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Maximal cycles in graphs of large girth

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Abstract. In this paper we give a lower bound of the circumference of a graph in terms of girth and the number of edges. It is shown that a graph of girth at least $g \geq 4$ with n vertices and at least $m \geq n$ edges contains a cycle of length at least $(g-2)m/(n-2)$. In particular, for the case $g = 4$, an analogous result for 2-edge-connected weighted graphs is given. Moreover, the extremal case is characterized in both results.

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§1. Introduction

In this paper we consider only finite simple graphs. The existence of long cycles in graphs have been paid much attention in the previous literature. The following is one of the classical result among such studies.

Theorem 1 (Erdős and Gallai, [2])**.** *Let G be a graph with n vertices. If $|E(G)| \geq n$, then G contains a cycle of length at least $2|E(G)|/(n-1)$.*

In this paper we extend this result by considering the graphs with large girth. Here $\theta_{r,k}$ denotes the graph obtained by exchanging each P_3 joining two vertices of degree r in $K_{2,r}$ for the path of length k (See Figure 1). Note that $\theta_{r,2} = K_{2,r}$ and $\theta_{2,k} = C_{2k}$ holds.

Theorem 2. *Let G be a graph with n vertices. If $|E(G)| \geq n$ and the girth of G is at least g, then G contains a cycle of length at least $(g-2)|E(G)|/(n-2)$. Moreover, if G does not contain any cycle of length more than $(g-2)|E(G)|/(n-2)$, then g is even and $G \simeq \theta_{r,g/2}$ for some r.*

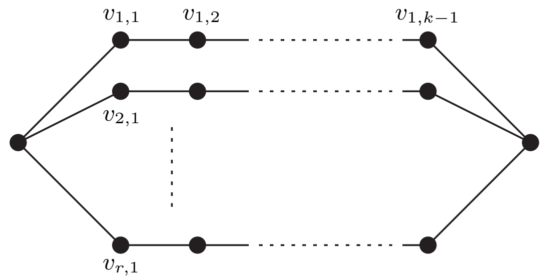

Figure 1: $\theta_{r,k}$.

In particular, for the case $g = 4$, we give an analogous result for *weighted graphs*. An *edge-weighted graph*, or simply a weighted graph, is one provided with an *edge-weighting function* w from the edge set to nonnegative real numbers. For an edge e of a weighted graph G, we call $w(e)$ the *weight* of e. The *weight* of a subgraph H of G is defined by the sum of the weights of the edges in H, denoted by $w(H)$. Moreover, the *weighted degree* $d_G^w(v)$ of a vertex v of G is the sum of the weights of the edges incident with v in G. An unweighted graph can be regarded as an weighted graph in which each edge has weight 1. In this sense, the following theorem generalizes Theorem 1 for 2-edge-connected graphs.

Theorem 3 (Bondy and Fan, [1]). *Let G be a 2-edge-connected weighted graph with n vertices. Then G contains a cycle of weight at least $2w(G)/(n-1)$.*

The following, which is a common generalization of Theorems 2 and 3 for 2-edge-connected graphs of girth at least 4, is our new result as well.

Theorem 4. *Let G be a 2-edge-connected weighted graph with n vertices. If the girth of G is at least 4, then G contains a cycle of weight at least $2w(G)/(n-2)$. Moreover, if $w(G) > 0$ and G does not contain any cycle of weight more than $2w(G)/(n-2)$, then i) $G \simeq K_{2,r}$ for some r, and ii) if $r \geq 3$, then each vertex of degree 2 has weighted degree $w(G)/(n-2)$.*

We prove Theorem 2 in Section 3 and Theorem 4 in the next section. In the rest of this section, we prepare terminology and notation used in this paper. Let G be a graph. For a vertex set X of G, we denote by $G[X]$ the subgraph of G induced by X. For $v \in V(G)$, we simply write $G - v$ instead of $G - \{v\}$ when there is no fear of confusion. The degree of v is denoted by $d(v)$, and the minimum degree of G is denoted by $\delta(G)$. An *endblock* is a block that has at most one cut vertex (we call a 2-connected graph itself an endblock as well). A block which is isomorphic to K_2 is called *trivial*.

If $G \simeq \theta_{r,k}$ for some $r, k \geq 2$, a pair of vertices (x, y) is called *an opposite pair* if both of x and y have degree r in G and the distance between x and y are k in G. Note that there is a unique opposite pair if $r \geq 3$, and there are k such pairs if $r = 2$. We call an weighted graph G with n vertices and girth at least 4 *extremal* if G does not contain any cycle of weight more than $2w(G)/(n-2)$.

§2. Proof of Theorem 4

Before we prove Theorem 4, we investigate the structure of weighted graphs in the extremal case.

Proposition 5. *Let G be an extremal weighted graph such that $G \simeq K_{2,n-2}$. Then*

a) *if $n \geq 5$, then each vertex of degree 2 has weighted degree $w(G)/(n-2)$,*

b) *if $n \geq 5$, then there exists a path of weight at least $2w(G)/(n-2)$ joining x and y for any two vertices x and y of degree 2 and*

c) *if $n \geq 4$, then there exists a path of weight $2w(G)/(n-2) - w(xy)$ joining x and y for any $xy \in E(G)$.*

Proof. First consider the case $n \geq 5$. Let V_2 be the vertices of degree 2 in G and let $s, t \in V(G) \setminus V_2$ with $s \neq t$. Moreover, let u_1 be the vertex of maximum weighted degree among V_2, and let u_2 be the second maximum one. Since $\sum_{v \in V_2} d_G^w(v) = w(G)$, $d_G^w(u_1) \geq w(G)/(n-2)$ holds. If $d_G^w(u_1) > w(G)/(n-2)$, then since the average weighted degree of the vertices in V_2 is $w(G)/(n-2)$ and $|V_2| \geq 3$, it follows that $d_G^w(u_1) + d_G^w(u_2) > 2w(G)/(n-2)$. Thus the weight of the cycle $u_1 s u_2 t u_1$ is more than $2w(G)/(n-2)$, contradicting the fact that G is extremal. Hence we have $d_G^w(u_1) = w(G)/(n-2)$, which implies a).

Let x, y, u be three distinct vertices in V_2. Then we can take two paths $P_1 = xsuty$ and $P_2 = xtusy$, and it follows from a) that $w(P_1) + w(P_2) = d_G^w(x) + d_G^w(y) + 2d_G^w(u) = 4w(G)/(n-2)$. Thus either P_1 or P_2 has weight at least $2w(G)/(n-2)$, which implies b).

Next consider the case $n \geq 4$. Let $xy \in E(G)$ and let $C = xyzvx$ be a cycle of G. If $n \geq 5$, it follows from a) that $w(C) = 2w(G)/(n-2)$, and if $n = 4$, then $w(C) = w(G) = 2w(G)/(n-2)$. In either case, the weight of the path $yzux$ is $2w(G)/(n-2) - w(xy)$, which implies c). □

Proof of Theorem 4. We use induction on n. The assertion immediately holds for the case $n = 4$. Thus we assume $n \geq 5$ and the theorem holds for every weighted graph with at most $n-1$ vertices. Moreover, the assertion immediately holds for the case $w(G)=0$, thus we assume that $w(G)>0$. Let C be the cycle of the heaviest weight in G. Note that $w(G)>0$ implies $w(C)>0$.

Assume that G has a cutvertex x. Let D_1 be a component of $G-x$, let $G_1 = G[V(D_1)\cup\{x\}]$ and let $G_2 = G[V(G)-V(D_1)]$. Note that both G_1 and G_2 are 2-edge-connected and $E(G_1)\cap E(G_2)=\emptyset$ holds by definition. By the induction hypothesis, there exists a cycle C_i of weight at least $2w(G_i)/(|V(G_i)|-2)$ in G_i, for $i=1$ and 2. Thus it follows that

$$\begin{aligned} 2w(G) &= 2w(G_1)+2w(G_2) \\ &\leq (|V(G_1)|-2)w(C_1)+(|V(G_2)|-2)w(C_2) \\ &\leq (n-3)w(C) \\ &< (n-2)w(C), \end{aligned}$$

which implies the assertion.

Next assume that G is 3-connected. In this case we use the following theorem.

Theorem 6 ([3])**.** *Let G be a 2-connected triangle-free weighted graph. If $d^w_G(v)\geq d$ for every $v\in V(G)$, then G contains a cycle of weight at least $2d$.*

By Theorem 6, we obtain the assertion if $d^w_G(v) > w(G)/(n-2)$ for every $v\in V(G)$. Thus we may assume that there exists a vertex $x\in V(G)$ such that $d^w_G(x)\leq w(G)/(n-2)$. Since $G-x$ is 2-connected, it follows from the induction hypothesis that there exists a cycle of weight at least

$$(2.1)\qquad \frac{2w(G-x)}{|V(G-x)|-2} \geq \frac{2}{n-3}\cdot\frac{(n-3)w(G)}{n-2} = \frac{2w(G)}{n-2}$$

in $G-x$. If the equality does not hold in (2.1), then we obtain the assertion, thus we assume that the equality holds in (2.1). Then $d^w_G(x)=w(G)/(n-2)$ and $G-x$ is extremal. Again by the induction hypothesis, it follows that $G-x\simeq K_{2,n-3}$. Since G is 3-connected, x is adjacent to all the vertices of degree 2 in $G-x$. Since the girth of G is at least 4, $G\simeq K_{3,n-3}$. Let $\mathcal{H}$ be the set of all the subgraphs of G which are isomorphic to $K_{3,3}$, then $|\mathcal{H}| = \binom{n-3}{3}$. Moreover, since each edge of G is contained in $\binom{n-4}{2}$ graphs in $\mathcal{H}$, $\sum_{H\in\mathcal{H}} w(H) = \binom{n-4}{2}w(G)$. Therefore we can find $H^*\in\mathcal{H}$ such that $w(H^*)\geq\binom{n-4}{2}w(G)/\binom{n-3}{3} = 3w(G)/(n-3)$. Since we can take three cycles C_1, C_2 and C_3 of length 6 in H^* so that each edge in H^* is contained in exactly two of C_1, C_2 and C_3 (i.e., $\{C_1, C_2, C_3\}$ is a cycle double cover of H^*), one of C_1, C_2 and C_3 must have weight at least $2w(H^*)/3 = 2w(G)/(n-3)$. Hence $w(C)\geq 2w(G)/(n-3) > 2w(G)/(n-2)$, which implies the assertion.

Therefore we may assume that the connectivity of G is exactly two. In the following we assume to the contrary that the assertion does not hold, i.e., we assume that $w(C) \le 2w(G)/(n-2)$, and if $w(C) = 2w(G)/(n-2)$, then either i) or ii) does not hold.

Claim 1. *For every cutset $\{u, v\}$, $uv \notin E(G)$.*

Proof. Assume that $uv \in E(G)$. Let D_1 be a component of $G - \{u, v\}$, let $D_2 = G - (\{u, v\} \cup V(D_1))$ and let $G_i = G[V(D_i) \cup \{u, v\}]$ for $i = 1, 2$. Then each G_i is 2-connected. Let C_i be the heaviest cycle in G_i, then it follows from the induction hypothesis that

$$\begin{aligned} (n-2)w(C) &\ge (|V(G_1)| - 2)w(C_1) + (|V(G_2)| - 2)w(C_2) \\ &\ge 2w(G_1) + 2w(G_2) \\ &= 2(w(G) + w(uv)) \\ &\ge 2w(G). \end{aligned}$$

Thus the equality hold in the above. Then we have $w(uv) = 0$, $w(C) = w(C_i)$ and G_i is extremal for $i = 1, 2$. Moreover, by the induction hypothesis, it follows that $G_1 \simeq K_{2,r_1}$, $G_2 \simeq K_{2,r_2}$ holds for some r_1 and r_2. By c) of Proposition 5, there exists a path P_i of weight $2w(G_i)/(|V(G_i)| - 2) = w(C_i)$ joining u and v in G_i for $i = 1, 2$. Then the weight of the cycle induced by P_1 and P_2 is $w(C_1) + w(C_2) = 2w(C) > w(C)$, which contradicts the choice of C. □

Claim 2. $\delta(G) \ge 3$.

Proof. Assume that there exists a vertex v of degree 2. In the case where there exists a vertex u such that $\{u, v\}$ is a cutset of G, let D_1 be a component of $G - \{u, v\}$, let $D_2 = G - (\{u, v\} \cup V(D_1))$ and let $G_i = G[V(D_i) \cup \{u, v\}]$ for $i = 1, 2$. Since G is 2-connected, we can take two neighbors, say x_1 and x_2, of v so that $x_i \in G_i$ for $i = 1, 2$. Moreover, since $|V(G)| \ge 5$, either G_1 or G_2 has at least 4 vertices. Without loss of generality, we may assume that $|V(G_1)| \ge 4$. Since $\{u, x_1\}$ is a cutset of G, it follows from Claim 1 that $ux_1 \notin E(G)$. Thus the edge vx_1 is not contained in any cycle of length 4 in G. Let G' be the weighted graph obtained from G by deleting the vertex v and adding the edge x_1x_2. Moreover, we assign weights to the edges of G' so that $w_{G'}(x_1x_2) = w_G(vx_1) + w_G(vx_2)$ and $w_{G'}(e) = w_G(e)$ for every $e \ne x_1x_2$. Then G' is 2-connected, the girth of G' is at least 4 and $|V(G')| = |V(G)| - 1$. By the induction hypothesis, there exists a cycle C' in G' such that

$$w(C') \ge \frac{2w(G')}{|V(G')| - 2} = \frac{2w(G)}{n-3} > \frac{2w(G)}{n-2}.$$

Since we can find a cycle of weight $w(C')$ in G whether the edge x_1x_2 is contained in C' or not, we have $w(C) > 2w(G)/(n-2)$, a contradiction.

Thus we may assume that $G - v$ is 2-connected. First consider the case where $d^w_G(v) \le w(G)/(n-2)$. By the induction hypothesis, we can find a cycle C' in $G - v$ such that

$$w(C') \ge \frac{2w(G-v)}{n-3} \ge \frac{2}{n-3} \cdot \frac{n-3}{n-2} w(G) = \frac{2}{n-2} w(G).$$

Since $w(C') \le w(C) \le 2w(G)/(n-2)$, it follows that $d^w_G(v) = w(G)/(n-2)$ and both G and $G - v$ are extremal. Since either i) or ii) does not hold by assumption, it follows from a) of Proposition 5 that $G \not\simeq K_{2,n-2}$, which implies $G - v \not\simeq K_{2,2}$. On the other hand, by the induction hypothesis, we have $G - v \simeq K_{2,n-3}$. Hence $n \ge 6$ holds. Since $G \not\simeq K_{2,n-2}$ and the girth of G is at least 4, v is adjacent to two vertices, say y_1 and y_2, of degree 2 in $G - v$. By b) of Proposition 5, there exists a path P of weight at least $2w(G-v)/(n-3) = 2w(G)/(n-2)$ joining y_1 and y_2 in $G - v$. Then the weight of the cycle vy_1Py_2v is $3w(G)/(n-2) > 2w(G)/(n-2)$, a contradiction.

Next consider the case where $d^w_G(v) > w(G)/(n-2)$. By the induction hypothesis, there exists a cycle C' of weight at least $2w(G-v)/(n-3) = 2(w(G) - d^w_G(v))/(n-3)$ in $G - v$. By taking two paths P_1, P_2 joining v and $V(C')$ which are disjoint except at v, we can find a cycle in G of weight at least

$$\begin{aligned} w(P_1) + w(P_2) + \frac{1}{2}w(C') &\ge d^w_G(v) + \frac{1}{2}w(C') \\ &\ge d^w_G(v) + \frac{w(G) - d^w_G(v)}{n-3} \\ &> \frac{n-4}{n-3} \cdot \frac{w(G)}{n-2} + \frac{w(G)}{n-3} \\ &= \frac{2w(G)}{n-2}, \end{aligned}$$

a contradiction. □

Let $\{u, v\}$ be a cutset of G, let D_1 be a component of $G - \{u, v\}$, let $D_2 = G - (\{u, v\} \cup V(D_1))$ and let $G_i = G[V(D_i) \cup \{u, v\}]$ for $i = 1, 2$. Since G is 2-connected, any endblock of G_i contains either u or v. Thus each G_i contains at most two endblocks, i.e., the block-cutvertex tree of G_i is a path. Let $\mathcal{B}_i$ be the set of blocks of G_i for $i = 1, 2$, and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \{B_1, B_2, \ldots, B_m\}$. Moreover, for a non-trivial block B_i, let C_i be the heaviest cycle in B_i.

Assume that each block in $\mathcal{B}$ is non-trivial. Then by the induction hypothesis,

$$(n-2)w(C) \ge (n-m)w(C) \ge \sum_{i=1}^{m} (|V(B_i)| - 2)w(C_i) \ge \sum_{i=1}^{m} 2w(B_i) = 2w(G),$$

and hence it follows that G and all the blocks in $\mathcal{B}$ are extremal. Since $w(G) > 0$, there exists $B_l \in \mathcal{B}$ such that $w(B_l) > 0$. Then by the induction hypothesis, $B_l \simeq K_{2,r}$ for some $r \geq 2$. Since the block-cutvertex tree of each G_i is a path, at most two vertices of B_l can have a neighbor in $G - V(B_l)$. Thus at least one vertex in B_l has degree 2, which contradicts Claim 2.

Thus there exists a trivial block B_k in $\mathcal{B}$. Let $V(B_k) = \{x, y\}$, then without loss of generality, we may assume that $x \in V(B_{k-1})$ and $y \in V(B_{k+1})$. By Claim 2, both B_{k-1} and B_{k+1} are non-trivial, and since the block-cutvertex tree of each G_i is a path, B_{k-1} (resp. B_{k+1}) contains a unique vertex x' (resp. y') such that $\{x, x'\}$ (resp. $\{y, y'\}$) is a cutset of G. By Claim 1, we have $xx', yy' \notin E(G)$, and thus the edge xy is not contained in any cycle of length 4 in G. Let G' be the weighted graph obtained from G by contracting the edge xy such that and $w_{G'}(e) = w_G(e)$ for every $e \in E(G')$. Then G' is a 2-connected weighted graph of girth at least 4. By the induction hypothesis, there exists a cycle C' of weight at least $2w(G')/(n-3)$ in G'. If C' is not a cycle in G, then $E(C') \cup \{xy\}$ induces a cycle in G, and its weight is

$$w(C') + w(xy) \geq \frac{2w(G')}{n-3} + w(xy) = \frac{2w(G)}{n-3} + \frac{(n-5)w(xy)}{n-3} > \frac{2w(G)}{n-2},$$

a contradiction. Thus C' is a cycle in G. If $w(xy) < w(G)/(n-2)$, then

$$w(C') > \frac{2\Big(w(G) - \dfrac{w(G)}{n-2}\Big)}{n-3} = \frac{2w(G)}{n-2},$$

a contradiction. Thus we have $w(xy) \geq w(G)/(n-2)$. By taking two disjoint paths joining $\{x, y\}$ and $V(C')$, we can find a cycle of weight at least

$$\begin{aligned} w(xy) + \frac{1}{2}w(C') &\geq w(xy) + \frac{w(G) - w(xy)}{n-3} \\ &= \frac{(n-4)w(xy)}{n-3} + \frac{w(G)}{n-3} \\ &\geq \frac{(n-4)w(G)}{(n-3)(n-2)} + \frac{w(G)}{n-3} \\ &= \frac{2w(G)}{n-2}. \end{aligned}$$

Since $w(C) \leq 2w(G)/(n-2)$, the equality holds in the above. Thus G' is extremal. By the induction hypothesis, we have $G' \simeq K_{2,n-3}$. Then G must contain a vertex of degree 2, which contradicts Claim 2. This completes the proof of Theorem 4. □

§3. Proof of Theorem 2

We use induction on n. If $n = g$, then the assertion immediately holds. Thus we assume that $n \geq g+1$ and the theorem holds for graphs with at most $n-1$ vertices. Let C be the longest cycle of G, then $|V(C)| \geq g$ holds. If G contains a vertex x of degree at most 1, then since $|E(G-x)| \geq |V(G-x)|$, it follows from the induction hypothesis that

$$\begin{aligned}(g-2)|E(G)| &= (g-2)|E(G-x)| + g - 2 \\ &\leq (|V(G-x)| - 2)|V(C)| + g - 2 \\ &= (n-3)|V(C)| + g - 2 \\ &< (n-2)|V(C)|,\end{aligned}$$

which implies $|V(C)| > (g-2)|E(G)|/(n-2)$. Thus we have $\delta(G) \geq 2$.

Assume that G is connected and has a cutvertex x. Let D_1 be a component of $G - x$, let $G_1 = G[V(D_1) \cup \{x\}]$ and let $G_2 = G[V(G) - V(D_1)]$. Since $\delta(G) \geq 2$, $|E(G_i)| \geq |V(G_i)|$ holds for $i = 1, 2$. By the induction hypothesis, there exists a cycle C_i of length at least $(g-2)|E(G_i)|/(|V(G_i)| - 2)$ in G_i, for $i = 1$ and 2. Thus it follows that

$$\begin{aligned}(g-2)|E(G)| &= (g-2)|E(G_1)| + (g-2)|E(G_2)| \\ &\leq (|V(G_1)| - 2)|V(C_1)| + (|V(G_2)| - 2)|V(C_2)| \\ &\leq (n-3)|V(C)| \\ &< (n-2)|V(C)|,\end{aligned}$$

which implies the assertion. By the similar argument, we obtain the assertion in the case G is disconnected. Thus we may assume that G is 2-connected. Moreover, we may assume that $g \geq 5$, since Theorem 4 implies the assertion if $g = 4$.

Let us assume for the moment that G is 3-connected. By the following theorem, we obtain the assertion if $\delta(G) > |E(G)|/(n-2)$.

Theorem 7 ([4]). *Let G be a 2-connected graph with girth $g \geq 5$ and $\delta(G) \geq 3$, then G contains a cycle of length at least $(g-2)\delta(G)$.*

Thus we may assume that there exists a vertex $x \in V(G)$ such that $d(x) \leq |E(G)|/(n-2)$. Since $G - x$ is 2-connected, it follows from the induction hypothesis that there exists a cycle of length at least

$$\frac{(g-2)|E(G-x)|}{n-3} \geq \frac{(g-2)|E(G)|}{n-2}$$

in $G - x$. Moreover, the equality holds in the above only when $G - x \simeq \theta_{r,g/2}$ for some r. However, since $d(x) \geq 3$ and $g \geq 5$, this cannot happen. Hence we obtain the assertion.

Therefore we may assume that the connectivity of G is exactly two. Let $\{u, v\}$ be a cutset of G, let D_1 be a component of $G - \{u, v\}$, let $D_2 = G - (\{u, v\} \cup V(D_1))$ and let $G_i = G[V(D_i) \cup \{u, v\}]$ for $i = 1, 2$. If $uv \in E(G)$, then each G_i is 2-connected, that is, $|E(G_i)| \geq |V(G_i)|$. Let c_i be the length of the longest cycle in G_i, then

$$\begin{aligned}(g-2)|E(G)| &= (g-2)|E(G_1)| + (g-2)|E(G_2)| - (g-2) \\ &\leq (|V(G_1)|-2)c_1 + (|V(G_2)|-2)c_2 - (g-2) \\ &\leq (|V(G_1)| + |V(G_2)| - 4)|V(C)| - (g-2) \\ &= (n-2)|V(C)| - (g-2) \\ &< (n-2)|V(C)|,\end{aligned}$$

which implies the assertion. Thus we may assume that $uv \notin E(G)$.

Since G is 2-connected, any endblock of G_i contains either u or v. Thus each G_i contains at most two endblocks, i.e., the block-cutvertex tree of each G_i is a path. Let k_i (resp. m_i) be the number of trivial (resp. non-trivial) blocks of G_i, let $H_1, \ldots, H_{m_1}$ be the non-trivial blocks of G_1 and let $H_{m_1+1}, \ldots, H_{m_1+m_2}$ be the non-trivial blocks of G_2. Moreover, let $k = k_1 + k_2$, $m = m_1 + m_2$ and $n_i = |V(H_i)|$. If $m = 0$, then G is a cycle, and thus the assertion holds. Hence we may assume $m \geq 1$. Then it follows from the induction hypothesis that

$$\begin{aligned}(n-2)|V(C)| &= \left(\sum_{i=1}^{m}(n_i - 1) + k - 2\right)|V(C)| \\ &= \sum_{i=1}^{m}(n_i-2)|V(C)| + (m+k-2)|V(C)| \\ &\geq \sum_{i=1}^{m}(n_i-2)|V(C_i)| + (m+k-2)|V(C)| \\ &\geq \sum_{i=1}^{m}(g-2)|E(H_i)| + (m+k-2)|V(C)| \\ &= (g-2)(|E(G)|-k) + (m+k-2)|V(C)| \\ &= (g-2)|E(G)| + (|V(C)|-(g-2))k + (m-2)|V(C)|, \qquad (3.1)\end{aligned}$$

where C_i is the longest cycle of H_i for each i. We divide the rest of the proof according to the value of m and k.

Case 1. $m \geq 2$.

In this case, it follows from (3.1) and the fact $|V(C)| - (g-2) \geq 2$ that $|V(C)| \geq (g-2)|E(G)|/(n-2)$. Assume that the equality holds. Then by (3.1), we have $m = 2$, $k = 0$, $|V(C_i)| = |V(C)|$ and $H_i \simeq \theta_{r_i, g/2}$ for some r_i.

For $i = 1, 2$, let P_i be the longest path joining u and v in H_i. Then since $H_i \simeq \theta_{r_i,g/2}$, $|E(P_i)| \geq g/2$ holds. On the other hand, it follows from the fact $|V(C_i)| = |V(C)|$ that $|V(C)| = g$, which implies $|E(P_i)| = g/2$ for $i = 1, 2$. Hence (u, v) is a opposite pair of H_i. This yields $G \simeq \theta_{r_1+r_2,g/2}$, and thus the assertion holds.

Case 2. $m = 1$.

Note that $k \geq 2$ follows from the assumption $m = 1$ of this case. Since the block-cutvertex tree of each G_i is a path, the edges in $E(G) \setminus E(H_1)$ induce a path, say P, of length k. Let s and t be the endvertices of P.

Subcase 2.1. $k \geq g/2$.

It follows from (3.1) that

$$\begin{aligned} (n-2)|V(C)| - (g-2)|E(G)| &\geq (|V(C)| - (g-2))k - |V(C)| \\ &\geq \frac{g}{2}|V(C)| - \frac{g(g-2)}{2} - |V(C)| \\ &= \frac{g-2}{2}(|V(C)| - g) \\ &\geq 0, \end{aligned}$$

which implies $|V(C)| \geq (g-2)|E(G)|/(n-2)$. Assume that the equality holds. Then we have $k = g/2$ and $g = |V(C)| = (g-2)|E(G)|/(n-2)$. Since $|E(H_1)| = |E(G)| - k = |E(G)| - g/2$ and $|V(H_1)| = |V(G)| - (k-1) = n - (g/2 - 1)$, it follows from the induction hypothesis that

$$\begin{aligned} |V(C_1)| &\geq \frac{(g-2)|E(H_1)|}{|V(H_1)| - 2} \\ &= \frac{(g-2)\left(|E(G)| - \frac{g}{2}\right)}{n - \left(\frac{g}{2} - 1\right) - 2} \\ &= \frac{(g-2)\left(|E(G)| - \frac{(g-2)|E(G)|}{2(n-2)}\right)}{n - 2 - \left(\frac{g-2}{2}\right)} \\ &= \frac{g-2}{n - 2 - \left(\frac{g-2}{2}\right)} \cdot \frac{(2(n-2) - (g-2))|E(G)|}{2(n-2)} \\ &= \frac{(g-2)|E(G)|}{(n-2)} = g. \end{aligned}$$

Since $|V(C_1)| \leq |V(C)| = g$, we have $|V(C_1)| = g$. Moreover, again by the induction hypothesis, it follows that $H_1 \simeq \theta_{r,g/2}$ for some r. Since $|V(C)| = g$,

(s,t) must be the opposite pair of H_1. Therefore, $G \simeq \theta_{r_1+1,g/2}$, which implies the assertion.

Subcase 2.2. $k < g/2$.

Since H_1 is a non-trivial block, we can take two internally disjoint paths P_1 and P_2 joining s and t in H_1. Then $sPtP_is$ is a cycle in G of length at least g, and hence $|E(P_i)| \geq g - |E(P)| = g - k$ hold for $i = 1, 2$. This yields $|V(C)| \geq |E(P_1)| + |E(P_2)| \geq 2(g-k)$, and thus it follows from (3.1) and the fact $k \geq 2$ that

$$\begin{aligned}(n-2)|V(C)| - (g-2)|E(G)| &\geq (|V(C)| - (g-2))k - |V(C)| \\ &= (k-1)|V(C)| - (g-2)k \\ &\geq 2(k-1)(g-k) - (g-2)k \\ &= (k-2)(g-2k). \qquad (3.2)\end{aligned}$$

Since $2 \leq k < g/2$, it follows that $|V(C)| \geq (g-2)|E(G)|/(n-2)$, and the equality holds only when $k = 2$.

We complete the proof of Theorem 2 by showing that $|V(C)| \neq (g-2)|E(G)|/(n-2)$ in this case. Assume to the contrary that $|V(C)| = (g-2)|E(G)|/(n-2)$. Then we obtain $|V(C)| = 2(g-k) = 2(g-2)$ by (3.2). Thus we have $|E(G)| = 2(n-2)$, $|E(H_1)| = |E(G)| - 2$ and $|V(H_1)| = |V(G)| - 1$. It follows from the induction hypothesis that

$$\begin{aligned}2(g-2) = |V(C)| \geq |V(C_1)| &\geq \frac{(g-2)|E(H_1)|}{|V(H_1)|-2} \\ &\geq \frac{(g-2)(|E(G)|-2)}{n-3} \\ &= \frac{(g-2)(2(n-2)-2)}{n-3} \\ &= 2(g-2).\end{aligned}$$

Therefore $|V(C_1)| = 2(g-2)$ and $H_1 \simeq \theta_{r,g/2}$ for some r. Hence we obtain $2(g-2) = |V(C_1)| = g$, which contradicts $g \geq 5$. □

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