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Hochschild cohomology ring of a cluster-tilted algebra of type $\mathbb{D}_4$

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Abstract. In this paper we describe the Hochschild cohomology rings for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of Dynkin type $\mathbb{D}_4$. In particular it is shown that the Hochschild cohomology rings modulo nilpotence for these algebras are isomorphic to the polynomial ring $K[x]$. As an application we prove that the cluster-tilted algebra of type $\mathbb{D}_4$ contained in this class satisfies the finiteness conditions **(Fg1)** and **(Fg2)** introduced in [EHSST].

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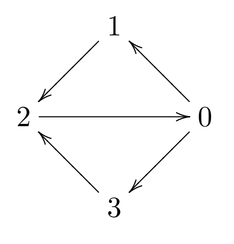
§1. Introduction

Let $\mathcal{Q}$ be the following quiver with four vertices 0, 1, 2, 3 and five arrows:



Let e_i be the trivial path corresponding to the vertex i for $i = 0, 1, 2$, and let f_1 the trivial path corresponding to the vertex 3. For our convenience, f_i also denotes the trivial path corresponding to i for $i = 0, 2$. Hence we may write $e_j = f_j$ for $j = 0, 2$. Let a_i be the arrow from i to $i+1$ for $i = 0, 1$, and let a_2 the arrow from 2 to 0. Moreover let b_0 the arrow from 0 to 3, and b_1 the arrow from 3 to 2. For our convenience b_2 also denotes the arrow from 2 to 0. Hence we may write $a_2 = b_2$.

Throughout this paper, we always consider the indices i of e_i, f_i, a_i and b_i as modulo 3. Hence it follows that, for all integers i, a_i starts at e_i and ends with e_{i+1}, whereas b_i starts at f_i and ends with f_{i+1}. We write paths from left to right.

Let K be an algebraically closed field, and let n be a non-negative integer. We denote by I_n the ideal in the path algebra $K\mathcal{Q}$ generated by the elements

$$(a_0a_1a_2)^n a_0a_1 - b_0b_1, \quad (a_ia_{i+1}a_{i+2})^n a_ia_{i+1}, \quad b_ib_{i+1} \quad \text{for } i = 1, 2.$$

Denote the algebra $K\mathcal{Q}/I_n$ by Λ_n. Then the set

$$\{(a_ia_{i+1}a_{i+2})^j,\ (a_ia_{i+1}a_{i+2})^j a_i,\ (a_ia_{i+1}a_{i+2})^k a_ia_{i+1},\ f_2,\ b_0,\ b_1,\ b_0b_1 \mid i = 0, 1, 2;\ j = 0, 1, \ldots, n;\ k = 0, 1, \ldots, n-1\}$$

is a K-basis of Λ_n, so that $\dim_K \Lambda_n = 9n + 10$. Also we easily see that Λ_n is a special biserial algebra and is not a selfinjective algebra. The purpose of this paper is to study the Hochschild cohomology of Λ_n.

In [BHL], Bastian, Holm and Ladkani introduced some finite quivers, called "standard forms," to investigate a derived equivalence classification for cluster-tilted algebras of Dynkin type $\mathbb{D}$. We notice that the quiver $\mathcal{Q}$ is one of these standard forms. Moreover, if $n = 0$, then the algebra Λ_0 is a Koszul cluster-tilted algebra of type $\mathbb{D}_4$ (see [ABS, F]), and hence is an algebra of finite representation type (see [BMR]). Also, Λ_0 appears in [BHL] as a representative of some derived equivalence class of cluster-tilted algebras of type $\mathbb{D}$.

In [F], we constructed a minimal projective bimodule resolution of Λ_n for all $n \geq 0$, and gave an explicit K-basis of the Hochschild cohomology groups of Λ_n. In this paper we use this K-basis to describe generators and relations of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_n)$ of Λ_n, where the product is given by the Yoneda product. In [EHSST], the authors proved that if a finite-dimensional algebra satisfies certain reasonable finiteness conditions, denoted by **(Fg1)** and **(Fg2)**, then the support varieties have a lot of analogous properties of those for finite group algebras (see also [Sn]). In particular it is proved in [EHSST, Theorem 2.5] that if these conditions are satisfied, then the algebra is Gorenstain and a module has trivial support variety if and only if it has finite projective dimension. In this paper, we show that the cluster-tilted algebra Λ_0 of type $\mathbb{D}_4$ satisfies **(Fg1)** and **(Fg2)**, and consider a condition for the support variety of a Λ_0-module to be trivial.

In [Sn], Snashall asked the following question: When is the Hochschild cohomology ring modulo nilpotence of a finite-dimensional algebra finitely generated as an algebra? It is known that the Hochschild cohomology rings modulo nilpotence for the classes of the following algebras are finitely generated as algebras: group algebras of finite groups ([E, V]), monomial algebras ([GSS2]), selfinjective algebras of finite representation type ([GSS1]), and algebras of finite global dimension ([H]). But any definitive answer to this question has not been obtained yet. Our main theorem shows that both the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_n)$ and the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\Lambda_n)/\mathcal{N}$ for $n \geq 0$ are finitely generated as algebras. Note that an example of the Hochschild cohomology ring modulo nilpotence which is not finitely generated appears in the papers [Sn, X].

This paper is organized as follows. In Section 2, we compute the products in the graded subring $\mathrm{HH}^{6*}(\Lambda_n) := \bigoplus_{i\geq 0} \mathrm{HH}^{6i}(\Lambda_n)$ of $\mathrm{HH}^*(\Lambda_n)$, and give generators and relations of $\mathrm{HH}^{6*}(\Lambda_n)$ (Proposition 2.1). In Section 3, we compute the products in the even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) := \bigoplus_{i\geq 0} \mathrm{HH}^{2i}(\Lambda_n)$, and find generators and relations of $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$, explicitly (Theorem 1). In Section 4, we describe all products in the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_n)$, and then as a main result we give the presentation of $\mathrm{HH}^*(\Lambda_n)$ by generators and relations for all $n \geq 0$ (Theorem 2). Moreover we determine the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(\Lambda_n)/\mathcal{N}$ for all $n \geq 0$. In section 5, as an application, we prove that the Ext algebra $E(\Lambda_0)$ of Λ_0 is finitely generated as a $\mathrm{HH}^{6*}(\Lambda_0)$-module, and consequently it is shown that Λ_0 satisfies **(Fg1)** and **(Fg2)** (Theorem 3). Finally we describe the support varieties for all indecomposabole modules over Λ_0 (Corollary 5.1), and determine the structures of the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type $\mathbb{D}_4$ (Corollary 5.2).

Throughout this paper, we denote the enveloping algebra $\Lambda_n^{\mathrm{op}} \otimes_K \Lambda_n$ of Λ_n by Λ_n^{e} (hence each Λ_n-Λ_n-bimodule corresponds to a right Λ_n^{e}-module and voice versa), and write $\otimes_K$ as $\otimes$, for simplicity. We always denote the minimal projective bimodule resolution of Λ_n given in [F] by $(Q^\bullet, \partial)$. For any $i \geq 0$ and right Λ^{e}-module homomorphism $\lambda : Q^i \to \Lambda_n$, we again write the element in $\mathrm{HH}^i(\Lambda_n) := \mathrm{Ext}^i_{\Lambda_n^{\mathrm{e}}}(\Lambda_n, \Lambda_n)$ represented by λ as λ, for simplicity.

§2. The subring $\mathrm{HH}^{6*}(\Lambda_n)$

In this section we investigate the products in the graded subring

$$\mathrm{HH}^{6*}(\Lambda_n) := \bigoplus_{i\geq 0} \mathrm{HH}^{6i}(\Lambda_n) = \bigoplus_{i\geq 0} \mathrm{Ext}^{6i}_{\Lambda_n^{\mathrm{e}}}(\Lambda_n, \Lambda_n)$$

of $\mathrm{HH}^*(\Lambda_n)$, and then find generators and relations of $\mathrm{HH}^{6*}(\Lambda_n)$.

We start by recalling the Yoneda product $\times$ in $\mathrm{HH}^*(\Lambda_n) := \bigoplus_{i\geq 0} \mathrm{HH}^i(\Lambda_n) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_{\Lambda_n^{\mathrm{e}}}(\Lambda_n, \Lambda_n)$. Let $\phi : Q^i \to \Lambda_n$ and $\psi : Q^j \to \Lambda_n$, $(i,\, j \geq 0)$ be right Λ_n^{e}-module homomorphisms. Then we have liftings $\sigma_0, \sigma_1, \ldots, \sigma_i$ of ψ, namely, there is the following commutative diagram of right Λ_n^{e}-modules:

$$
\begin{array}{ccccccccccccc}
Q^{i+j} & \xrightarrow{\partial^{i+j}} & Q^{i+j-1} & \xrightarrow{\partial^{i+j-1}} & \cdots & \xrightarrow{\partial^{j+2}} & Q^{j+1} & \xrightarrow{\partial^{j+1}} & Q^{j} & & & & \\
\downarrow{\scriptstyle \sigma^{i}} & & \downarrow{\scriptstyle \sigma^{i-1}} & & & & \downarrow{\scriptstyle \sigma^{1}} & & \downarrow{\scriptstyle \sigma^{0}} & \searrow{\scriptstyle \psi} & & & \\
Q^{i} & \xrightarrow{\partial^{i}} & Q^{i-1} & \xrightarrow{\partial^{i-1}} & \cdots & \xrightarrow{\partial^{2}} & Q^{1} & \xrightarrow{\partial^{1}} & Q^{0} & \xrightarrow{\partial^{0}} & \Lambda_n & \longrightarrow & 0 \\
& \searrow{\scriptstyle \phi} & & & & & & & & & & & \\
& & \Lambda_n & & & & & & & & & &
\end{array}
$$

We define the product $\phi \times \psi$ of the homogeneous elements $\phi \in \mathrm{HH}^i(\Lambda_n)$ and $\psi \in \mathrm{HH}^j(\Lambda_n)$ by $\phi\sigma^i \in \mathrm{HH}^{i+j}(\Lambda_n)$. Then the Yoneda product $\times$ in $\mathrm{HH}^*(\Lambda_n)$ is defined by linearly extending these to the products in $\mathrm{HH}^*(\Lambda_n)$.

Now, for simplicity, we denote the basis elements of $\mathrm{HH}^0(\Lambda_n)$ and $\mathrm{HH}^{6j}(\Lambda_n)$ $(j \geq 1)$ given in [F, Proposition 4.9] as follows:

$$
\begin{aligned}
X_{0,0} &:= \alpha_0^0 + \alpha_1^0 + \alpha_2^0 + \beta : Q^0 \to \Lambda_n; \\
X_{0,m} &:= \alpha_0^m + \alpha_1^m + \alpha_2^m : Q^0 \to \Lambda_n \text{ for } m = 1, \ldots, n, \text{ if } n > 0; \\
X_{6j,0} &:= \phi_0^0 + \phi_1^0 + \phi_2^0 - \psi : Q^{6j} \to \Lambda_n; \\
X_{6j,m} &:= \phi_0^m + \phi_1^m + \phi_2^m : Q^{6j} \to \Lambda_n \text{ for } m = 1, \ldots, n, \text{ if } n > 0.
\end{aligned}
$$

Then we have by [F, Lemma 4.1] that: for $l = 0, 1, 2$

$$
X_{0,0} : \begin{cases} e_l \otimes e_l & \mapsto \ e_l \\ f_1 \otimes f_1 & \mapsto \ f_1, \end{cases} \qquad
X_{0,m} : \begin{cases} e_l \otimes e_l & \mapsto \ (a_l a_{l+1} a_{l+2})^m \\ f_1 \otimes f_1 & \mapsto \ 0, \end{cases}
$$

$$
X_{6j,0} : \begin{cases} e_l \otimes e_l & \mapsto \ e_l \\ e_1 \otimes f_1 & \mapsto \ 0 \\ f_1 \otimes e_1 & \mapsto \ 0 \\ f_1 \otimes f_1 & \mapsto \ -f_1, \end{cases} \qquad
X_{6j,m} : \begin{cases} e_l \otimes e_l & \mapsto \ (a_l a_{l+1} a_{l+2})^m \\ e_1 \otimes f_1 & \mapsto \ 0 \\ f_1 \otimes e_1 & \mapsto \ 0 \\ f_1 \otimes f_1 & \mapsto \ 0. \end{cases}
$$

Remark 2.1. *It is known that the map* $\mathrm{HH}^0(\Lambda_n) \to Z(\Lambda_n)$, $h \mapsto h((\sum_{l=0}^2 e_l \otimes e_l) + f_1 \otimes f_1)$ *is an isomorphism of algebras, where the products in* $\mathrm{HH}^0(\Lambda_n)$ *are given by the Yoneda products. Then using this isomorphism we have*

$$
X_{0,s} \times X_{0,t} = \begin{cases} X_{0,s+t} & \text{if } s + t \leq n, \\ 0 & \text{if } s + t > n. \end{cases} \tag{2.1}
$$

for integers s and t with $0 \leq s, t \leq n$. In particular, we see that $X_{0,0}$ is the identity of $\mathrm{HH}^0(\Lambda_n)$.

For $u \geq 1$, we define the map $\sigma^0_{6u,0} : Q^{6u} \to Q^0$ of Λ_n-Λ_n-bimodules by

$$\sigma^0_{6u,0} : \begin{cases} e_l \otimes e_l & \mapsto \; e_l \otimes e_l \qquad \text{for } l = 0,1,2 \\ e_1 \otimes f_1 & \mapsto \; 0 \\ f_1 \otimes e_1 & \mapsto \; 0 \\ f_1 \otimes f_1 & \mapsto \; -f_1 \otimes f_1, \end{cases}$$

and $\sigma^i_{6u,0} : Q^{i+6u} \to Q^i$ by the identity map id_{Q^i} for $i \geq 1$.

Now by direct observations we have the following lemma.

Lemma 2.1. *We have* $X_{6u,0} = \partial^0 \sigma^0_{6u,0}$ *and* $\partial^i \sigma^i_{6u,0} = \sigma^{i-1}_{6u,0} \partial^{i+6}$ *for all* $u \geq 1$ *and* $i \geq 1$. *Hence* $\sigma^i_{6u,0}$ $(i \geq 0)$ *give liftings of* $X_{6u,0}$.

From the lemma above we can describe the products in $\mathrm{HH}^{6*}(\Lambda_n)$.

Lemma 2.2. *We have the following products*:

(a) $X_{6u,0} \times X_{0,s} = X_{6u,s}$ *for integers* $u \geq 1$ *and* s *with* $0 \leq s \leq n$.

(b) $X_{6u,0} \times X_{6v,0} = X_{6(u+v),0}$ *for integers* $u \geq 1$ *and* $v \geq 1$.

Hence, for integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, *and* s_2 *with* $0 \leq s_1, s_2 \leq n$,

$$X_{6u_1,s_1} \times X_{6u_2,s_2} = \begin{cases} X_{6(u_1+u_2),s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. (a) Let u be a positive integer. Then for $0 \leq s \leq n$ we get

$$X_{0,s}\sigma^0_{6u,0} : \begin{cases} e_l \otimes e_l & \mapsto \; (a_l a_{l+1} a_{l+2})^s \quad \text{for } l = 0,1,2 \\ e_1 \otimes f_1 & \mapsto \; 0 \\ f_1 \otimes e_1 & \mapsto \; 0 \\ f_1 \otimes f_1 & \mapsto \; 0. \end{cases}$$

Therefore $X_{0,s}\sigma^0_{6u,0} = X_{6u,s}$, which gives the desired product.

(b) Clearly $X_{6v,0}\sigma^{6v}_{6u,0} = X_{6(u+v),0}$ holds for $u \geq 1$ and $v \geq 1$, so that we have the desired product.

The last equality easily follows from (2.1), (a) and (b). □

Now we can find generators and relations of $\mathrm{HH}^{6*}(\Lambda_n)$. Here we note that $\mathrm{HH}^{6*}(\Lambda_n)$ is a commutative graded subring of $\mathrm{HH}^*(\Lambda_n)$.

Proposition 2.1. *There is the following isomorphism of graded rings*:

(a) *If* $n = 0$, *then* $\mathrm{HH}^{6*}(\Lambda_n) \simeq K[y_6]$, *where* $\deg y_6 = 6$.

(b) *If* $n > 0$, *then* $\mathrm{HH}^{6*}(\Lambda_n) \simeq K[y_0, y_6]/(y_0^{n+1})$, *where* $\deg y_0 = 0$ *and* $\deg y_6 = 6$.

Proof. We put $y_6 = X_{6,0}$. If $n = 0$, then by Lemma 2.2 we get the desired isomorphism in (a).

Now suppose that $n > 0$, and put $y_0 = X_{0,1}$. Then we see by (2.1) and Lemma 2.2 that $X_{6u,s} = y_0^s \times y_6^u$ for $u \geq 0$ and $s = 0, 1, \ldots, n$, so that $\{y_0, y_6\}$ is a generator of $\mathrm{HH}^{6*}(\Lambda_n)$. Also by (2.1) we have the relation $y_0^{n+1} = 0$. Therefore we get the desired isomorphism in (b). □

§3. The even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$

In this section we compute all products of basis elements in even degrees, and then determine the structure of the even Hochschild cohomology ring

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) = \bigoplus_{i \geq 0} \mathrm{HH}^{2i}(\Lambda_n).$$

Note that $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ is a commutative graded subring of $\mathrm{HH}^*(\Lambda_n)$. Throughout this section we keep the notations from Section 2.

For simplicity we denote the basis elements of $\mathrm{HH}^{6j+2}(\Lambda_n)$ and $\mathrm{HH}^{6j+4}(\Lambda_n)$ given in [F, Proposition 4.9] as follows: for $j \geq 0$

$$X_{6j+2,m} := \theta_0^m + \theta_1^m + \theta_2^m : Q^{6j+2} \to \Lambda_n \text{ for } m = 0, 1, \ldots, n-1, \text{ if } n > 0;$$
$$X_{6j+4,m} := \mu_0^m + \mu_1^m + \mu_2^m : Q^{6j+4} \to \Lambda_n \text{ for } m = 0, 1, \ldots, n-1, \text{ if } n > 0;$$
$$X_{6j+4,n} := \mu_0^n + \mu_1^n + \mu_2^n : Q^{6j+4} \to \Lambda_n \text{ if } \operatorname{char} K \mid 3n+2.$$

Note that, by [F, Lemma 4.1], for $s = 0, 1, \ldots, n-1$

$$X_{6j+2,s} : \begin{cases} e_l \otimes e_{l+2} & \mapsto (a_l a_{l+1} a_{l+2})^s a_l a_{l+1} \quad \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+2} & \mapsto 0 \quad \text{for } r = 1, 2, \end{cases}$$

and, for $t = 0, 1, \ldots, n$,

$$X_{6j+4,t} : \begin{cases} e_l \otimes e_{l+1} & \mapsto (a_l a_{l+1} a_{l+2})^t a_l \quad \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} & \mapsto 0 \quad \text{for } r = 0, 1. \end{cases}$$

hold.

3.1. Liftings of $X_{6u+2,0}$ and $X_{6u+4,0}$

To compute Yoneda products in $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ we find liftings of $X_{6u+2,0}$ and $X_{6u+4,0}$ for $u \geq 0$.

For $u \geq 0$ we define a homomorphism $\sigma^0_{6u+2,0} : Q^{6u+2} \to Q^0$ as Λ_n-Λ_n-bimodules by

$$\sigma^0_{6u+2,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto & a_l a_{l+1} \otimes e_{l+2} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} & \mapsto & 0 & \text{for } r = 1,2. \end{cases}$$

Also, for $u \geq 0$ and $i \geq 1$, define homomorphisms $\sigma^i_{6u+2,0} : Q^{6u+i+2} \to Q^i$ as Λ_n-Λ_n-bimodules by the following: For $j \geq 0$

$$\sigma^{3j+1}_{6u+2,0} : \begin{cases} e_l \otimes e_l & \mapsto & a_l a_{l+1} \otimes e_l & \text{for } l = 0,1,2 \\ e_1 \otimes f_1 & \mapsto & a_1 a_2 \otimes f_1 \\ f_1 \otimes e_1 & \mapsto & 0 \\ f_1 \otimes f_1 & \mapsto & 0, \end{cases}$$

$$\sigma^{3j+2}_{6u+2,0} : \begin{cases} e_l \otimes e_{l+1} & \mapsto & a_l a_{l+1} \otimes e_{l+1} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+1} & \mapsto & \begin{cases} a_0 a_1 \otimes f_1 & \text{for } r = 0 \\ 0 & \text{for } r = 1, \end{cases} \end{cases}$$

$$\sigma^{3j+3}_{6u+2,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto & a_l a_{l+1} \otimes e_{l+2} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} & \mapsto & \begin{cases} 0 & \text{for } r = 1 \\ a_2 a_0 \otimes f_1 & \text{for } r = 2. \end{cases} \end{cases}$$

Then by direct computations we have the following lemma.

Lemma 3.1. *We have $X_{6u+2,0} = \partial^0 \sigma^0_{6u+2,0}$ and $\partial^i \sigma^i_{6u+2,0} = \sigma^{i-1}_{6u+2,0} \partial^{6u+i+2}$ for all $u \geq 0$ and $i \geq 1$. Thus the map $\sigma^i_{6u+2,0} : Q^{6u+i+2} \to Q^i$ $(i \geq 0)$ is a lifting of $X_{6u+2,0}$.*

Next for $u \geq 0$ we define a homomorphism $\sigma^0_{6u+4,0} : Q^{6u+4} \to Q^0$ as Λ_n-Λ_n-bimodules by

$$\sigma^0_{6u+4,0} : \begin{cases} e_l \otimes e_{l+1} & \mapsto & a_l \otimes e_{l+1} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+1} & \mapsto & 0 & \text{for } r = 0,1. \end{cases}$$

Moreover for $u \geq 0$ and $i \geq 1$ we define homomorphisms $\sigma^i_{6u+4,0} : Q^{6u+i+4} \to$

Q^i as Λ_n-Λ_n-bimodules by: For $j \geq 0$

$$\sigma^{3j+1}_{6u+4,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \quad a_l \otimes e_{l+2} \qquad \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} & \mapsto \quad \begin{cases} 0 & \text{for } r = 1 \\ a_2 \otimes f_1 & \text{for } r = 2, \end{cases} \end{cases}$$

$$\sigma^{3j+2}_{6u+4,0} : \begin{cases} e_l \otimes e_l & \mapsto \quad a_l \otimes e_l \quad \text{for } l = 0,1,2 \\ e_1 \otimes f_1 & \mapsto \quad a_1 \otimes f_1 \\ f_1 \otimes e_1 & \mapsto \quad 0 \\ f_1 \otimes f_1 & \mapsto \quad 0, \end{cases}$$

$$\sigma^{3j+3}_{6u+4,0} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \quad a_l \otimes e_{l+1} \qquad \text{for } l = 0,1,2 \\ f_r \otimes f_{r+1} & \mapsto \quad \begin{cases} a_0 \otimes f_1 & \text{for } r = 0 \\ 0 & \text{for } r = 1. \end{cases} \end{cases}$$

We also have the following lemma.

Lemma 3.2. *We have* $X_{6u+4,0} = \partial^0 \sigma^0_{6u+4,0}$ *and* $\partial^i \sigma^i_{6u+4,0} = \sigma^{i-1}_{6u+4,0} \partial^{6u+i+4}$ *for all* $u \geq 0$ *and* $i \geq 1$*. Hence the map* $\sigma^i_{6u+4,0} : Q^{6u+i+4} \to Q^i \quad (i \geq 0)$ *is a lifting of* $X_{6u+4,0}$.

3.2. The products in $\mathrm{HH}^{6u}(\Lambda_n) \times \mathrm{HH}^{6v+2}(\Lambda_n)$

Now we investigate the products of elements in $\mathrm{HH}^{6u}(\Lambda_n)$ and $\mathrm{HH}^{6v+2}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.3. *Suppose that* $n > 0$ *(so that* $\mathrm{HH}^{6j+2}(\Lambda_n) \neq 0$*). We have the following products*:

(a) *For any integers* $u \geq 0$ *and* s *with* $0 \leq s \leq n$,

$$X_{6u+2,0} \times X_{0,s} = \begin{cases} X_{6u+2,s} & \text{if } 0 \leq s \leq n-1 \\ 0 & \text{if } s = n. \end{cases}$$

(b) $X_{6u,0} \times X_{2,0} = X_{6u+2,0}$ *for any integer* $u \geq 1$.

Consequently, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 *with* $0 \leq s_1 \leq n$, *and* s_2 *with* $0 \leq s_2 < n$, *we have*

$$X_{6u_1,s_1} \times X_{6u_2+2,s_2} = \begin{cases} X_{6(u_1+u_2)+2,s_1+s_2} & \text{if } 0 \leq s_1+s_2 \leq n-1 \\ 0 & \text{if } s_1+s_2 \geq n. \end{cases}$$

Proof. (a) Let u and s be integers with $u \geq 0$ and $0 \leq s \leq n$. Then we have

$$X_{0,s}\sigma^0_{6u+2,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} (a_l a_{l+1} a_{l+2})^s a_l a_{l+1} & \text{if } 0 \leq s \leq n-1, \\ & \text{ or if } s = n \text{ and } l = 0 \\ 0 & \text{if } s = n \text{ and } l \neq 0 \end{cases} \\ f_r \otimes f_{r+2} & \mapsto 0 \end{cases}$$

for $l = 0, 1, 2$ and $r = 1, 2$. This shows that

$$X_{0,s}\sigma^0_{6u+2,0} = \begin{cases} X_{6u+2,s} & \text{if } 0 \leq s \leq n-1 \\ \eta & \text{if } s = n. \end{cases}$$

However $\eta \in \operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6u+2}, \Lambda_n)$ by [F, Lemma 4.5 (b)], and so $X_{0,n}\sigma^0_{6u+2,0} = 0$ in $\mathrm{HH}^{6u+2}(\Lambda_n)$. Thus we get the desired equality.

(b) Clearly $X_{2,0}\sigma^2_{6u,0} = X_{6u+2,0}$ for all $u > 0$. So we have the required equality in (b).

The last equality follows from (a), (b), and Lemma 2.2. □

3.3. The products in $\mathrm{HH}^{6u}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Now we describe the products of elements in $\mathrm{HH}^{6u}(\Lambda_n)$ and $\mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.4. *Suppose that $n > 0$ or $\operatorname{char} K \mid 3n+2$ (hence $\mathrm{HH}^{6j+4}(\Lambda_n) \neq 0$). We have the following products*:

(a) *For any integers $u \geq 0$ and $0 \leq s \leq n$,*

$$X_{6u+4,0} \times X_{0,s} = \begin{cases} X_{6u+4,s} & \text{if } 0 \leq s \leq n-1, \text{ or if } s = n \text{ and } \operatorname{char} K \mid 3n+2 \\ 0 & \text{if } s = n \text{ and } \operatorname{char} K \nmid 3n+2. \end{cases}$$

(b) $X_{6u,0} \times X_{4,0} = X_{6u+4,0}$ *for any integer $u \geq 1$.*

So, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 \leq n$ (if $\operatorname{char} K \mid 3n+2$) or $0 \leq s_2 \leq n-1$ (if $\operatorname{char} K \nmid 3n+2$), we have

$$X_{6u_1,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+4,s_1+s_2} & \text{if } s_1 + s_2 \leq n-1, \\ & \text{ or if } \operatorname{char} K \mid 3n+2 \text{ and } s_1 + s_2 = n \\ 0 & \text{if } \operatorname{char} K \nmid 3n+2 \text{ and } s_1 + s_2 = n, \\ & \text{ or if } s_1 + s_2 > n. \end{cases}$$

Proof. (a) Let u and s be integers with $u \geq 0$ and $1 \leq s \leq n$. We get

$$X_{0,s}\sigma^0_{6u+4,0} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \quad (a_l a_{l+1} a_{l+2})^s a_l \quad \text{for } l = 0,1,2 \\ f_r \otimes f_{r+1} & \mapsto \quad 0 \quad \text{for } r = 0,1, \end{cases}$$

Hence $X_{0,s}\sigma^0_{6u+4,0} = \mu^s_0 + \mu^s_1 + \mu^s_2$. Therefore if $0 \leq s \leq n-1$, or if $s = n$ and $\operatorname{char} K \mid 3n+2$, then $X_{0,s}\sigma^0_{6u+4,0} = X_{6u+4,s}$. Also if $s = n$ and $\operatorname{char} K \nmid 3n+2$, then $X_{0,n}\sigma^0_{6u+4,0} \in \operatorname{Im}\operatorname{Hom}_{\Lambda^e_n}(\partial^{6j+4}, \Lambda_n)$ by [F, Lemma 4.5 (d)]. Therefore (a) is proved.

(b) We have $X_{4,0}\sigma_{6u,0} = X_{6u+4,0}$ for all $u \geq 0$. This yields the required equality in (b).

The last equality follows from (a), (b), and Lemma 2.2. □

3.4. The products in $\mathrm{HH}^{6u+2}(\Lambda_n) \times \mathrm{HH}^{6v+2}(\Lambda_n)$

Now we describe the products in $\mathrm{HH}^{6u+2}(\Lambda_n) \times \mathrm{HH}^{6v+2}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.5. *Suppose that $n > 0$. We get*

$$X^2_{2,0} = \begin{cases} X_{4,1} & \textit{if } n = 1 \textit{ and } \operatorname{char} K \mid 3n+2, \textit{ or if } n > 1 \\ 0 & \textit{if } n = 1 \textit{ and } \operatorname{char} K \nmid 3n+2. \end{cases}$$

Thus, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, and s_2 with $0 \leq s_1, s_2 \leq n-1$, we have

$$X_{6u_1+2,s_1} \times X_{6u_2+2,s_2} = \begin{cases} X_{6(u_1+u_2)+4,s_1+s_2+1} & \textit{if } s_1 + s_2 < n-1, \\ & \quad \textit{or if } \operatorname{char} K \mid 3n+2 \textit{ and } s_1 + s_2 = n-1 \\ 0 & \textit{if } s_1 + s_2 > n-1, \\ & \quad \textit{or if } \operatorname{char} K \nmid 3n+2 \textit{ and } s_1 + s_2 = n-1. \end{cases}$$

Proof. We have $X_{2,0}\sigma^2_{2,0} = \mu^1_0 + \mu^1_1 + \mu^1_2$. Therefore if $n = 1$ and $\operatorname{char} K \nmid 3n+2$, then $X_{2,0}\sigma^2_{2,0} \in \operatorname{Im}\operatorname{Hom}_{\Lambda^e_n}(\partial^4, \Lambda_n)$ by [F, Lemma 4.5 (d)]. So $X_{2,0}\sigma^2_{2,0} = 0$ in $\mathrm{HH}^4(\Lambda_n)$. On the other hand if $n = 1$ and $\operatorname{char} K \mid 3n+2$ or if $n > 1$, then $X_{2,0}\sigma^2_{2,0} = X_{4,1}$. This shows that the first equality holds.

The second equality follows from the first equality and Lemmas 2.2, 3.3, and 3.4. □

3.5. The products in $\mathrm{HH}^{6u+2}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Now we investigate the products of elements in $\mathrm{HH}^{6u+2}(\Lambda_n)$ and $\mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.6. *Suppose that $n > 0$. Then we get $X_{2,0} \times X_{4,0} = X_{6,1}$. Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n-1$, and s_2 with $0 \leq s_2 \leq n$ (if $\mathrm{char}\, K \mid 3n+2$) or $0 \leq s_2 \leq n-1$ (if $\mathrm{char}\, K \nmid 3n+2$), we have*

$$X_{6u_1+2,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2+1),s_1+s_2+1} & \text{if } s_1+s_2 \leq n-1, \\ 0 & \text{if } s_1+s_2 \geq n. \end{cases}$$

Proof. We have $X_{2,0}\sigma^2_{4,0} = \phi^1_0 + \phi^1_1 + \phi^1_2 = X_{6,1}$. Also, by this equality and Lemmas 2.2, 3.3, and 3.4, we have the second equality. □

3.6. The products in $\mathrm{HH}^{6u+4}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Finally we consider the products of elements in $\mathrm{HH}^{6u+4}(\Lambda_n)$ and $\mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 3.7. *Suppose that $n > 0$ or $\mathrm{char}\, K \mid 3n+2$. We have*

$$X^2_{4,0} = \begin{cases} 0 & \text{if } n = 0 \text{ and } \mathrm{char}\, K \mid 3n+2, \\ X_{8,0} & \text{if } n > 0. \end{cases}$$

Thus, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, and s_2 with $0 \leq s_1, s_2 \leq n$ (if $\mathrm{char}\, K \mid 3n+2$) or $0 \leq s_1, s_2 \leq n-1$ (if $\mathrm{char}\, K \nmid 3n+2$), we have

$$X_{6u_1+4,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2+1)+2,s_1+s_2} & \text{if } s_1+s_2 < n \\ 0 & \text{if } s_1+s_2 \geq n. \end{cases}$$

Proof. We have

$$X_{4,0}\sigma^4_{4,0} : \begin{cases} e_l \otimes e_{l+2} \mapsto a_l a_{l+1} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1,2. \end{cases}$$

Thus if $n = 0$ and $\mathrm{char}\, K \mid 3n+2$, then $X_{4,0}\sigma^4_{4,0} = \eta \in \mathrm{Im}\,\mathrm{Hom}_{\Lambda_0^e}(\partial^8, \Lambda_0)$ by [F, Lemma 4.5 (b)], so that $X_{4,0}\sigma^4_{4,0} = 0$ in $\mathrm{HH}^8(\Lambda_0)$. Moreover if $n > 0$ then $X_{4,0}\sigma^4_{4,0} = \theta^0_0 + \theta^0_1 + \theta^0_2 = X_{8,0}$. Therefore the first equality holds.

The second equality follows from the first equality and Lemmas 2.2, 3.3, and 3.4. □

3.7. Generators and relations of $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$

Now we can provide generators and relations of the even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$ of Λ_n.

Theorem 1. *We have the following isomorphism of commutative graded algebras*:

(a) *The case* char $K \mid 3n+2$:
(1) *If* $n = 0$ (*hence* char $K = 2$), *then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_0) \simeq K[y_4, y_6]/(y_4^2),$$

where $\deg y_i = i \quad (i = 4, 6)$.
(2) *If* $n > 0$, *then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) \simeq K[y_0, y_2, y_4, y_6]/(y_0^{n+1},\ y_0^n y_2,\ y_2^2 - y_0 y_4,\ y_4^2 - y_2 y_6),$$

where $\deg y_i = i \quad (i = 0, 2, 4, 6)$.

(b) *The case* char $K \nmid 3n+2$:
(1) *If* $n = 0$, *then*

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda_0) \simeq K[y_6], \quad \textit{where } \deg y_6 = 6.$$

(2) *If* $n = 1$, *then*

$$\begin{aligned}\mathrm{HH}^{\mathrm{ev}}(\Lambda_1) \simeq K&[y_0, y_2, y_4, y_6] \\ &/(y_0^2,\ y_0 y_2,\ y_0 y_4,\ y_2^2,\ y_2 y_4 - y_0 y_6,\ y_4^2 - y_2 y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 2, 4, 6)$.
(3) *If* $n > 1$, *then*

$$\begin{aligned}\mathrm{HH}^{\mathrm{ev}}(\Lambda_n) \simeq K&[y_0, y_2, y_4, y_6] \\ &/(y_0^{n+1},\ y_0^n y_2,\ y_0^n y_4,\ y_2^2 - y_0 y_4,\ y_2 y_4 - y_0 y_6,\ y_4^2 - y_2 y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 2, 4, 6)$.

Proof. We put

$$y_0 := X_{0,1},\ y_2 := X_{2,0},\ y_4 := X_{4,0},\ y_6 := X_{6,0}.$$

If char $K \mid 3n+2$ and $n = 0$ (hence char $K = 2$), then note that $\mathrm{HH}^{6u+s}(\Lambda_0) = K$ and $\mathrm{HH}^{6u+2}(\Lambda_0) = 0$ hold for all $u \geq 0$ and $s = 0, 4$. Since $X_{6u,0} = y_6^u$ and $X_{6u+4,0} = y_4 y_6^u$ hold for all $u \geq 0$, $\mathrm{HH}^{\mathrm{ev}}(\Lambda_0)$ is multiplicatively generated by y_4 and y_6. Moreover the equation $y_4^2 = 0$ holds.

If $\operatorname{char} K \nmid 3n+2$ and $n = 0$, then $\mathrm{HH}^{\mathrm{ev}}(\Lambda_0) = \mathrm{HH}^{6*}(\Lambda_0)$. So by Proposition 2.1 we have the desired isomorphism.

If $n > 0$, from Lemmas 2.2 and Lemmas 3.3 through 3.7, we have the following relations:

$$\begin{aligned} y_0^{n+1} &= 0, \quad y_0^n y_2 = 0, \\ y_2^2 &= \begin{cases} y_0 y_4 & \text{if } n = 1 \text{ and } \operatorname{char} K \mid 3n+2, \text{ or if } n > 1 \\ 0 & \text{if } n = 1 \text{ and } \operatorname{char} K \nmid 3n+2, \end{cases} \\ y_0^n y_4 &= \begin{cases} X_{4,n} & \text{if } \operatorname{char} K \mid 3n+2 \\ 0 & \text{if } \operatorname{char} K \nmid 3n+2, \end{cases} \\ y_4^2 &= y_2 y_6. \end{aligned}$$

Furthermore we have that

$$\begin{aligned} X_{6u,s} &= y_6^u y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n, \\ X_{6u+2,s} &= y_6^u y_2 y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+4,s} &= y_6^u y_4 y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s < n, \\ X_{6u+4,n} &= y_6^u y_4 y_0^s && \text{for } u \geq 0 \text{ (if } \operatorname{char} K \mid 3n+2). \end{aligned}$$

Therefore we take $\{y_0, y_2, y_4, y_6\}$ as algebra generators of $\mathrm{HH}^{\mathrm{ev}}(\Lambda_n)$. This completes the proof. □

§4. The Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_n)$

In this section we describe all products of basis elements of whole Hochshild cohomology ring $\mathrm{HH}^*(\Lambda_n)$, and then describe its ring structure of the Hochschild, completely. Throughout this section, we keep the notations from Sections 2 and 3.

For simplicity, we denote the basis elements of $\mathrm{HH}^{6j+1}(\Lambda_n)$ given in [F, Lemma 4.1]) as follows: for $j \geq 0$

$$\begin{aligned} X_{6j+1,0} &:= \mu_0^0 + (n+1)\nu_0 : Q^{6j+1} \to \Lambda_n; \\ X_{6j+1,m} &:= \mu_0^m : Q^{6j+1} \to \Lambda_n \text{ for } m = 1, \dots, n, \text{ if } n > 0, \end{aligned}$$

which is given by

$$X_{6j+1,0}:\begin{cases} e_l\otimes e_{l+1} & \mapsto \begin{cases} a_0 & \text{for } l=0\\ 0 & \text{for } l=1,2\end{cases}\\ f_r\otimes f_{r+1} & \mapsto \begin{cases} (n+1)b_0 & \text{for } r=0\\ 0 & \text{for } r=1,\end{cases}\end{cases}$$

$$X_{6j+1,m}:\begin{cases} e_l\otimes e_{l+1} & \mapsto \begin{cases} (a_la_{l+1}a_{l+2})^m a_l & \text{for } l=0\\ 0 & \text{for } l=1,2\end{cases}\\ f_r\otimes f_{r+1} & \mapsto 0 \qquad \text{for } r=0,1,\end{cases}$$

respectively. Similarly, we denote the basis elements of $\mathrm{HH}^{6j+3}(\Lambda_n)$ given in [F, Lemma 4.1]) as follows: for $j\geq 0$

$$X_{6j+3,0}:=\phi_0^0+\phi_1^0+\phi_2^0+\psi : Q^{6j+3}\to\Lambda_n,\ \text{if char}\,K\mid 3n+2;$$
$$X_{6j+3,m}:=\phi_0^m+\phi_1^m+\phi_2^m : Q^{6j+3}\to\Lambda_n$$
$$\text{for } m=1,\dots,n,\ \text{if } n>0 \text{ and char}\,K\mid 3n+2;$$
$$X_{6j+3,m}:=\phi_0^m : Q^{6j+3}\to\Lambda_n \text{ for } m=1,\dots,n,\ \text{if } n>0 \text{ and char}\,K\nmid 3n+2.$$

If char $K\mid 3n+2$, then $X_{6j+3,m}$ is given by

$$X_{6j+3,m}:\begin{cases} e_l\otimes e_l & \mapsto (a_la_{l+1}a_{l+2})^m \quad \text{for } l=0,1,2\\ f_1\otimes f_1 & \mapsto \begin{cases} f_1 & \text{if } m=0\\ 0 & \text{if } m\geq 1\end{cases}\\ e_1\otimes f_1 & \mapsto 0\\ f_1\otimes e_1 & \mapsto 0,\end{cases}$$

where $m=0,1,\dots,n$. If char $K\nmid 3n+2$ and $n>0$, then $X_{6j+3,m}$ is given by

$$X_{6j+3,m}:\begin{cases} e_l\otimes e_l & \mapsto \begin{cases} (a_0a_1a_2)^m & \text{for } l=0\\ 0 & \text{for } l=1,2\end{cases}\\ f_1\otimes f_1 & \mapsto 0\\ e_1\otimes f_1 & \mapsto 0\\ f_1\otimes e_1 & \mapsto 0,\end{cases}$$

where $m=1,\dots,n$. Moreover, if $n>0$, we denote the basis elements of $\mathrm{HH}^{6j+5}(\Lambda_n)$ as follows: for $j\geq 0$ and $m=0,1,\dots,n-1$

$$X_{6j+5,m}:=\theta_0^m : Q^{6j+5}\to\Lambda_n,$$

which is given by

$$X_{6j+5,m}:\begin{cases} e_l\otimes e_{l+2} & \mapsto \begin{cases}(a_0a_1a_2)^m a_0a_1 & \text{for } l=0\\ 0 & \text{for } l=1,2\end{cases}\\ f_r\otimes f_{r+2} & \mapsto\ 0 \qquad\qquad\qquad \text{for } r=1,2.\end{cases}$$

4.1. An initial part of liftings of $X_{1,0}$, $X_{3,0}$, $X_{3,1}$, and $X_{5,0}$

We start by giving an initial part of liftings for $X_{1,0}$, $X_{3,0}$, $X_{3,1}$, and $X_{5,0}$.

Definition 4.1. *For $u\geq 0$ and $j=0,1$, we define homomorphisms $\sigma^j_{6u+1,0}$: $Q^{6u+1+j}\to Q^j$ as Λ_n-Λ_n-bimodules by the following formulas*:

$$\sigma^0_{6u+1,0}:\begin{cases} e_l\otimes e_{l+1}\mapsto \begin{cases} a_0\otimes e_1 & \textit{for } l=0\\ 0 & \textit{for } l=1,2\end{cases}\\ f_r\otimes f_{r+1}\mapsto \begin{cases}(n+1)b_0\otimes f_1 & \textit{for } r=0\\ 0 & \textit{for } r=1,\end{cases}\end{cases}$$

$$\sigma^1_{6u+1,0}:\begin{cases} e_l\otimes e_{l+2}\mapsto \begin{cases} \sum\limits_{k=0}^{n}(k+1)(a_0a_1a_2)^k a_0\otimes(a_2a_0a_1)^{n-k}\\ +\sum\limits_{k=0}^{n-1}(k+1)(a_0a_1a_2)^k a_0a_1\otimes a_0a_1(a_2a_0a_1)^{n-k-1}\\ +\sum\limits_{k=1}^{n}k(a_0a_1a_2)^k\otimes a_1(a_2a_0a_1)^{n-k}-(n+1)b_0\otimes e_2\\ \hfill \textit{for } l=0\\ \sum\limits_{k=1}^{n}k(a_1a_2a_0)^k\otimes a_2(a_0a_1a_2)^{n-k}\\ +\sum\limits_{k=1}^{n}k(a_1a_2a_0)^k a_1\otimes(a_0a_1a_2)^{n-k}\\ +\sum\limits_{k=1}^{n-1}k(a_1a_2a_0)^k a_1a_2\otimes a_1a_2(a_0a_1a_1)^{n-k-1}\\ \hfill \textit{for } l=1\\ \sum\limits_{k=0}^{n-1}(k+1)(a_2a_0a_1)^k a_2a_0\otimes a_2a_0(a_1a_2a_0)^{n-k-1}\\ +\sum\limits_{k=1}^{n}k(a_2a_0a_1)^k\otimes a_0(a_1a_2a_0)^{n-k}\\ +\sum\limits_{k=1}^{n}k(a_2a_0a_1)^k a_2\otimes(a_1a_2a_0)^{n-k} \quad \textit{for } l=2\end{cases}\\ f_r\otimes f_{r+2}\mapsto 0 \qquad \textit{for } r=1,2.\end{cases}$$

By direct computations, we have the following lemma.

Lemma 4.1. *We have*

$$X_{6u+1,0} = \partial^0 \sigma^0_{6u+1,0} \text{ and } \partial^1 \sigma^1_{6u+1,0} = \sigma^0_{6u+1,0} \partial^2$$

for $u \geq 0$. *Hence* $\sigma^i_{6u+1,0}$ $(i = 0, 1)$ *is an initial part of a lifting of* $X_{6u+1,0}$.

Definition 4.2. *If* $\operatorname{char} K \mid 3n+2$, *for* $u \geq 0$ *and* $j = 0, 1, 2, 3$, *we define homomorphisms* $\sigma^j_{6u+3,0} : Q^{6u+3+j} \to Q^j$ *as* Λ_n-Λ_n*-bimodules by the following formulas*:

$$\sigma^0_{6u+3,0} : \begin{cases} e_l \otimes e_l & \mapsto \quad e_l \otimes e_l \quad \text{for } l = 0,1,2 \\ f_1 \otimes f_1 & \mapsto \quad f_1 \otimes f_1 \\ e_1 \otimes f_1 & \mapsto \quad 0 \\ f_1 \otimes e_1 & \mapsto \quad 0, \end{cases}$$

$$\sigma^1_{6u+3,0} : \begin{cases} e_l \otimes e_{l+1} & \mapsto \begin{cases} e_l \otimes e_{l+1} \qquad \text{if } n = 0 \text{ and } l = 0,1,2 \\ e_0 \otimes (a_1a_2a_0)^n \\ \quad + \sum_{k=0}^{3n-2} (k+2) a_0a_1a_2 \cdots a_k \otimes a_{k+2} \cdots a_{3n} \\ \qquad + (3n+1)(a_0a_1a_2)^n \otimes e_1 \\ \qquad\qquad \text{if } n \geq 1 \text{ and } l = 0 \\ e_1 \otimes (a_2a_0a_1)^n \\ \quad + \sum_{k=1}^{3n-1} (k+1) a_1a_2 \cdots a_k \otimes a_{k+2} \cdots a_{3n+1} \\ \qquad + (3n+1)(a_1a_2a_0)^n \otimes e_2 \\ \qquad\qquad \text{if } n \geq 1 \text{ and } l = 1 \\ e_2 \otimes (a_0a_1a_2)^n \\ \quad + \sum_{k=2}^{3n} k a_2a_3 \cdots a_k \otimes a_{k+2} \cdots a_{3n+2} \\ \qquad + (3n+1)(a_2a_0a_1)^n \otimes e_0 \\ \qquad\qquad \text{if } n \geq 1 \text{ and } l = 2 \end{cases} \\ f_r \otimes f_{r+1} & \mapsto \quad (-1)^r f_r \otimes f_{r+1} \qquad \text{for } r = 0,1, \end{cases}$$

$$\sigma^2_{6u+3,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \quad e_l \otimes e_{l+2} \qquad \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} & \mapsto \quad \begin{cases} -f_1 \otimes (a_0a_1a_2)^n & \text{for } r = 1 \\ -(a_2a_0a_1)^n \otimes f_1 & \text{for } r = 2, \end{cases} \end{cases}$$

$$\sigma^3_{6u+3,0}:\begin{cases} e_l\otimes e_l & \mapsto \begin{cases} e_l\otimes e_l & \text{if } n=0 \text{ and } l=0,1,2\\ e_0\otimes(a_0a_1a_2)^n+ & \\ \quad\sum_{k=0}^{3n-2}(k+2)a_0a_1a_2\cdots a_k\otimes a_{k+1}\cdots a_{3n-1} & \\ \qquad +(3n+1)(a_0a_1a_2)^n\otimes e_0 & \text{if } n\geq 1 \text{ and } l=0\\ e_1\otimes(a_1a_2a_0)^n+ & \\ \quad\sum_{k=1}^{3n-1}(k+1)a_1a_2\cdots a_k\otimes a_{k+1}\cdots a_{3n} & \\ \qquad +(3n+1)(a_1a_2a_0)^n\otimes e_1 & \text{if } n\geq 1 \text{ and } l=1\\ e_2\otimes(a_2a_0a_1)^n+ & \\ \quad\sum_{k=2}^{3n}ka_2a_3\cdots a_k\otimes a_{k+1}\cdots a_{3n+1} & \\ \qquad +(3n+1)(a_2a_0a_1)^n\otimes e_2 & \text{if } n\geq 1 \text{ and } l=2\end{cases}\\ f_1\otimes f_1 & \mapsto \begin{cases} f_1\otimes f_1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases}\\ e_1\otimes f_1 & \mapsto e_1\otimes f_1\\ f_1\otimes e_1 & \mapsto -f_1\otimes e_1.\end{cases}$$

Then we have the following lemma.

Lemma 4.2. *If* $\operatorname{char} K \mid 3n+2$, *we have that* $X_{6u+3,0}=\partial^0\sigma^0_{6u+3,0}$ *and* $\partial^i\sigma^i_{6u+3,0}=\sigma^{i-1}_{6u+3,0}\partial^{i+3}$ *for all* $u\geq 0$ *and* $i=1,2,3$. *Hence* $\sigma^i_{6u+3,0}$ $(i=0,1,2,3)$ *is an initial part of a lifting of* $X_{6u+3,0}$.

Definition 4.3. *If* $\operatorname{char} K\nmid 3n+2$ *and* $n>0$, *for* $u\geq 0$, *we define a homomorphism* $\sigma^0_{6u+3,1}:Q^{6u+3}\to Q^0$ *as* Λ_n-Λ_n*-bimodules by the following formulas*:

$$\sigma^0_{6u+3,1}:\begin{cases} e_l\otimes e_l & \mapsto \begin{cases} a_0a_1a_2\otimes e_0 & \text{for } l=0\\ 0 & \text{for } l=1,2\end{cases}\\ f_1\otimes f_1 & \mapsto 0\\ e_1\otimes f_1 & \mapsto 0\\ f_1\otimes e_1 & \mapsto 0,\end{cases}$$

Clearly, the following lemma holds.

Lemma 4.3. *If* $\operatorname{char} K\nmid 3n+2$ *and* $n>0$, *the equation* $X_{6u+3,1}=\partial^0\sigma^0_{6u+3,1}$ *holds for all* $u\geq 0$. *Hence* $\sigma^0_{6u+3,1}$ *is an initial part of a lifting of* $X_{6u+3,1}$.

Definition 4.4. *If $n > 0$, for $u \geq 0$, we define a homomorphism $\sigma^0_{6u+5,0} : Q^{6u+5} \to Q^0$ as Λ_n-Λ_n-bimodules by the following formula*:

$$\sigma^0_{6u+5,0} : \begin{cases} e_l \otimes e_{l+2} & \mapsto \begin{cases} a_0 a_1 \otimes e_2 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} & \mapsto \quad 0 \qquad\qquad \text{for } r = 1, 2. \end{cases}$$

Lemma 4.4. *If $n > 0$, we have that $X_{6u+5,0} = \partial^0 \sigma^0_{6u+5,0}$ for all $u \geq 0$. Hence $\sigma^0_{6u+5,0}$ is an initial part of a lifting of $X_{6u+5,0}$.*

4.2. The products in $\mathrm{HH}^{6u}(\Lambda_n) \times \mathrm{HH}^{6v+1}(\Lambda_n)$

First we compute the products of elements in $\mathrm{HH}^{6u}(\Lambda_n)$ and $\mathrm{HH}^{6v+1}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.5. *We have the following products*:

(a) $X_{0,s} \times X_{6u+1,0} = X_{6u+1,s}$ *for any integers $u \geq 0$ and s with $0 \leq s \leq n$.*

(b) $X_{6u,0} \times X_{1,0} = X_{6u+1,0}$ *for any integer $u \geq 1$.*

Therefore for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, and s_2 with $0 \leq s_1, s_2 \leq n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+1,s_2} = \begin{cases} X_{6(u_1+u_2)+1,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma^0_{6u+1,0} : \begin{cases} e_l \otimes e_{l+1} \mapsto \begin{cases} (a_0 a_1 a_2)^s a_0 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+1} \mapsto \begin{cases} (n+1) b_0 & \text{if } s = r = 0 \\ 0 & \text{otherwise,} \end{cases} \end{cases}$$

we have that $X_{0,s}\sigma^0_{6u+1,0} = X_{6u+1,s}$. Furthermore, it is clear that $X_{1,0}\sigma^1_{6u,0} = X_{6u+1,0}$ for all $u > 0$. Thus (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. □

4.3. The products in $\mathrm{HH}^{6u}(\Lambda_n) \times \mathrm{HH}^{6v+3}(\Lambda_n)$

In the following, we calculate the products of elements in $\mathrm{HH}^{6u}(\Lambda_n)$ and $\mathrm{HH}^{6v+3}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.6. *Let* $\operatorname{char} K \mid 3n+2$. *We have the following products*:

(a) $X_{0,s} \times X_{6u+3,0} = X_{6u+3,s}$ *for any integers* $u \geq 0$ *and* s *with* $0 \leq s \leq n$.

(b) $X_{6u,0} \times X_{3,0} = X_{6u+3,0}$ *for any integer* $u \geq 1$.

Therefore for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, *and* s_2 *with* $0 \leq s_1, s_2 \leq n$, *we have*

$$X_{6u_1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+3,s_1+s_2} & \text{if } s_1+s_2 \leq n \\ 0 & \text{if } s_1+s_2 > n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma^0_{6u+3,0} : \begin{cases} e_l \otimes e_l \mapsto (a_l a_{l+1} a_{l+2})^s & \text{for } l = 0,1,2 \\ f_1 \otimes f_1 \mapsto \begin{cases} f_1 & \text{if } s = 0 \\ 0 & \text{if } 0 < s \leq n \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have that $X_{0,s}\sigma^0_{6u+3,0} = X_{6u+3,s}$. Moreover, we have $X_{3,0}\sigma^3_{6u,0} = X_{6u+3,0}$ for all $u > 0$. So (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. □

Lemma 4.7. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *We have the following products*:

(a) *For any integers* $u \geq 0$ *and* s *with* $0 \leq s \leq n$,

$$X_{0,s} \times X_{6u+3,1} = \begin{cases} X_{6u+3,s+1} & \text{if } 0 \leq s < n \\ 0 & \text{if } s = n. \end{cases}$$

(b) $X_{6u,0} \times X_{3,1} = X_{6u+3,1}$ *for any integer* $u \geq 1$.

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 *with* $0 \leq s_1 \leq n$, *and* s_2 *with* $0 < s_2 \leq n$, *we have*

$$X_{6u_1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if } s_1+s_2 < n \\ 0 & \text{if } s_1+s_2 \geq n. \end{cases}$$

Proof. Since for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma^0_{6u+3,1} : \begin{cases} e_l \otimes e_l \mapsto \begin{cases} (a_0 a_1 a_2)^{s+1} & \text{for } l = 0 \\ 0 & \text{for } l = 1,2 \end{cases} \\ f_1 \otimes f_1 \mapsto 0 \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have that $X_{0,s}\sigma^0_{6u+3,1} = X_{6u+3,s+1}$. Moreover, for all $u > 0$, we have $X_{3,1}\sigma^3_{6u,0} = X_{6u+3,1}$. Thus (a) and (b) are proved. The last equation follows from (a), (b), and Lemma 2.2. □

4.4. The products in $\mathrm{HH}^{6u}(\Lambda_n) \times \mathrm{HH}^{6v+5}(\Lambda_n)$

Lemma 4.8. *Let $n > 0$. We have the following products:*

(a) *For any integers $u \geq 0$ and s with $0 \leq s \leq n$,*

$$X_{0,s} \times X_{6u+5,0} = \begin{cases} X_{6u+5,s} & \text{if } 0 \leq s < n \\ 0 & \text{if } s = n. \end{cases}$$

(b) *$X_{6u,0} \times X_{5,0} = X_{6u+5,0}$ for any integer $u \geq 1$.*

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have

$$X_{6u_1,s_1} \times X_{6u_2+5,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

Proof. Since, for integers u and s with $u \geq 0$ and $0 \leq s \leq n$,

$$X_{0,s}\sigma^0_{6u+5,0} : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} (a_0a_1a_2)^s a_0a_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2, \end{cases}$$

we have that $X_{0,s}\sigma^0_{6u+5,0} = \theta^s_0 = X_{6u+5,s}$ for $0 \leq s < n$ and $X_{0,n}\sigma^0_{6u+5,0} = \eta \in \operatorname{Im} \operatorname{Hom}_{\Lambda^e_n}(\partial^{6u+5}, \Lambda_n)$ by [F, Lemma 4.5 (e)]. Therefore (a) is proved. As for (b), we have $X_{5,0}\sigma^5_{6u,0} = X_{6u+5,0}$ for all $u > 0$. The last equation follows from (a), (b), and Lemma 2.2. □

4.5. The products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+1}(\Lambda_n)$

Lemma 4.9. *We have $X^2_{1,0} = 0$. Hence for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, and s_2 with $0 \leq s_1, s_2 \leq n$, we have $X_{6u_1+1,s_1} \times X_{6u_2+1,s_2} = 0$.*

Proof. We have

$$X_{1,0}\sigma^1_{1,0} : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} \dfrac{n(n+1)}{2}(a_0a_1a_2)^n a_0a_1 & \text{for } l = 0 \\ 0 & \text{for } l = 1, 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1, 2. \end{cases}$$

Since η is an element in $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^e}(\partial^2, \Lambda_n)$ (see [F, Lemma 4.5 (b)]), we have $X_{1,0}^2 = (n(n+1))/2\ \eta = 0$ in $\mathrm{HH}^2(\Lambda_n)$. The second equation follows from the first equation and Lemmas 2.2 and 4.5. □

4.6. The products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+2}(\Lambda_n)$

Next we describe the products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$ for $u \geq 0$ and $v \geq 0$.

Lemma 4.10. *Let $n > 0$. For any integer $u \geq 0$, we have the following product*:

$$X_{6u+1,0} \times X_{2,0} = \begin{cases} 3^{-1} X_{6u+3,1} & \text{if } \operatorname{char} K \mid 3n+2 \\ X_{6u+3,1} & \text{if } \operatorname{char} K \nmid 3n+2. \end{cases}$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have

$$\begin{aligned} &X_{6u_1+1,s_1} \times X_{6u_2+2,s_2} \\ &= \begin{cases} 3^{-1} X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if } \operatorname{char} K \mid 3n+2 \text{ and } s_1+s_2 < n \\ X_{6(u_1+u_2)+3,s_1+s_2+1} & \text{if } \operatorname{char} K \nmid 3n+2 \text{ and } s_1+s_2 < n \\ 0 & \text{if } s_1+s_2 \geq n. \end{cases} \end{aligned}$$

Proof. Since, for any integer $u \geq 0$,

$$X_{6u+1,0}\sigma_{2,0}^1 : \begin{cases} e_l \otimes e_l \mapsto \begin{cases} 0 & \text{for } l = 0, 2 \\ a_1 a_2 a_0 & \text{for } l = 1 \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0 \\ f_1 \otimes f_1 \mapsto 0, \end{cases}$$

we have

$$\begin{aligned} &X_{6u+1,0} \times X_{2,0} = \phi_1^1 \\ &= \begin{cases} \phi_0^1 + \phi_1^1 + \phi_2^1 + (\phi_0^1 - \phi_2^1) & \text{if } \operatorname{char} K \mid 3n+2 \text{ and } \operatorname{char} K = 2 \\ 3^{-1}(\phi_0^1 + \phi_1^1 + \phi_2^1) + 3^{-1}(\phi_0^1 - \phi_2^1) + 3^{-1} \cdot 2(\phi_1^1 - \phi_0^1) & \\ \qquad\qquad \text{if } \operatorname{char} K \mid 3n+2 \text{ and } \operatorname{char} K \neq 2 & \\ \phi_0^1 + (\phi_1^1 - \phi_0^1) & \text{if } \operatorname{char} K \nmid 3n+2. \end{cases} \end{aligned}$$

Since $\phi_0^1 - \phi_2^1$ and $\phi_1^1 - \phi_0^1$ are in $\operatorname{Im}\operatorname{Hom}_{\Lambda_n^e}(\partial^{6u+3}, \Lambda_n)$ (see [F, Lemma 4.5 (c)]), the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.3, 4.6, and 4.7. □

Remark 4.1. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *Then we have* $X_{6u+3,s} = X_{6,0}^u X_{0,1}^{s-1} X_{1,0} X_{2,0}$ *for* $u \geq 0$ *and* $0 < s \leq n$. *Hence* $\mathrm{HH}^{6u+3}(\Lambda_n)$ *is generated by the products of* $X_{6,0}, X_{0,1}, X_{1,0}, X_{2,0}$ *for* $u \geq 0$.

4.7. The products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+3}(\Lambda_n)$

Lemma 4.11. *Let* $\operatorname{char} K \mid 3n+2$. *For any integer* $u \geq 0$, *we have the following product*:

$$X_{6u+1,0} \times X_{3,0} = \begin{cases} X_{6u+4,n} & \textit{if } \operatorname{char} K = 2 \textit{ and } n \equiv 0 \ (\textit{mod } 4) \\ 0 & \textit{otherwise} \end{cases}$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1, *and* s_2 *with* $0 \leq s_1, s_2 \leq n$, *we have*

$$X_{6u_1+1,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+4,n} & \textit{if } \operatorname{char} K = 2,\ n \equiv 0 \ (\textit{mod } 4) \\ & \quad \textit{and } s_1 = s_2 = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since, for an integer $u \geq 0$,

$$X_{6u+1,0}\sigma_{3,0}^1 : \begin{cases} e_l \otimes e_{l+1} \mapsto \begin{cases} \dfrac{(3n+2)(n+1)}{2}(a_0a_1a_2)^n a_0 & \text{for } l = 0 \\ \dfrac{3n(n+1)}{2}(a_1a_2a_0)^n a_1 & \text{for } l = 1 \\ \dfrac{(3n+1)n}{2}(a_2a_0a_1)^n a_2 & \text{for } l = 2 \end{cases} \\ f_r \otimes f_{r+1} \mapsto \begin{cases} (n+1)b_0 & \text{for } r = 1 \\ 0 & \text{for } r = 2 \end{cases} \end{cases}$$

it follows that

$$\begin{aligned} X_{6u+1,0} &\times X_{3,0} \\ &= \frac{(3n+2)(n+1)}{2}\mu_0^n + \frac{3n(n+1)}{2}\mu_1^n + \frac{(3n+1)n}{2}\mu_2^n + (n+1)\nu_0 \\ &= \begin{cases} -(n+1)(\mu_1^n - \mu_2^n - \nu_0) & \text{if } n \text{ odd} \\ \dfrac{3n+2}{2}(\mu_0^n + \mu_1^n + \mu_2^n) - (n+1)(\mu_1^n - \mu_2^n - \nu_0) & \text{if } n \text{ even} \end{cases} \end{aligned}$$

hold. Note that $\mu_1^n - \mu_2^n - \nu_0 \in \operatorname{Im} \operatorname{Hom}_{\Lambda_n^e}(\partial^{6u+4}, \Lambda_n)$ by [F, Lemma 4.5 (d)]. Therefore we have that

$$X_{6u+1,0} \times X_{3,0} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \dfrac{3n+2}{2} X_{6u+4,n} & \text{if } n \text{ even.} \end{cases}$$

If n is even, then $\operatorname{char} K = 2$ or $\operatorname{char} K \mid (3n+2)/2$. Thus we have

$$\frac{3n+2}{2} = \begin{cases} 1 & \text{if } \operatorname{char} K = 2 \text{ and } n \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the first equation is proved. The second equality follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.6. □

Corollary 4.1. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *For any integer* $u \geq 0$, *we have* $X_{6u+1,0} \times X_{3,1} = 0$. *Hence, for any integers* $u_1 \geq 0$, $u_2 \geq 0$, s_1 *with* $0 \leq s_1 \leq n$, *and* s_2 *with* $0 < s_2 \leq n$, *we have* $X_{6u_1+1,s_1} \times X_{6u_2+3,s_2} = 0$.

4.8. The products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Lemma 4.12. *Suppose that* $n > 0$ *or* $\operatorname{char} K \mid 3n+2$. *For any integer* $u \geq 0$, *we have the following products*:

$$X_{6u+1,0} \times X_{4,0} = \begin{cases} 0 & \textit{if } \operatorname{char} K = 2 \textit{ and } n = 0 \\ X_{6u+5,0} & \textit{otherwise.} \end{cases}$$

Hence for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 *with* $0 \leq s_1 \leq n$, *and* s_2 *with* $0 \leq s_2 \leq n$ (*if* $\operatorname{char} K \mid 3n+2$) *or* $0 \leq s_2 < n$ (*if* $\operatorname{char} K \nmid 3n+2$), *we have*

$$X_{6u_1+1,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \textit{if } n > 0 \textit{ and } s_1 + s_2 < n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since, for any integer $u \geq 0$,

$$X_{6u+1,0}\sigma^1_{4,0} : \begin{cases} e_l \otimes e_{l+2} \mapsto \begin{cases} 0 & \text{for } l = 0, 1 \\ a_2 a_0 & \text{for } l = 2 \end{cases} \\ f_r \otimes f_{r+2} \mapsto 0 \qquad \text{for } r = 1, 2, \end{cases}$$

we have that

$$X_{6u+1,0} \times X_{4,0} = \begin{cases} 0 & \text{if } \operatorname{char} K \mid 3n+2 \text{ and } n = 0 \\ \theta_2^0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0 & \text{if } \operatorname{char} K = 2 \text{ and } n = 0 \\ \theta_0^0 - (\theta_0^0 - \theta_2^0) & \text{otherwise.} \end{cases}$$

Since $\theta_0^0 - \theta_2^0$ is in Im $\mathrm{Hom}_{\Lambda_n^e}(\partial^{6u+5}, \Lambda_n)$ (see [F, Lemma 4.5 (e)]), the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.8. □

Remark 4.2. *If $n > 0$, then by Lemma 4.8 and 4.12 we have $X_{6u+5,s} = X_{6,0}^u X_{0,1}^s X_{1,0} X_{4,0}$ holds for $u \geq 0$ and $0 \leq s < n$. Thus $\mathrm{HH}^{6u+5}(\Lambda_n)$ is generated by the products of $X_{6,0}, X_{1,0}, X_{4,0}, X_{0,1}$ for $u > 0$. Note that the equation $X_{0,1}^n X_{1,0} X_{4,0} = 0$ holds.*

4.9. The products in $\mathrm{HH}^{6u+1}(\Lambda_n) \times \mathrm{HH}^{6v+5}(\Lambda_n)$

By the previous sections, we have the following corollary:

Corollary 4.2. *Let $n > 0$. For any integer $u \geq 0$, we have $X_{6u+1,0} \times X_{5,0} = 0$. Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 \leq n$, and s_2 with $0 \leq s_2 < n$, we have $X_{6u_1+1,s_1} \times X_{6u_2+5,s_2} = 0$.*

4.10. The products in $\mathrm{HH}^{6u+2}(\Lambda_n) \times \mathrm{HH}^{6v+3}(\Lambda_n)$

Lemma 4.13. *Let $\mathrm{char}\, K \mid 3n+2$ and $n > 0$ (hence $\mathrm{char}\, K \neq 3$). For any integer $u \geq 0$, we have the following product:*

$$X_{6u+2,0} \times X_{3,0} = 3X_{6u+5,0}$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 with $0 \leq s_1 < n$, and s_2 with $0 \leq s_2 \leq n$, we have

$$X_{6u_1+2,s_1} \times X_{6u_2+3,s_2} = \begin{cases} 3X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

Proof. Since

$$X_{6u+2,0}\sigma_{3,0}^2 = \begin{cases} e_l \otimes e_{l+2} \mapsto a_l a_{l+1} & \text{for } l = 0,1,2 \\ f_r \otimes f_{r+2} \mapsto 0 & \text{for } r = 1,2, \end{cases}$$

it follows that

$$\begin{aligned} X_{6u+2,0} \times X_{3,0} &= \theta_0^0 + \theta_1^0 + \theta_2^0 \\ &= (\theta_1^0 - \theta_0^0) - (\theta_0^0 - \theta_2^0) + 3\theta_0^0 \\ &= 3\theta_0^0. \end{aligned}$$

holds in $\mathrm{HH}^{6u+5}(\Lambda_n)$. Notice that $\theta_1^0 - \theta_0^0$ and $\theta_0^0 - \theta_2^0$ are in Im $\mathrm{Hom}_{\Lambda_n^e}(\partial^{6u+5}, \Lambda_n)$ (see [F, Lemma 4.5 (e)]). The second equality follows from the first equation and Lemmas 2.2, 3.3, 4.6, and 4.8. □

Corollary 4.3. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *For any integer* $u \geq 0$, *we have*

$$X_{6u+2,0} \times X_{3,1} = \begin{cases} 0 & \text{if } n = 1 \\ X_{6u+5,1} & \text{if } n \geq 2. \end{cases}$$

Therefore, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 *with* $0 \leq s_1 < n$, *and* s_2 *with* $0 < s_2 \leq n$, *we have*

$$X_{6u_1+2,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2)+5,s_1+s_2} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

4.11. The products in $\mathrm{HH}^{6u+2}(\Lambda_n)$ and $\mathrm{HH}^{6v+5}(\Lambda_n)$

By the previous sections we get the following corollary:

Corollary 4.4. *Let* $n > 0$. *For any integer* $u \geq 0$, *we have* $X_{6u+2,0} \times X_{5,0} = X_{6u+7,1}$. *Hence, for any integers* $u_1 \geq 0$, $u_2 \geq 0$, s_1, *and* s_2 *with* $0 \leq s_1, s_2 \leq n$, *we have*

$$X_{6u_1+2,s_1} \times X_{6u_2+5,s_2} = \begin{cases} X_{6(u_1+u_2+1)+1,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

4.12. The products in $\mathrm{HH}^{6u+3}(\Lambda_n) \times \mathrm{HH}^{6v+3}(\Lambda_n)$

We next describe the products in $\mathrm{HH}^{6u+3}(\Lambda_n) \times \mathrm{HH}^{6v+3}(\Lambda_n)$ for $u, v \geq 0$. First we consider the case where $\operatorname{char} K \mid 3n+2$.

Lemma 4.14. *Let* $\operatorname{char} K \mid 3n+2$. *For any integer* $u \geq 0$, *we have the following product*:

$$X_{3,0}^2 = \begin{cases} X_{6,n} & \text{if } \operatorname{char} K = 2 \text{ and } n \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for any integers $u_1 \geq 0$, $u_2 \geq 0$, s_1 *and* s_2 *with* $0 \leq s_1, s_2 \leq n$, *we have*

$$X_{6u_1+3,s_1} \times X_{6u_2+3,s_2} = \begin{cases} X_{6(u_1+u_2+1),n} & \text{if } \operatorname{char} K = 2,\ n \equiv 0 \pmod 4 \\ & \quad \text{and } s_1 = s_2 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since

$$X_{3,0}\sigma^3_{3,0}: \begin{cases} e_l \otimes e_l \mapsto \dfrac{(3n+1)(3n+2)}{2}(a_l a_{l+1} a_{l+2})^n & \text{for } l = 0,1,2 \\ f_1 \otimes f_1 \mapsto \begin{cases} f_1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \\ e_1 \otimes f_1 \mapsto 0 \\ f_1 \otimes e_1 \mapsto 0, \end{cases}$$

we have

$$\begin{aligned} X^2_{3,0} &= \begin{cases} \phi^0_0 + \phi^0_1 + \phi^0_2 + \psi & \text{if } n = 0 \\ \dfrac{(3n+1)(3n+2)}{2}\left(\phi^0_0 + \phi^0_1 + \phi^0_2\right) & \text{if } n \geq 1 \end{cases} \\ &= \frac{(3n+1)(3n+2)}{2} X_{6,n}. \end{aligned}$$

In the case n odd, we have $X^2_{3,0} = 0$. If n is even, then $\operatorname{char} K = 2$ or $\operatorname{char} K \mid (3n+2)/2$. Thus we have that

$$\begin{aligned} \frac{(3n+1)(3n+2)}{2} = -\frac{3n+2}{2} &= \frac{3n+2}{2} \\ &= \begin{cases} 1 & \text{if } \operatorname{char} K = 2 \text{ and } n \equiv 0 \pmod 4 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore the first equation is proved. The second equation follows from the first equation and Lemmas 2.2 and 4.6. □

Similarly, by direct computations and the previous sections, we also have the following:

Corollary 4.5. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *We have* $X^2_{3,1} = 0$. *Therefore, for any integers* $u_1 \geq 0$, $u_2 \geq 0$, s_1, *and* s_2 *with* $0 < s_1, s_2 \leq n$, *we have* $X_{6u_1+3,s_1} \times X_{6u_2+3,s_2} = 0$.

4.13. The products in $\mathrm{HH}^{6u+3}(\Lambda_n) \times \mathrm{HH}^{6v+4}(\Lambda_n)$

Lemma 4.15. *Let* $\operatorname{char} K \mid 3n+2$. *For any integer* $u \geq 0$, *we have* $X_{6u+4,0} \times X_{3,0} = 3X_{6u+7,0}$. *Hence, for any integers* $u_1 \geq 0$, $u_2 \geq 0$, s_1 *and* s_2 *with* $0 \leq s_1, s_2 \leq n$, *we have*

$$X_{6u_1+3,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+7,s_1+s_2} & \textit{if } s_1 + s_2 \leq n \\ 0 & \textit{if } s_1 + s_2 > n. \end{cases}$$

Proof. For integer $u \geq 0$, we have the following:

$$X_{6u+3,0}\sigma_{4,0}^{3} : \begin{cases} e_l \otimes e_{l+1} \mapsto a_l & \text{for } l = 0, 1, 2 \\ f_r \otimes f_{r+1} \mapsto 0 & \text{for } r = 0, 1. \end{cases}$$

Therefore we have that

$$\begin{aligned} X_{6u+4,0} \times X_{3,0} &= \mu_0^0 + \mu_1^0 + \mu_2^0 \\ &= 3(\mu_0^0 + (n+1)\nu_0) + (\mu_1^0 - \mu_0^0) - (\mu_0^0 - \mu_2^0 + \nu_0). \end{aligned}$$

Then we get $X_{6u+4,0} \times X_{3,0} = 3X_{6u+7,0}$, since $\mu_1^0 - \mu_0^0$ and $\mu_0^0 - \mu_2^0 + \nu_0$ are in $\mathrm{Im}\,\mathrm{Hom}_{\Lambda_n^e}(\partial^{6u+7}, \Lambda_n)$ (see [F, Lemma 4.5 (a)]). Therefore the first equation is proved. The second equation follows from the first equation and Lemmas 2.2, 3.4, 4.5, and 4.6. □

Similarly, by direct computations and the previous sections, we also have the following:

Corollary 4.6. *Let* $\operatorname{char} K \nmid 3n+2$ *and* $n > 0$. *For any integer* $u \geq 0$, *we have* $X_{6u+4,0} \times X_{3,1} = X_{6u+7,1}$. *Therefore, for any integers* $u_1 \geq 0,\ u_2 \geq 0$, s_1, *and* s_2 *with* $0 < s_1 \leq n,\ 0 \leq s_2 < n$, *we have*

$$X_{6u_1+3,s_1} \times X_{6u_2+4,s_2} = \begin{cases} X_{6(u_1+u_2)+7,s_1+s_2} & \text{if } s_1 + s_2 \leq n \\ 0 & \text{if } s_1 + s_2 > n. \end{cases}$$

4.14. The products in $\mathrm{HH}^{6u+i}(\Lambda_n) \times \mathrm{HH}^{6v+5}$ for $i = 3, 4, 5$

By the previous sections we get the following corollaries:

Corollary 4.7. *Let* $n \geq 0$. *Then the following statements hold.*

(a) *Let* $\operatorname{char} K \mid 3n+2$ (*hence* $\operatorname{char} K \neq 3$). *Then, for any integer* $u \geq 0$,

$$X_{6u+3,0} \times X_{5,0} = 0 \quad \text{and} \quad X_{6u+4,0} \times X_{5,0} = 3^{-1}X_{6u+9,1}.$$

Hence, for any integers $u_1 \geq 0,\ u_2 \geq 0$, s_1 *and* s_2 *with* $0 \leq s_1, s_2 < n$,

$$X_{6u_1+3,s_1} \times X_{6u_2+5,s_2} = 0 \quad \text{and}$$

$$X_{6u_1+4,s_1} \times X_{6u_2+5,s_2} = \begin{cases} 3^{-1}X_{6(u_1+u_2)+9,s_1+s_2+1} & \text{if } s_1 + s_2 < n \\ 0 & \text{if } s_1 + s_2 \geq n. \end{cases}$$

(b) *Let* char $K \nmid 3n+2$. *Then, for any integer* $u \geq 0$,

$$X_{6u+3,1} \times X_{5,0} = 0 \quad \textit{and} \quad X_{6u+4,0} \times X_{5,0} = X_{6u+9,1}.$$

Hence, for any integers $u_1 \geq 0,\ u_2 \geq 0$,

$$X_{6u_1+3,s_1} \times X_{6u_2+5,s_2} = 0$$

for s_1 *with* $0 < s_1 \leq n$, *and* s_2 *with* $0 \leq s_2 < n$, *and*

$$X_{6u_1+4,s_1} \times X_{6u_2+5,s_2} = \begin{cases} X_{6(u_1+u_2)+9,s_1+s_2+1} & \textit{if } s_1+s_2 < n \\ 0 & \textit{if } s_1+s_2 \geq n \end{cases}$$

for s_1 *and* s_2 *with* $0 \leq s_1, s_2 < n$.

(c) $X_{5,0}^2 = 0$. *Hence for any integers* $u_1 \geq 0,\ u_2 \geq 0,\ s_1$ *and* s_2 *with* $0 \leq s_1, s_2 < n$.

4.15. Generators and relations of $\mathrm{HH}^*(\Lambda_n)$

By summarizing Sections 2 and 3 and Sections 4.1 through 4.16, we have the following theorem.

Theorem 2. *The Hochschild cohomology ring of* Λ_n *is commutative, and it is given as follows*:

(a) *The case* char $K \mid 3n+2$*:*

(1) *If* char $K = 2$ *and* $n = 0$, *then*

$$\mathrm{HH}^*(\Lambda_0) \simeq K[y_1, y_3]/(y_1^2),$$

where $\deg y_i = i \quad (i = 1, 3)$.

(2) *If* char $K = 2,\ n \equiv 0 \pmod 4$ *and* $n \neq 0$, *then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_n) \simeq K[&y_0, y_1, y_2, y_3, y_4, y_6] \\ &/(y_0^{n+1},\ y_1^2,\ y_0^n y_2,\ y_1y_2 - y_0y_3,\ y_1y_3 - y_0^n y_4, \\ &\quad y_2^2 - y_0y_4,\ y_1y_4 - y_2y_3,\ y_2y_4 - y_0y_6,\ y_3^2 - y_0^n y_6, \\ &\quad y_1y_6 - y_3y_4,\ y_4^2 - y_2y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 1, 2, 3, 4, 6)$.

(3) *If* char $K = 2,\ n \not\equiv 0 \pmod 4$, *or if* char $K \neq 2$, *then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_n) \simeq K[&y_0, y_1, y_2, y_3, y_4, y_6] \\ &/(y_0^{n+1},\ y_1^2,\ y_0^n y_2,\ 3y_1y_2 - y_0y_3,\ y_1y_3, \\ &\quad y_2^2 - y_0y_4,\ 3y_1y_4 - y_2y_3,\ y_2y_4 - y_0y_6,\ y_3^2, \\ &\quad 3y_1y_6 - y_3y_4,\ y_4^2 - y_2y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 1, 2, 3, 4, 6)$.

(b) *The case* char $K \nmid 3n+2$*:*

(1) *If* $n = 0$*, then*

$$\mathrm{HH}^*(\Lambda_0) \simeq K[y_1, y_6]/(y_1^2),$$

where $\deg y_i = i \quad (i = 1, 6)$.

(2) *If* $n = 1$*, then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_1) \simeq K[&y_0, y_1, y_2, y_4, y_6] \\ &/(y_0^2,\ y_1^2,\ y_0y_2,\ y_0y_4,\ y_2^2,\ y_2y_4 - y_0y_6,\ y_4^2 - y_2y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 1, 2, 4, 6)$.

(3) *If* $n > 1$*, then*

$$\begin{aligned}\mathrm{HH}^*(\Lambda_n) \simeq K[y_0, y_1, y_2, y_4, y_6]/(&y_0^{n+1},\ y_1^2,\ y_0^n y_2,\ y_0^n y_4, \\ &y_2^2 - y_0y_4,\ y_2y_4 - y_0y_6,\ y_4^2 - y_2y_6),\end{aligned}$$

where $\deg y_i = i \quad (i = 0, 1, 2, 4, 6)$.

Proof. (a) Suppose char $K \mid 3n+2$. We put

$$y_0 := X_{0,1},\ y_1 := X_{1,0},\ y_2 := X_{2,0},\ y_3 := X_{3,0},\ y_4 := X_{4,0},\ y_6 := X_{6,0}.$$

(1): If $n = 0$ (hence char $K = 2$), then note that $\mathrm{HH}^{3u+s}(\Lambda_0) = K$ and $\mathrm{HH}^{3u+2}(\Lambda_0) = 0$ hold for all $u \geq 0$ and $s = 0, 1$. Since $X_{4,0} = y_1y_3$ and $y_6 = y_3^2$ hold, we have $X_{3u,0} = y_3^u$ and $X_{3u+1,0} = y_3^u y_1$ hold for all $u \geq 0$. Thus $\mathrm{HH}^*(\Lambda_0)$ is multiplicatively generated by y_1 and y_3, and the equation $y_1^2 = 0$ holds. Therefore we have the desired isomorphism.

(2) and (3): If $n > 0$, from Sections 2 and 3 and Sections 4.1 through 4.16 we have the following equations:

$$\begin{aligned}
&y_0^{n+1} = 0,\ y_1^2 = 0,\ y_0^n y_2 = 0,\ 3y_1y_2 = y_0y_3,\\
&y_1y_3 = \begin{cases} y_0^n y_4 & \text{if char } K = 2,\ n \equiv 0 \pmod 4 \\ 0 & \text{otherwise,} \end{cases}\\
&y_2^2 = y_0y_4,\ 3y_1y_4 = y_2y_3,\ y_2y_4 = y_0y_6,\\
&y_3^2 = \begin{cases} y_0^n y_6 & \text{if char } K = 2,\ n \equiv 0 \pmod 4 \\ 0 & \text{otherwise,} \end{cases}\\
&3y_1y_6 = y_3y_4,\ y_4^2 = y_2y_6.
\end{aligned}$$

Note that the equations $y_0^n y_2 = 0$ and $3y_1y_4 = y_2y_3$ yield the equation $y_0^n y_1 y_4 = 0$. Moreover we have that

$$X_{6u,s} = y_6^u y_0^s \qquad \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,$$

$$\begin{aligned}
X_{6u+1,s} &= y_6^u y_0^s y_1 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,\\
X_{6u+2,s} &= y_6^u y_0^s y_2 && \text{for } u \geq 0 \text{ and } 0 \leq s < n,\\
X_{6u+3,s} &= y_6^u y_0^s y_3 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,\\
X_{6u+4,s} &= y_6^u y_0^s y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,\\
X_{6u+5,s} &= y_6^u y_0^s y_1 y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n.
\end{aligned}$$

Thus it is shown that the relations are enough, and therefore we can take $\{y_0, y_1, y_2, y_3, y_4, y_6\}$ as algebra generators of $\mathrm{HH}^*(\Lambda_n)$. Note that $\mathrm{HH}^*(\Lambda_n)$ is a commutative algebra, since $\mathrm{HH}^*(\Lambda_n)$ is graded commutative and y_1^2, y_1y_3, y_3^2 are zero if char $K \neq 2$. Hence we have the results.

(b) Suppose char $K \nmid 3n+2$. We put

$$y_0 := X_{0,1},\ y_1 := X_{1,0},\ y_2 := X_{2,0},\ y_4 := X_{4,0},\ y_6 := X_{6,0}.$$

(1): If $n = 0$ then note that $\mathrm{HH}^{6u+s}(\Lambda_0) = K$ and $\mathrm{HH}^{6u+t}(\Lambda_0) = 0$ hold for all $u \geq 0$, $s = 0, 1$, and $t = 2, 3, 4, 5$. Then we have $y_1^2 = 0$, and the equations $X_{6u,0} = y_6^u$ and $X_{6u+1,0} = y_6^u y_1$ hold for $u \geq 0$. Hence $\mathrm{HH}^*(\Lambda_0)$ is multiplicatively generated by y_1, y_6, and we have the result.

(2) and (3): If $n > 0$, from Sections 2 and 3 and Sections 4.1 through 4.16 we have the following equations:

$$y_0^{n+1} = 0,\ y_1^2 = 0,\ y_0^n y_2 = 0,\ y_0^n y_4 = 0,\ y_2^2 = \begin{cases} 0 & \text{for } n = 1\\ y_0 y_4 & \text{for } n > 1,\end{cases}$$
$$y_2 y_4 = y_0 y_6,\ y_4^2 = y_2 y_6.$$

Furthermore we have

$$\begin{aligned}
X_{6u,s} &= y_6^u y_0^s && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,\\
X_{6u+1,s} &= y_6^u y_0^s y_1 && \text{for } u \geq 0 \text{ and } 0 \leq s \leq n,\\
X_{6u+2,s} &= y_6^u y_0^s y_2 && \text{for } u \geq 0 \text{ and } 0 \leq s < n,\\
X_{6u+3,s} &= y_6^u y_0^{s-1} y_1 y_2 && \text{for } u \geq 0 \text{ and } 0 < s \leq n,\\
X_{6u+4,s} &= y_6^u y_0^s y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n,\\
X_{6u+5,s} &= y_6^u y_0^s y_1 y_4 && \text{for } u \geq 0 \text{ and } 0 \leq s < n.
\end{aligned}$$

Hence it is shown that the relations are enough, and therefore we can take $\{y_0, y_1, y_2, y_4, y_6\}$ as algebra generators of $\mathrm{HH}^*(\Lambda_n)$. Moreover $\mathrm{HH}^*(\Lambda_n)$ is commutative, since $\mathrm{HH}^*(\Lambda_n)$ is graded commutative and $y_1^2 = 0$. □

Finally, by Theorem 2, we have the following structure of the Hochschild cohomology ring modulo nilpotence of Λ_n:

Corollary 4.8. *There is the following of isomorphism of commutative graded algebras*:

$$\mathrm{HH}^*(\Lambda_n)/\mathcal{N} \simeq K[x], \quad \textit{where } \deg x = \begin{cases} 3 & \textit{if } n = 0 \textit{ and } \mathrm{char}\, K = 2 \\ 6 & \textit{otherwise.} \end{cases}$$

Hence for $n \geq 0$, $\mathrm{HH}^*(\Lambda_n)/\mathcal{N}$ is finitely generated as an algebra.

Proof. If $n = 0$, the statement is clear. Also, if $\mathrm{char}\, K \mid 3n+2$ and $n > 0$, then y_0, y_1, y_2, y_3, and y_4 are nilpotent elements, and moreover if $\mathrm{char}\, K \nmid 3n+2$ and $n > 0$, then y_0, y_1, y_2, and y_4 are nilpotent elements. This completes the proof. □

§5. Applications

Throughout this section we suppose that $n = 0$, that is, we only deal with the cluster-tilted algebra Λ_0 of type $\mathbb{D}_4$, so that denote Λ_0 by Λ, for simplicity. Also we keep the notation from the previous sections.

In this section, as an application, we show that Λ satisfies the finiteness conditions **(Fg1)** and **(Fg2)**, and describe the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type $\mathbb{D}_4$.

5.1. (Fg1) and (Fg2)

We start by recalling the finiteness conditions **(Fg1)** and **(Fg2)** of [EHSST]. Let A be a finite-dimensional algebra, and let $E(A)$ denote the Ext algebra of A

$$E(A) := \mathrm{Ext}_A^*(A/\mathfrak{r}_A, A/\mathfrak{r}_A) = \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(A/\mathfrak{r}_A, A/\mathfrak{r}_A),$$

where $\mathfrak{r}_A$ is the Jacobson radical of A. We then see that the functor $A/\mathfrak{r}_A \otimes_A -$ naturally induces a homomorphism $\phi_A : \mathrm{HH}^*(A) \to E(A)$ of graded algebras. For a graded subalgebra S of $\mathrm{HH}^*(A)$ we will consider $E(A)$ as a S-module by using ϕ_A. Then **(Fg1)** and **(Fg2)** are as follows:

(Fg1) There is a graded subalgebra H of $\mathrm{HH}^*(A)$ such that:

(i) H is a commutative noetherian ring.

(ii) $H^0 = \mathrm{HH}^0(A) = Z(A)$.

(Fg2) $E(A)$ is finitely generated as a H-module.

Recall that the graded centre $Z_{\mathrm{gr}}(E(A))$ of $E(A)$ is the subring

$$Z_{\mathrm{gr}}(E(A)) := \Big(x \in \mathrm{Ext}^i_A(A/\mathfrak{r}_A, A/\mathfrak{r}_A) \\ \Big| \ i \geq 0; \text{ and } xy = (-1)^{ij} yx \text{ for all } \in \mathrm{Ext}^j_A(A/\mathfrak{r}_A, A/\mathfrak{r}_A)\ (\forall j \geq 0)\Big).$$

We first show the following lemma.

Lemma 5.1. *The following statements hold*:

(a) $E(\Lambda) = K\mathcal{Q}/(a_0a_1 + b_0b_1)$.

(b) *The element* $w := \left(\sum_{i=0}^{2}(a_ia_{i+1}a_{i+2})^2\right) + (b_1b_2b_0)^2 \in \mathrm{Ext}^6_\Lambda(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$ *belongs to* $Z_{\mathrm{gr}}(E(\Lambda))$.

(c) $\phi_\Lambda(y_6)(= \phi_\Lambda(X_{6,0})) = w$.

Proof. (a) By [F] Λ is a Koszul algebra. Hence it follows by [GM, Theorem 2.2] (see also [So]) that $E(\Lambda) = K\mathcal{Q}/I^\perp$, where $I^\perp := (a_0a_1 + b_0b_1)$.

(b) It is straightforward to check that w commutes with all arrows a_i, b_i and trivial paths e_i, f_i, and therefore $w \in Z_{\mathrm{gr}}(E(\Lambda))$.

(c) This easily follows from the definition of ϕ_Λ. □

Note that since Λ is a Koszul algebra, the image of ϕ_Λ is exactly $Z_{\mathrm{gr}}(E(\Lambda))$ by [BGSS, Theorem 4.1].

Now we can prove the main result in this section.

Theorem 3. *$E(\Lambda)$ is finitely generated as a* $\mathrm{HH}^{6*}(\Lambda)$*-module. Accordingly Λ satisfies* **(Fg1)** *and* **(Fg2)**.

Proof. We verify that $E(\Lambda)$ is a $\mathrm{HH}^{6*}(\Lambda)$-module generated by the set

$$U = \{e_i,\ f_1,\ a_i,\ b_j,\ a_ia_{i+1},\ b_jb_{j+1},\ a_ia_{i+1}a_{i+2},\ a_1a_2b_0,\ b_1b_2a_0,\ b_1b_2b_0, \\ a_ia_{i+1}a_{i+2}a_i,\ b_jb_{j+1}b_{j+2}b_j,\ a_ia_{i+1}a_{i+2}a_ia_{i+1},\ b_jb_{j+1}b_{j+2}b_jb_{j+1}, \\ (a_ia_{i+1}a_{i+2})^2,\ a_1a_2a_0a_1a_2b_0,\ b_1b_2b_0b_1b_2a_0,\ (b_jb_{j+1}b_{j+2})^2 \\ \mid i = 0,1,2;\ j = 0,1\}.$$

Noting that $a_0a_1 = -b_0b_1$ in $E(\Lambda)$, it can be seen that U gives a K-basis of $\bigoplus_{l=0}^{6} \mathrm{Ext}^l_\Lambda(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$, and moreover the set $\{a_iw,\ b_jw \mid i = 0,1,2;\ j = 0,1\}$ gives a K-basis of $\mathrm{Ext}^7_\Lambda(\Lambda/\mathfrak{r}_\Lambda, \Lambda/\mathfrak{r}_\Lambda)$. Then it is straightforward to check that all homogeneous elements in $E(\Lambda)$ can be written in the form $\sum_{p\in U} k_p p w^t$ for some $k_p \in K$ $(p \in U)$ and $t \geq 0$, and so $E(\Lambda)$ is finitely generated as a right $\mathrm{HH}^{6*}(\Lambda)$-module.

Also it follows by Proposition 2.1 that $\mathrm{HH}^{6*}(\Lambda)$ is isomorphic to the polynomial ring $K[y_6]$ and hence is a commutative noetherian ring. Therefore Λ satisfies **(Fg1)** and **(Fg2)**. □

It is well-known that there are 12 isomorphism classes of indecomposable right modules for the path algebra of a Dynkin quiver of type $\mathbb{D}_4$ (see, for example, [ASS, Chapter VII, Theorem 5.10]). Hence, by [BMR], Λ has 12 isomorphism classes of indecomposable right Λ-modules. In fact, there are precisely the following indecomposable right Λ-modules up to isomorphism:

$$\begin{aligned} &e_i\Lambda/e_i\mathfrak{r}_\Lambda \quad (i=0,1,2), \quad f_1\Lambda/f_1\mathfrak{r}_\Lambda, \quad e_0\Lambda/a_0\mathfrak{r}_\Lambda, \quad f_0\Lambda/b_0\mathfrak{r}_\Lambda, \\ &e_j\Lambda \quad (j=1,2), \quad f_1\Lambda, \quad e_0\Lambda/e_0\mathfrak{r}_\Lambda^2, \quad e_0\mathfrak{r}_\Lambda, \quad e_0\Lambda. \end{aligned}$$

Then we directly see that an indecomposable right Λ-module has finite projective dimension if and only if it is an injective module or a projective module. On the other hand, since Λ satisfies **(Fg1)** and **(Fg2)**, by [EHSST, Theorem 2.5] a right Λ-module has finite projective dimension if an only if it has trivial variety. Therefore we have got the following corollary.

Corollary 5.1. *For an indecomposable right Λ-module M, the following are equivalent*:

(a) *The support variety of M is trivial.*

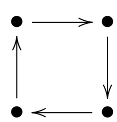
(b) *M is a projective module or an injective module.*

5.2. The Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $\mathbb{D}_4$

We end this paper by determining the Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type $\mathbb{D}_4$.

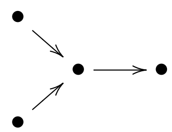
We know from [BHL] that there are three derived equivalence classes of cluster-tilted algebras of type $\mathbb{D}_4$, and moreover, as their representatives, we can take Λ and the following algebras:

(a) The selfinjective algebra $A_1 = K\Gamma_1/I_1$ of finite representation type, where Γ_1 is the cyclic quiver



and I_1 is the ideal generated by all paths of length 3.

(b) The hereditary algebra $A_2 = K\Gamma_2$, where Γ_2 is the Dynkin quiver



of type $\mathbb{D}_4$.

Then by [GSS1] we get $\mathrm{HH}^*(A_1)/\mathcal{N} \simeq K[x]$ whereas, by [H], $\mathrm{HH}^*(A_2)/\mathcal{N} \simeq \mathrm{HH}^*(A_2) \simeq K$. (Note that the structure of $\mathrm{HH}^*(A_1)$ is described in [BLM, EH].) Hence by Corollary 4.8 we have the following:

Corollary 5.2. *The Hochschild cohomology rings modulo nilpotence for all cluster-tilted algebras of type $\mathbb{D}_4$ are finitely generated as algebras, and are isomorphic to K or $K[x]$.*

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