} & (h_0 - h_2x^3y^6) + (h_1 + h_2x^3y^{12})x + h_2y^9y \\ & + (h_1y^6 + h_2)x^2 - h_1y^3y^2 + h_2y^3xy^2 + h_2y^9x^2y^2. \end{aligned}$$

Note that the coefficients of 1, x, y, x^2, y^2, xy^2, x^2y^2 (as an R^p -linear combination of x^iy^j for $i, j \in \{0, 1, 2\}$) belong to hR^p, because $\{x^iy^j\}_{0\leq i,j\leq 2}$ is a p-basis of R over R^p. Since $h_1y^6 + h_2, -h_1y^3 \in hR^p$, we see $h_2 \in hR^p$, and so $h_0 - h_2x^3y^6 \in hR^p$ resp. $h_1 + h_2x^3y^{12} \in hR^p$ implies $h_0 \in hR^p$ resp. $h_1 \in hR^p$. Therefore $h_0/h, h_1/h, h_2/h \in R^p$. Hence $g = h_0/h + (h_1/h)f_2 + (h_2/h)f_2^2 \in R_2'$. Thus $R_2' \cap hR \subset hR_2'$ holds for any $h \in R^p - \{0\}$. Put $D' := (1 - 2xy^6)\partial/\partial x + 4x^2y^8\partial/\partial y$. Then, since $\{D, D'\}$ forms a basis for $\operatorname{Der}_{R^p}(R)$ and $D'(f_2) \neq 0$, we see that $\{D\}$ forms a basis for $\mathrm{Der}_{R_2'}(R)$. Clearly $D^3 \in \mathrm{Der}_{R_2'}(R)$. Hence, $D^3 = aD$ for some $a \in R$. Since $Dx = y^4$ and $D^3x = y^{13}$, we see $a = y^9$, i.e.,

$$D^3 = y^9 D.$$

(From this fact we can also see that R has no p-basis over R_2' (see [8]).) By a straightforward computation we obtain

$$D^2 = y^8\frac{\partial^2}{\partial x^2} + y^4(1+2xy^6)\frac{\partial^2}{\partial x\partial y} + (1+2xy^6)^2\frac{\partial^2}{\partial y^2} - y^3(1+2xy^6)\frac{\partial}{\partial x} + y^{10}\frac{\partial}{\partial y}.$$

To prove that R_2' is factorial, suppose that R_2' is not factorial. By Lemma 4.1 of [10] there exist non-zero polynomials h, ξ of R such that $D(h) = h\xi$, and we have $D^3(h) = h(\xi^3 + D^2(\xi))$ as in the proof of Theorem 3.1. Hence

$$D^2(\xi) = -\xi^3 + y^9\xi.$$

Clearly $\deg \xi \leq 6$. Let ξ_k $(0 \leq k \leq 6)$ and $[D^2(\xi)]_r$ $(0 \leq r \leq 18)$ be as in the proof of Theorem 3.1, and moreover we express ξ as $\sum_{0\leq i+j\leq 6} c_{i,j}x^iy^j$ $(c_{i,j} \in K)$. Now we consider the equation $D^2(\xi) = -\xi^3 + y^9\xi$. Since $x^2y^{12}\partial^2\xi_6/\partial y^2 = [D^2(\xi)]_{18} = -\xi_6^3$, we have $2c_{4,2}x^6y^{12} + 2c_{1,5}x^3y^{15} = -\xi_6^3$. It follows that $c_{6,0} = c_{5,1} = c_{4,2} = c_{3,3} = c_{2,4} = c_{0,6} = 0$. Since $y^4\partial^2\xi_4/\partial x\partial y - y^3\partial\xi_4/\partial x = [D^2(\xi)]_6 = -\xi_2^3$, we get $c_{2,0} = 0$. Since $y^4\partial^2\xi_2/\partial x\partial y + \partial^2\xi_6/\partial y^2 - y^3\partial\xi_2/\partial x = [D^2(\xi)]_4 = 0$, we have $c_{1,1}y^4 + 2c_{1,5}xy^3 - y^3(2c_{2,0}x + c_{1,1}y) = 0$, and so $c_{1,5} = c_{2,0} = 0$. Hence $\xi_6 = 0$. Note that $\partial^2\xi_3/\partial y^2 = \partial^2\xi_4/\partial y^2 = \partial^2\xi_5/\partial y^2 = 0$, since $\partial^2\xi_3/\partial y^2 = [D^2(\xi)]_1 = 0$, $\partial^2\xi_4/\partial y^2 = [D^2(\xi)]_2 = 0$ and $x^2y^{12}\partial^2\xi_5/\partial y^2 = [D^2(\xi)]_{17} = 0$. The left-hand side of $[D^2(\xi)]_{15} = -\xi_5^3 + y^9\xi_6 = -\xi_5^3$ is of the form

$$2xy^{10}\frac{\partial^2\xi_6}{\partial x\partial y} + x^2y^{12}\frac{\partial^2\xi_3}{\partial y^2} + xy^9\frac{\partial\xi_6}{\partial x} + y^{10}\frac{\partial\xi_6}{\partial y}.$$

It follows that $\xi_5 = 0$. Since $y^4\partial^2\xi_5/\partial x\partial y + xy^6\partial^2\xi_2/\partial y^2 - y^3\partial\xi_5/\partial x = [D^2(\xi)]_7 = 0$, we get $\partial^2\xi_2/\partial y^2 = 0$. So we have $\xi_0 = 0$, because $\partial^2\xi_2/\partial y^2 = [D^2(\xi)]_0 = -\xi_0^3$. Since $y^8\partial^2\xi_3/\partial x^2 + xy^6\partial^2\xi_4/\partial y^2 = [D^2(\xi)]_9 = -\xi_3^3 + y^9\xi_0 = -\xi_3^3$, we obtain $2c_{2,1}y^9 = -\xi_3^3$. This implies $\xi_3 = 0$. The left-hand side of $[D^2(\xi)]_{12} = -\xi_4^3 + y^9\xi_3 = -\xi_4^3$ is of the form

$$y^8\frac{\partial^2\xi_6}{\partial x^2} + 2xy^{10}\frac{\partial^2\xi_3}{\partial x\partial y} + xy^9\frac{\partial\xi_3}{\partial x} + y^{10}\frac{\partial\xi_3}{\partial y}.$$

It follows that $\xi_4 = 0$. So, by considering the equation $y^4\partial^2\xi_4/\partial x\partial y - y^3\partial\xi_4/\partial x = [D^2(\xi)]_6 = -\xi_2^3$ again, we obtain $\xi_2 = 0$. Since $\partial^2\xi_5/\partial y^2 -$

$y^3\partial\xi_1/\partial x = [D^2(\xi)]_3 = -\xi_1^3$, we have $c_{1,0}y^3 = \xi_1^3$. This implies $\xi_1 = 0$. Consequently we see $\xi = 0$, which is a contradiction. Thus R_2' is factorial.

Now, set $u := x - y^\alpha$ and $v := y$ where α is a positive integer such that $\alpha \not\equiv 0 \pmod 3$. Then the system u and v is a system of variables of R, and we have $f_1 = u + v^\alpha - v^2 + (u+v^\alpha)^2 v^3$ and $f_2 = u + v^\alpha - v^5 + (u+v^\alpha)^2 v^6$. Clearly $\deg_{\{u,v\}} f_1 = 2\alpha + 3$ and $\deg_{\{u,v\}} f_2 = 2\alpha + 6$. Hence the assertion holds. □

Next we give examples of a non-regular factorial subring $K[x^p, y^p, f]$ ($\deg f = 5$) of the polynomial ring $K[x, y]$.

Example 3.5. *Assume that* $p = 3$. *Let* $f_0 = x - y^2 + x^2y^3$ *and* $f_1 = x - y^4 + x^2y^3$, *and let* f_t *be* $(1-t)f_0 + tf_1$ *for any* $t \in K$. *Let* $D_t = \{(1-t)y - ty^3\}\partial/\partial x - (1 - xy^3)\partial/\partial y$ *and* $K_t = \operatorname{Ker} D_t$. *Then* $K_t = R^p[f_t]$, *and* K_t *is factorial, but is not a polynomial ring in two variables over* K. *Moreover, the following properties hold:*

(1) K_t *is regular if and only if* $t = 0$ *or* 1.

(2) R *has a* p*-basis over* K_t *if and only if* $t = 1$.

Proof. Note that f_0 is the same as f_1 in the proof of Theorem 3.4. Hence, $R^p[f_0]$ is regular and factorial, but is not a polynomial ring in two variables over K, and R has no p-basis over $R^p[f_0]$. Consider the system $u = x$ and $v = y - x^2$ of variables of R. Then $f_1 = x - (y - x^2)y^3 = u - v(v + u^2)^3$ is the same as f_1 in the proof of Theorem 3.1. Hence, $R^p[f_1]$ is regular and factorial, but is not a polynomial ring in two variables over K, and R has a p-basis over $R^p[f_1]$.

To show that $K_t = R^p[f_t]$, by Lemma 3.3 we only check that $R^p[f_t] \cap hR \subset hR^p[f_t]$ holds for any $h \in R^p - \{0\}$. Take any $h_0 + h_1 f_t + h_2 f_t^2 \in R^p[f_t] \cap hR$ with $h_0, h_1, h_2 \in R^p$, and write

$$\begin{aligned} h_0 + h_1 f_t + h_2 f_t^2 = h_0 &+ h_2\{-x^3y^3 + t(t-1)y^6\} + (h_1 + h_2 x^3 y^6)x \\ &+ \{-th_1 + (t-1)^2 h_2\}y^3 y + (h_1 y^3 + h_2)x^2 \\ &+ th_2 y^3 xy + \{(t-1)h_1 + t^2 h_2 y^6\}y^2 + th_2 y^6 x^2 y \\ &+ (1-t)h_2 xy^2 + (1-t)h_2 y^3 x^2 y^2. \end{aligned}$$

Then, by looking at the coefficients of 1, x, x^2, y^2 and xy^2 (as an R^p-linear combination of $x^i y^j$ for $i, j \in \{0, 1, 2\}$), we know that $h_0 + h_2\{-x^3y^3 + t(t-1)y^6\}$, $h_1 + h_2 x^3 y^6$, $h_1 y^3 + h_2$, $(t-1)h_1 + t^2 h_2 y^6$, $(1-t)h_2$ belong to hR^p as in the proof of Theorem 3.4. When $t = 1$, we have $h_1 + h_2 x^3 y^6, h_2 y^6 \in hR^p$, and hence $h_1 \in hR^p$. Since $h_1 y^3 + h_2, h_0 - h_2 x^3 y^3 \in hR^p$, it follows that h_0 and h_2 also belong to hR^p. Similarly, we have $h_0, h_1, h_2 \in hR^p$ when $t \neq 1$, since $(1-t)h_2$, $h_0 + h_2\{-x^3y^3 + t(t-1)y^6\}$ and $h_1 + h_2 x^3 y^6$ belong to hR^p. Hence $h_0 + h_1 f_t + h_2 h_t^2 \in hR^p[f_t]$.

From now on assume that $t \neq 0, 1$. The equations $\partial f_t/\partial x = 1 - xy^3 = 0$ and $\partial f_t/\partial y = (1-t)y - ty^3 = 0$ have common zeros. Lemma 2.1 (1) says that K_t is not regular. Hence K_t is not a polynomial ring in two variables over K, and R has no p-basis over K_t by Theorem 15.7 of [6] (cf. [4]). Clearly $D_t^3 \in \mathrm{Der}_{K_t}(R) \subset \mathrm{Der}_{R^p}(R)$, so that D_t^3 is of the form $a_t\partial/\partial x + b_t\partial/\partial y$ ($a_t, b_t \in R$). By an easy computation we have $a_t = D_t^3(x) = (1-t)y^3\{(1-t)y - ty^3\}$ and $b_t = D_t^3(y) = -(1-t)y^3(1-xy^3)$. It follows that

$$D_t^3 = (1-t)y^3 D_t.$$

By a straightforward computation we obtain

$$\begin{aligned} D_t^2 = {} & \{(1-t)y - ty^3\}^2\frac{\partial^2}{\partial x^2} + \{(1-t)y - ty^3\}(1-xy^3)\frac{\partial^2}{\partial x\partial y} \\ & + (1-xy^3)^2\frac{\partial^2}{\partial y^2} - (1-t)(1-xy^3)\frac{\partial}{\partial x} + y^3\{(1-t)y - ty^3\}\frac{\partial}{\partial y}. \end{aligned}$$

To prove that K_t is factorial, suppose that K_t is not factorial. Then, by Lemma 4.1 of [10] there exist non-zero polynomials h_t, ξ_t of R such that $D_t(h_t) = h_t\xi_t$, and moreover $\deg \xi_t \leq 3$. We obtain $D_t^2(\xi_t) = -\xi_t^3 + (1-t)y^3\xi_t$ as in the proof of Theorem 3.4. To consider this equation, we prepare the notations $\xi_{t\,k}$ ($0 \leq k \leq 3$) and $[D_t^2(\xi_t)]_r$ ($0 \leq r \leq 9$) as in the proof of Theorem 3.1, and we express ξ_t as $\sum_{0\leq i+j\leq 3} c_{t\,i,j}x^iy^j$ ($c_{t\,i,j} \in K$). Since $x^2y^6\partial^2\xi_{t\,3}/\partial y^2 = [D_t^2(\xi_t)]_9 = -\xi_{t\,3}^3$, we get $c_{t\,3,0} = c_{t\,2,1} = c_{t\,0,3} = 0$. The left-hand side of $[D_t^2(\xi_t)]_8 = 0$ is of the form

$$x^2y^6\frac{\partial^2\xi_{t\,2}}{\partial y^2} + txy^6\frac{\partial^2\xi_{t\,3}}{\partial x\partial y} - xy^6\frac{\partial\xi_{t\,3}}{\partial y}.$$

Hence we obtain $c_{t\,0,2} = 0$. The left-hand side of $[D_t^2(\xi_t)]_6 = -\xi_{t\,2}^3 + y^3\xi_{t\,3}$ is of the form

$$t^2y^6\frac{\partial^2\xi_{t\,2}}{\partial x^2} - (1-t)xy^4\frac{\partial^2\xi_{t\,3}}{\partial x\partial y} + (1-t)xy^3\frac{\partial\xi_{t\,3}}{\partial x} + (1-t)y^4\frac{\partial\xi_{t\,3}}{\partial y} - ty^6\frac{\partial\xi_{t\,1}}{\partial y}.$$

It follows that $(1-t)c_{t\,1,2}xy^5 - tc_{t\,0,1}y^6 = -(c_{t\,2,0}^3x^6 + c_{t\,1,1}^3x^3y^3) + c_{t\,1,2}xy^5$. Hence $c_{t\,1,2} = c_{t\,0,1} = c_{t\,2,0} = c_{t\,1,1} = 0$, so that $\xi_{t\,2} = \xi_{t\,3} = 0$. From these facts we have

$$-(1-t)c_{t\,1,0}(1-xy^3) = D_t^2(\xi_t) = -\xi_{t\,0}^3 - c_{t\,1,0}x^3 + \xi_{t\,0}y^3 + c_{t\,1,0}xy^3.$$

It follows that $c_{t\,1,0} = \xi_{t\,0} = 0$. Hence we have $\xi_t = 0$, which is a contradiction. Thus K_t is factorial. The assertion follows from these facts. □

§4. Regular subrings $R^p[f]$ with $\deg f = 4$

Throughout this section, let $\mathbb{A}^2$ resp. $\mathbb{P}^2$ be the affine plane $\operatorname{Specm} R$ resp. the projective plane over K, and let $K[X, Y, Z]$ be the homogeneous coordinate ring of $\mathbb{P}^2$ and we denote by $[a, b, c]$ the point of $\mathbb{P}^2$ given by $X = a$, $Y = b$, $Z = c$. Let $\iota : \mathbb{A}^2 \to \mathbb{P}^2$ be the canonical embedding of $\mathbb{A}^2$ given by $(a, b) \mapsto [a, b, 1]$, and put $L_\infty := \mathbb{P}^2 - \iota(\mathbb{A}^2) = \{Z = 0\}$, $P := [0, 1, 0]$ and $Q := [1, 0, 0]$. Let $f(x, y)$ be a polynomial $\sum_{1\le i+j\le 4} c_{i,j}x^iy^j$ of R with degree 4 such that $\partial f/\partial x$, $\partial f/\partial y$ generate R as an R-module. Note that either $c_{1,0} \neq 0$ or $c_{0,1} \neq 0$. Let $F(X, Y, Z) \in K[X, Y, Z]$ be the homogeneous polynomial $\sum_{1\le i+j\le 4} c_{i,j}X^iY^jZ^{4-i-j}$ such that $f(x, y) = F(X, Y, Z)/Z^4$. Then

$$\frac{\partial F}{\partial X} = \sum_{1\le i+j\le 3} ic_{i,j}X^{i-1}Y^jZ^{4-i-j} + c_{4,0}X^3 + 2c_{2,2}XY^2 + c_{1,3}Y^3,$$

$$\frac{\partial F}{\partial Y} = \sum_{1\le i+j\le 3} jc_{i,j}X^iY^{j-1}Z^{4-i-j} + c_{3,1}X^3 + 2c_{2,2}X^2Y + c_{0,4}Y^3.$$

We put $H_X := c_{4,0}X^3 + 2c_{2,2}XY^2 + c_{1,3}Y^3$ and $H_Y := c_{3,1}X^3 + 2c_{2,2}X^2Y + c_{0,4}Y^3$.
Since $\partial f/\partial x$ and $\partial f/\partial y$ generate R as an R-module, $V(\partial f/\partial x)\cap V(\partial f/\partial y) = \emptyset$ and so $V(\partial F/\partial X) \cap V(\partial F/\partial Y) \subseteq L_\infty$. If $H_X \neq 0$ and $H_Y \neq 0$, we have $V(\partial F/\partial X) \cap V(\partial F/\partial Y) = V(H_X) \cap V(H_Y) \cap L_\infty$.

Lemma 4.1. *Assume that $p = 3$. Let $f \in R$ be such that $\deg f = 4$ and $R' := R^p[f]$ is regular. If the monomial x^2y^2 appears in f, then R' is not factorial, $R' = R^p[u + u^2v^2]$ for some system u and v of variables of R, and R has no p-basis over R'.*

Proof. Since $c_{2,2} \neq 0$, after a suitable K-linear change of the system x and y of variables of R, we are able to assume that $c_{2,2} = 1$ and $c_{4,0} = c_{0,4} = 0$. Moreover we may assume that $c_{2,1} = c_{1,2} = 0$ with a suitable affine change of the system x and y of variables of R. We will argue about 4 cases as below.

Case 1. Suppose that $c_{3,1} = c_{1,3} = 0$. Clearly $V(\partial F/\partial X) \cap V(\partial F/\partial Y) = \{P, Q\}$. Now we consider the intersection number $I(P, \partial F/\partial X \cap \partial F/\partial Y)$ of $\partial F/\partial X$ and $\partial F/\partial Y$ at P. Set

$$f_{X1} := \frac{1}{Y^3}\frac{\partial F}{\partial X} = c_{1,0}z^3 + 2c_{2,0}xz^2 + c_{1,1}z^2 + 2x,$$

$$f_{Y1} := \frac{1}{Y^3}\frac{\partial F}{\partial Y} = c_{0,1}z^3 + c_{1,1}xz^2 + 2c_{0,2}z^2 + 2x^2,$$

where $x = X/Y$ (we use the same symbol with an affine coordinate x of $\mathbb{A}^2$) and $z = Z/Y$. Then

$$I(P, \partial F/\partial X \cap \partial F/\partial Y) = I(P, f_{X1} \cap f_{Y1}) = I(P, f_{X1} \cap (f_{Y1} - x f_{X1})).$$

Since $f_{Y1} - x f_{X1} = z^2 f_{Y2}$ where $f_{Y2} = -c_{0,2} + c_{0,1}z + c_{2,0}x^2 - c_{1,0}xz$, we obtain

$$I(P, \partial F/\partial X \cap \partial F/\partial Y) = 2 + I(P, f_{X1} \cap f_{Y2}).$$

If $c_{0,2} = 0$, we have

$$I(P, f_{X1} \cap f_{Y2}) = \begin{cases} 1 & \text{if } c_{0,1} \neq 0, \\ 3 & \text{if } c_{0,1} = 0, \ c_{1,1} \neq 0, \\ 4 & \text{if } c_{0,1} = 0, \ c_{1,1} = 0. \end{cases}$$

On the other hand, if $c_{0,2} \neq 0$, we have $I(P, f_{X1} \cap f_{Y2}) = 0$. Hence we obtain

$$I(P, \partial F/\partial X \cap \partial F/\partial Y) = \begin{cases} 2 & \text{if } c_{0,2} \neq 0, \\ 3 & \text{if } c_{0,2} = 0, \ c_{0,1} \neq 0, \\ 5 & \text{if } c_{0,2} = 0, \ c_{0,1} = 0, \ c_{1,1} \neq 0, \\ 6 & \text{if } c_{0,2} = 0, \ c_{0,1} = 0, \ c_{1,1} = 0. \end{cases}$$

Similarly we have

$$I(Q, \partial F/\partial X \cap \partial F/\partial Y) = \begin{cases} 2 & \text{if } c_{2,0} \neq 0, \\ 3 & \text{if } c_{2,0} = 0, \ c_{1,0} \neq 0, \\ 5 & \text{if } c_{2,0} = 0, \ c_{1,0} = 0, \ c_{1,1} \neq 0, \\ 6 & \text{if } c_{2,0} = 0, \ c_{1,0} = 0, \ c_{1,1} = 0. \end{cases}$$

Bézout's theorem says $I(P, \partial F/\partial X \cap \partial F/\partial Y) + I(Q, \partial F/\partial X \cap \partial F/\partial Y) = 9$, so we see that $c_{2,0} = c_{0,2} = c_{1,1} = 0$, and either $c_{1,0} \neq 0$ and $c_{0,1} = 0$, or $c_{1,0} = 0$ and $c_{0,1} \neq 0$. Thus f is either $c_{1,0}x + c_{3,0}x^3 + c_{0,3}y^3 + x^2y^2$ $(c_{1,0} \neq 0)$ or $c_{0,1}y + c_{3,0}x^3 + c_{0,3}y^3 + x^2y^2$ $(c_{0,1} \neq 0)$.

Case 2. Suppose that $c_{3,1} \neq 0$ and $c_{1,3} = 0$. First note that $V(\partial F/\partial X) \cap V(\partial F/\partial Y) = \{P\}$. Set

$$f_{X1} := \frac{1}{Y^3}\frac{\partial F}{\partial X} = c_{1,0}z^3 + 2c_{2,0}xz^2 + c_{1,1}z^2 + 2x,$$

$$f_{Y1} := \frac{1}{Y^3}\frac{\partial F}{\partial Y} = c_{0,1}z^3 + c_{1,1}xz^2 + 2c_{0,2}z^2 + c_{3,1}x^3 + 2x^2,$$

where $x = X/Y$ and $z = Z/Y$. Then, since $I(P, f_{X1} \cap f_{Y1}) = I(P, \partial F/\partial X \cap \partial F/\partial Y) = 9$, we have $c_{0,2} = 0$ and so $f_{Y1} - x f_{X1} = c_{3,1}x^3 + c_{0,1}z^3 - c_{1,0}xz^3 +$

$c_{2,0}x^2z^2$. Hence $c_{0,1}=0$, because $I(P, f_{X1} \cap (f_{Y1} - xf_{X1})) = I(P, f_{X1} \cap f_{Y1}) = 9$. It follows that $c_{1,0} \neq 0$ and $f_{Y1} - xf_{X1} = xf_{Y2}$ where $f_{Y2} = c_{3,1}x^2 - c_{1,0}z^3 + c_{2,0}xz^2$. Clearly $I(P, f_{X1} \cap x) \leq 3$. Since $f_{Y2} + c_{3,1}xf_{X1} = z^2 f_{Y3}$ where $f_{Y3} = (c_{2,0} + c_{1,1}c_{3,1})x - c_{1,0}z + c_{1,0}c_{3,1}xz - c_{2,0}c_{3,1}x^2$, we have

$$I(P, f_{X1} \cap f_{Y2}) = I(P, f_{X1} \cap (f_{Y2} + c_{3,1}xf_{X1})) = 2 + I(P, f_{X1} \cap f_{Y3}) = 3,$$

and so $I(P, f_{X1} \cap f_{Y1}) = I(P, f_{X1} \cap x) + I(P, f_{X1} \cap f_{Y2}) \leq 6$. This is a contradiction. Hence this case never occurs.

Case 3. Suppose that $c_{3,1} = 0$ and $c_{1,3} \neq 0$. By the change of x and y, this case is reduced to the previous case. Thus this case does not occur.

Case 4. Suppose that $c_{3,1} \neq 0$ and $c_{1,3} \neq 0$. Since $H_X = Y^2(-X + c_{1,3}Y)$, $H_Y = X^2(c_{3,1}X - Y)$ and $V(H_X) \cap V(H_Y) \cap L_\infty \neq \emptyset$,we have $c_{3,1}c_{1,3} = 1$. Hence F is written as

$$\sum_{1 \leq i+j \leq 3} c_{i,j} X^i Y^j Z^{4-i-j} + XY\Big(\sqrt{c_{3,1}}\, X - \frac{1}{\sqrt{c_{3,1}}}\, Y\Big)^2.$$

Setting $X' := \sqrt{c_{3,1}}\, X - (1/\sqrt{c_{3,1}})\, Y$, the polynomial F is given by

$$F' = \sum_{1 \leq i+j \leq 3} c'_{i,j} (X')^i Y^j Z^{4-i-j} + \frac{1}{\sqrt{c_{3,1}}} (X')^3 Y + \frac{1}{c_{3,1}} (X')^2 Y^2.$$

Hence this case is reduced to Case 2 so that it never occurs.

We conclude that f is either $c_{1,0}x + c_{3,0}x^3 + c_{0,3}y^3 + x^2y^2$ ($c_{1,0} \neq 0$) or $c_{0,1}y + c_{3,0}x^3 + c_{0,3}y^3 + x^2y^2$ ($c_{0,1} \neq 0$). This implies that there exists a system u and v of variables of R such that $R' = R^p[u + u^2v^2]$. When $f = u + u^2v^2$, $R^p[f]$ is non-factorial by Lemma 2.4 and $(\partial f/\partial u)^2 = 1 + fv^2 \notin Q(R') \oplus Q(R')v$ by Lemma 2.3. According to Lemma 2.2 the later fact implies that R has no p-basis over R'. □

Lemma 4.2. *Assume that $p = 3$. Let $f \in R$ be such that $R' = R^p[f]$ is regular and $\deg f = 4$. If the monomial x^2y^2 does not appear in f, then there exists a system u and v of variables of R such that one of the following conditions holds:*

1. *R' is equal to the polynomial ring $K[u, v^3]$;*
2. *R' is a non-factorial ring, and is equal to $R^p[u+u^2v]$, or $R^p[u+cu^2+u^3v]$ for some $c \in K$.*

Moreover, R has a p-basis over R' in all cases.

Proof. Case A. Suppose that $c_{4,0} = c_{0,4} = 0$. If $c_{3,1} \neq 0$ and $c_{1,3} \neq 0$, we have $V(H_X) \cap V(H_Y) = \emptyset$. Hence either $c_{3,1} = 0$ or $c_{1,3} = 0$. So we may assume that $c_{3,1} = 1$ and $c_{1,3} = 0$. Moreover we may assume that $c_{0,1} = 0$ with a suitable affine change of the system x and y of variables of R, and so $c_{1,0} \neq 0$. Put $F_X := c_{1,0}Z^2 + 2c_{2,0}XZ + c_{1,1}YZ + 2c_{2,1}XY + c_{1,2}Y^2$. Then $\partial F/\partial X = ZF_X$. Since $I(P, Z \cap \partial F/\partial Y) = 3$, we have $V(F_X) \cap V(\partial F/\partial Y) = \{P\}$ and $c_{1,2} = 0$. Set

$$f_{X1} := \frac{1}{Y^2} F_X = c_{1,0}z^2 + 2c_{2,0}xz + c_{1,1}z + 2c_{2,1}x,$$
$$f_{Y1} := \frac{1}{Y^3}\frac{\partial F}{\partial Y} = c_{1,1}xz^2 + 2c_{0,2}z^2 + c_{2,1}x^2z + x^3,$$

where $x = X/Y$ and $z = Z/Y$.

Now we claim that $c_{2,1} = 0$. To show this, assume that $c_{2,1} \neq 0$. Since $I(P, f_{X1} \cap f_{Y1}) = 6$, we see that $c_{0,2} = 0$ and $f_{Y1} = x(c_{1,1}z^2 + c_{2,1}xz + x^2)$. Moreover we get $I(P, f_{X1} \cap (c_{1,1}z^2 + c_{2,1}xz + x^2)) \geq 4$. This implies that f_{X1} and $c_{1,1}z^2 + c_{2,1}xz + x^2$ have a tangent line in common at P. Hence $c_{1,1}(c_{1,1} - c_{2,1}^2) = 0$. If $c_{1,1} = 0$, we have $I(P, f_{X1} \cap (c_{2,1}xz + x^2)) = I(P, f_{X1} \cap x) + I(P, f_{X1} \cap (c_{2,1}z + x)) = 3$. This is contradictory to the fact $I(P, f_{X1} \cap (c_{2,1}xz + x^2)) \geq 4$. Hence $c_{1,1} = c_{2,1}^2$, it follows that $6 = I(P, f_{X1} \cap f_{Y1}) = I(P, f_{X1} \cap x) + I(P, f_{X1} \cap (c_{1,1}z^2 + c_{2,1}xz + x^2)) = 1 + 2I(P, f_{X1} \cap (c_{2,1}z - x))$, which is a contradiction. Thus $c_{2,1} = 0$.

From the claim we obtain $I(P, f_{X1} \cap f_{Y1}) = I(P, z \cap f_{Y1}) + I(P, (c_{1,0}z - c_{2,0}x + c_{1,1}) \cap f_{Y1}) = 3 + I(P, (c_{1,0}z - c_{2,0}x + c_{1,1}) \cap f_{Y1})$, and so $I(P, (c_{1,0}z - c_{2,0}x + c_{1,1}) \cap f_{Y1}) = 3$. Hence $c_{1,1} = 0$. If $c_{0,2} = 0$, we see that $R' = R^p[c_{1,0}x + c_{2,0}x^2 + x^3y]$ is regular and non-factorial by Lemma 2.4, and R has a p-basis over R' by Lemmas 2.2 and 2.3. On the other hand, if $c_{0,2} \neq 0$, we obtain $c_{2,0} = 0$ and so

$$\begin{aligned} f &= c_{1,0}x + c_{0,2}y^2 + c_{3,0}x^3 + c_{0,3}y^3 + x^3y \\ &= c_{1,0}x + c_{0,2}\Big(y - \frac{1}{c_{0,2}}x^3\Big)^2 + c_{3,0}x^3 + c_{0,3}y^3 - \frac{1}{c_{0,2}}x^6. \end{aligned}$$

Setting $y' := y - (1/c_{0,2})x^3$, the polynomial $f - c_{3,0}x^3 - c_{0,3}y^3 + (1/c_{0,2})x^6$ is given by $f' = c_{1,0}x + c_{0,2}(y')^2$. Thus $R' = R^p[f'\,] = K[f', (y')^p]$ is a polynomial ring.

Case B. From now on suppose that $c_{4,0} \neq 0$. By a suitable K-linear change of the system x and y of variables of R, we may assume that $c_{4,0} = 1$, $c_{0,4} = 0$. Moreover, we shall divide this case into three subcases.

Case B1. Suppose that $c_{3,1} = c_{1,3} = 0$. Then $\partial F/\partial Y = ZF_Y$ where $F_Y = c_{0,1}Z^2 + c_{1,1}XZ + 2c_{0,2}YZ + c_{2,1}X^2 + 2c_{1,2}XY$. Clearly $V(\partial F/\partial X) \cap V(F_Y) = \{P\}$. Set

$$f_{X1} := \frac{1}{Y^3}\frac{\partial F}{\partial X} = c_{1,0}z^3 + 2c_{2,0}xz^2 + c_{1,1}z^2 + 2c_{2,1}xz + c_{1,2}z + x^3,$$

$$f_{Y1} := \frac{1}{Y^2}F_Y = c_{0,1}z^2 + c_{1,1}xz + 2c_{0,2}z + c_{2,1}x^2 + 2c_{1,2}x,$$

where $x = X/Y$ and $z = Z/Y$. Since $I(P, f_{X1} \cap f_{Y1}) = 6$, we get $c_{1,2} = 0$. First we consider the case where $c_{2,1} \neq 0$. Then, by a suitable affine change of the system x and y of variables of R, we may assume that $c_{2,0} = c_{1,1} = 0$. If $c_{0,2} \neq 0$, we obtain $c_{0,2}f_{X1} - c_{2,1}xf_{Y1} = c_{1,0}c_{0,2}z^3 - c_{0,1}c_{2,1}xz^2 + (c_{0,2} - c_{2,1}^2)x^3$, and so $c_{0,2} = c_{2,1}^2$ and $c_{0,1} = 0$. Hence

$$\begin{aligned} f &= c_{1,0}x + c_{2,1}^2y^2 + c_{3,0}x^3 + c_{2,1}x^2y + c_{0,3}y^3 + x^4 \\ &= c_{1,0}x + (c_{2,1}y - x^2)^2 + c_{3,0}x^3 + c_{0,3}y^3. \end{aligned}$$

Setting $y' := c_{2,1}y - x^2$, the polynomial $f - c_{3,0}x^3 - c_{0,3}y^3$ is given by $f' = c_{1,0}x + (y')^2$. Hence, if $c_{0,2} \neq 0$, then $R' = R^p[f'\,] = K[f', (y')^p]$ is a polynomial ring. On the other hand, if $c_{0,2} = 0$, we obtain $c_{0,1} = 0$, so that

$$f = c_{1,0}x + c_{3,0}x^3 + c_{2,1}x^2y + c_{0,3}y^3 + x^4.$$

Setting $y' := c_{2,1}y + x^2$, the polynomial $f - c_{3,0}x^3 - c_{0,3}y^3$ is given by $f' = c_{1,0}x + x^2y'$. Thus $R' = R^p[f'\,]$ is regular and non-factorial, and R has a p-basis over R' (see (B) in §1). Next we consider the case where $c_{2,1} = 0$. Since $I(P, f_{X1} \cap z) = 3$, we have $c_{0,2} = c_{1,1} = 0$ and $c_{0,1} \neq 0$, so that

$$f = c_{1,0}x + c_{0,1}y + c_{2,0}x^2 + c_{3,0}x^3 + c_{0,3}y^3 + x^4.$$

Thus R' is the polynomial ring $K[f', (y')^p]$ where $f' = f - c_{3,0}x^3 - c_{0,3}y^3$ and $y' = c_{0,1}y + c_{2,0}x^2 + x^4$.

Case B2. Suppose that $c_{3,1} \neq 0$. Then $H_X = X^3 + c_{1,3}Y^3$ and $H_X = c_{3,1}X^3$. Hence $c_{1,3} = 0$. Setting $Y' := X + c_{3,1}Y$, the polynomial F is given by

$$F' = \sum_{1 \leq i+j \leq 3} c'_{i,j}X^i(Y')^jZ^{4-i-j} + X^3Y'.$$

Hence this case is reduced to Case A.

Case B3. Suppose that $c_{3,1} = 0$ and $c_{1,3} \neq 0$. Then F is written as

$$\sum_{1 \leq i+j \leq 3} c_{i,j}X^iY^jZ^{4-i-j} + X(X + \sqrt[3]{c_{1,3}}Y)^3.$$

Setting $X' := X + \sqrt[3]{c_{1,3}}Y$ and $Y' := X$, the polynomial F is given by

$$F' = \sum_{1 \leq i+j \leq 3} c'_{i,j}(X')^i(Y')^jZ^{4-i-j} + (X')^3Y'.$$

Hence this case is reduced to Case A. □

Lemma 4.1 and Lemma 4.2 are made up into the following statement:

Theorem 4.3. *Assume that* $p = 3$. *Let* $f \in R$ *be such that* $R' = R^p[f]$ *is regular and* $\deg f = 4$. *Then, there exists a system* u *and* v *of variables of* R *such that one of the following conditions holds:*

(1) R' *is equal to the polynomial ring* $K[u, v^3]$;

(2) R' *is a non-factorial ring, and is equal to* $R^p[u + u^2v^2]$ *or* $R^p[u + u^2v]$, *or* $R^p[u + cu^2 + u^3v]$ *for some* $c \in K$.

Moreover, R *has no* p*-basis over* R' *in the case of* $R' = R^p[u + u^2v^2]$, *while* R *has a* p*-basis over* R' *in the other case.*

Corollary 4.4. *Assume that* $p = 3$. *Let* $f \in R$ *be such that* $R' = R^p[f]$ *is regular and* $\deg f \leq 4$. *If* R' *is factorial, then it is a polynomial ring in two variables over* K.

Proof. This assertion immediately follows from Theorem 4.3 and (B) in §1. □

§5. Questions

Finally we present three questions under the condition that K has a prime characteristic p greater than 3.

Question 1. For each $d \in \mathbb{N}$ with $d \geq p + 2$ and $d \not\equiv 0 \pmod{p}$, does there exist a polynomial $f \in R$ with $\deg f = d$ such that $R^p[f]$ is regular and factorial, but is not a polynomial ring in two variables over K, and R has a p-basis over $R^p[f]$?

Question 2. For each $d \in \mathbb{N}$ with $d \geq p + 2$ and $d \not\equiv 0 \pmod{p}$, does there exist a polynomial $f \in R$ with $\deg f = d$ such that $R^p[f]$ is regular and factorial, but is not a polynomial ring in two variables over K, and R has no p-basis over $R^p[f]$?

Question 3. Let f be a polynomial of $R - R^p$ such that $\deg f \leq p+1$ and $R^p[f]$ is regular and factorial. Does it follow that $R^p[f]$ is a polynomial ring in two variables over K?

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Tomoaki Ono
Tokyo Metropolitan College of Industrial Technology
8-52-1, Minami-senju, Arakawa-ku, Tokyo 116-8523, Japan
E-mail: `tono@acp.metro-cit.ac.jp`