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1-cosymplectic hypersurfaces axiom and six-dimensional planar Hermitian submanifolds of the Octonian

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Dedicated to the memory of Professor Leonid Evgen'evich EVTUSHIK (1931-2013)

Abstract. Six-dimensional planar Hermitian submanifolds of the Octonian are considered. It is proved that if such a submanifold of the octave algebra satisfies the 1-cosymplectic hypersurfaces axiom, then it is Kählerian.

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§1. Introduction

The almost contact metric structures are remarkable and very important differential-geometrical structures. These structures are studied from the point of view of differential geometry as well as of modern theoretical physics. The geometry of almost contact metric structures on manifolds is an important part of contact geometry. Without going in the details of such an extensive subject, we mark out the recent fundamental monograph by Arkady Kholodenko in the field of contact geometry and its applications [13]. We also mark out the close connection of almost contact metric and almost Hermitian structures. For instance, an almost contact metric structure is induced on an arbitrary oriented hypersurface of an almost Hermitian manifold [19].

The almost Hermitian structures also belong to the most important and meaningful differential-geometrical structures. As it is well known, the existence of 3-vector cross products in the Octonian gives a lot of substantive examples of almost Hermitian manifolds. Really, every 3-vector cross product in the Octonian induces a 1-vector cross product (or, what is the same in this case, an almost Hermitian structure) on its six-dimensional oriented submanifold (see [10], [11], [14]). Such almost Hermitian structures (in particular, Hermitian, special Hermitian, nearly-Kählerian, Kählerian etc) were studied by a number of outstanding geometers such as Alfred Gray and Vadim Feodorovich Kirichenko. For example, a complete classification of Kählerian structures on six-dimensional submanifolds of the octave algebra has been obtained [14]. We also mark out the first important paper [17] on almost Hermitian manifolds satisfying hypersurfaces axioms of diverse kinds.

In the present note, we consider six-dimensional Hermitian planar submanifolds in the Octonian. We shall prove the following main result.

Theorem 1.1. *If a six-dimensional Hermitian planar submanifold of general type in the Octonian satisfies the 1-cosymplectic hypersurfaces axiom, then it is Kählerian.*

This short article is a continuation of the authors' researches in the area of planar Hermitian submanifolds of the Octonian (see [4], [5], [8], [9] and others).

§2. Preliminaries

Let us consider an almost Hermitian manifold, i.e. a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J. Moreover, the following condition must hold

$$\langle JX,\ JY \rangle = \langle X,\ Y \rangle, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}[16]. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^{∞}. We recall that the fundamental form (or Kählerian form) of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [16]. Its elements are the local frames adapted to the structure (local A-frames). They look as follows:

$$(p,\ \varepsilon_1,\ \ldots,\ \varepsilon_n,\ \varepsilon_{\hat{1}},\ \ldots,\ \varepsilon_{\hat{n}})$$

where

$$\varepsilon_a = \frac{1}{2}\,(e_a - i\,J\,e_a)\,, \quad \varepsilon_{\hat{a}} = \frac{1}{2}\,(e_a + i\,J\,e_a)\,.$$

Here the index a ranges from 1 to n, and we state $\hat{a} = a + n$.

Therefore, the matrixes of the operator of the complexified almost complex structure J^C, the complexified Riemannian metric g^C and the complexified fundamental form F^C written in an A-frame look as follows, respectively:

$$\left({J^C}^k_j\right) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right); \quad \left(g^C_{kj}\right) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right);$$

$$\left(F^C_{kj}\right) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right).$$

where I_n is the identity matrix; $k, j = 1, \ \ldots, \ 2n$.

An almost Hermitian manifold is called Hermitian, if its almost complex structure is integrable. The following identity characterizes the Hermitian structure [12], [16]:

$$\nabla_X(F)(Y,Z) - \nabla_{JX}(F)(JY,Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$. The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows [16]:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b,$$

$$d\omega_a = -\omega^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b,$$

where $\{B^{ab}_c\}$ and $\{B^c_{ab}\}$ are components of the Kirichenko tensors of M^{2n} [1], [7]; $a, b, c = 1, \ldots, n$.

We recall also that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi, \ \xi, \ \eta, \ g\}$ on this manifold, where ξ is a vector field, η is a covector field, Φ is a tensor of the type (1; 1) and $g = \langle\cdot, \ \cdot\rangle$ is the Riemannian metric [16], [18]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1, \ \Phi(\xi) = 0, \ \eta \circ \Phi = 0, \ \Phi^2 = -id + \xi \otimes \eta$$

,

$$\langle \Phi X, \Phi Y \rangle = \langle \Phi X, \Phi Y \rangle - \eta\,(X)\,\eta\,(Y)\,,$$

$$X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of smooth vector fields on N. As an example of an almost contact metric structure we can consider the cosymplectic structure that is characterized by the following condition:

$$\nabla\eta = 0, \quad \nabla\Phi = 0,$$

where ∇ is the Levi-Civita connection of the metric. It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to the product $M \times R$, where M is a Kählerian manifold [15].

§3. The proof of the theorem

At first, we remind that an almost Hermitian manifold M^{2n} satisfies the 1-cosymplectic hypersurfaces axiom, if there exists a foliation of this manifold consisting of 1-type cosymplectic hypersurfaces in this manifold. Consequently, we can take the A-frame in a neighborhood of an arbitrary point $p \in M^{2n}$.

Let $\mathbf{O} \equiv R^8$ be the Octonian. As it is well-known [10], [11], [14] two non-isomorphic three-fold vector cross products are defined on it by means of the relations:

$$P_1(X,\ Y,\ Z\ ) = -X(\bar{Y}Z) + \langle X,\ Y\rangle\ Z + \langle Y,\ Z\rangle\ X - \langle Z, X\rangle\ Y,$$

$$P_2(X,\ Y,\ Z\ ) = -(X\bar{Y})Z + \langle X,\ Y\rangle\ Z + \langle Y,\ Z\rangle\ X - \langle Z, X\rangle\ Y,$$

where $X, Y, Z \in O$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X \to \bar{X}$ is the conjugation operator. Moreover, any other three-fold vector cross product in the octave algebra is isomorphic to one of the above-mentioned two. If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_y, g = \langle\cdot,\ \cdot\rangle\}$ is determined by the relation

$$J_t(X) = P_t(X,\ e_1,\ e_2), \quad t = 1,\ 2,$$

where $\{e_1,\ e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at the point p, $X \in T_p(M^6)$[5].

We recall that the point $p \in M^6$ is called general [2], [3], [14], if $e_0 \notin T_p(M^6)$, where e_0 is the unit of the Octonian. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [14]. In what follows, all submanifolds M^6 that will be considered are assumed to be of general type. We also note that a six-dimensional submanifold in the Octonian is called planar if a hyperplane of $\mathbf{O}$ contains this submanifold [5], [8].

Let N be an arbitrary oriented hypersurface of a six-dimensional planar Hermitian submanifold $M^6 \subset \mathbf{O}$ of the Octonian, let σ be the second fundamental form of immersion of N into M^6. The Cartan structural equations of the almost contact metric structure on such a hypersurface look as follows [4]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta +$$
$$+ \left(\sqrt{2} B^{\alpha 3}{}_\beta + i\sigma^\alpha_\beta\right) \omega^\beta \wedge \omega^3 + \left(-\frac{1}{\sqrt{2}} B^{\alpha\beta}{}_3 + i\sigma^{\alpha\beta}\right) \omega_\beta \wedge \omega^3,$$
$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta +$$
$$+ \left(\sqrt{2} B_{\alpha 3}{}^\beta - i\sigma^\beta_\alpha\right) \omega_\beta \wedge \omega^3 + \left(-\frac{1}{\sqrt{2}} B_{\alpha\beta}{}^3 - i\sigma_{\alpha\beta}\right) \omega^\beta \wedge \omega^3,$$
$$d\omega^3 = \left(\sqrt{2} B^{3\alpha}{}_\beta - \sqrt{2} B_{3\beta}{}^\alpha - 2i\sigma^\alpha_\beta\right) \omega^\beta \wedge \omega_\alpha +$$
$$+ \left(B^{n\beta}{}_3 - i\sigma^\beta_3\right) \omega^3 \wedge \omega_\beta.$$

Here the indices $\alpha,\ \beta,\ \gamma$ range from 1 to 2. If the type number of the hypersurface is equal to 0 or 1, then the matrix of the second fundamental form of the immersion of N into M^6 looks as follows [4]:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c} 0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \\ \hline 0\dots 0 & \sigma_{33} & 0 \dots 0 \\ \hline 0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \end{array}\right), \quad p, s = 1, ..., 5.$$

Evidently, $\sigma_{33} = 0$ if and only if the type number is equal to zero, i.e. the hypersurface is totally geodesic. That is why we can rewrite the Cartan structural equations for this case:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + B^{\alpha\beta}{}_\gamma \omega^\gamma \wedge \omega_\beta + \sqrt{2}\, B^{\alpha 3}{}_\beta\, \omega^\beta \wedge \omega^3 + \left(-\frac{1}{\sqrt{2}} B^{\alpha\beta}{}_3\right) \omega_\beta \wedge \omega^3;$$

$$d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + B_{\alpha\beta}{}^\gamma \omega_\gamma \wedge \omega^\beta + \sqrt{2}\, B_{\alpha 3}{}^\beta\, \omega_\beta \wedge \omega^3 + \left(-\frac{1}{\sqrt{2}} B_{\alpha\beta}{}^3\right) \omega^\beta \wedge \omega^3;$$

$$d\omega^3 = \left(\sqrt{2} B^{3\alpha}{}_\beta - \sqrt{2} B_{3\beta}{}^\alpha\right) \omega^\beta \wedge \omega_\alpha + B_{3\beta}{}^3 \omega^3 \wedge \omega^\beta + B^{3\beta}{}_3\, \omega^3 \wedge \omega_\beta.$$

Taking into account that the Cartan structural equations of a cosymplectic structure look as follows [16]:

$$d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta,$$

$$d\omega_\alpha = -\omega_\alpha^\beta \wedge \omega_\beta,$$
$$d\omega^3 = 0\,,$$

we get the conditions whose simultaneous fulfillment is a criterion for the structure induced on N to be cosymplectic:

$$1)\ B^{\alpha\beta}{}_\gamma = 0;\ 2)\ B^{\alpha 3}{}_\beta = 0;\ 3)\ B^{\alpha\beta}{}_3 = 0;$$
$$4)\ B^{3\alpha}{}_\beta - B_{3\beta}{}^\alpha = 0;\ 5)\ B^{3\beta}{}_3 = 0. \qquad (1)$$

and the formulae, obtained by complex conjugation (no need to write them explicitly).

Next, let us use the expressions for Kirichenko tensors of six-dimensional Hermitian submanifolds of the Octonian [4], [5], [8], [9]:

$$B^{ab}{}_c = \frac{1}{\sqrt{2}}\varepsilon^{abh}\,D_{hc};\ B_{ab}{}^c = \frac{1}{\sqrt{2}}\varepsilon_{abh}\,D^{hc},$$

where

$$\varepsilon^{abc} = \varepsilon^{abc}_{123}, \varepsilon_{abc} = \varepsilon^{123}_{abc}$$

are the components of the third-order Kronecher tensor [14] and

$$D_{hc} = \pm T^8_{hc} + iT^7_{hc},\ D^{hc} = D_{\hat{h}\hat{c}} = \pm T^8_{\hat{h}\hat{c}} - iT^7_{\hat{h}\hat{c}}.$$

Here $\{T^\varphi_{hc}\}$ are the components of the configuration tensor (in A. Gray's notation) of the Hermitian submanifold $M^6 \subset \mathbf{O}$; the index φ ranges from 7 to 8 and the indices a, b, c, h range from 1 to 3 [4], [5], [14].

Consequently, from (1) we obtain:

$$B^{\alpha\beta}{}_\gamma = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta 3}\,D_{3\gamma} = 0 \Leftrightarrow D_{\gamma 3} = 0$$

$$B^{\alpha\beta}{}_3 = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta 3}\,D_{33} = 0 \Leftrightarrow D_{33} = 0.$$

Similarly,

$$B^{\alpha 3}{}_\beta = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{\alpha 3h}\,D_{h\beta} = 0\,.$$

We consider all possible cases:

$$\text{if } \alpha = 1,\ \beta = 1,\ \text{then} \frac{1}{\sqrt{2}}\varepsilon^{132}\,D_{21} = 0 \Leftrightarrow D_{12} = 0\,;$$

$$\text{if } \alpha = 1,\ \beta = 2,\ \text{then} \frac{1}{\sqrt{2}}\varepsilon^{132}\,D_{22} = 0 \Leftrightarrow D_{22} = 0\,;$$

$$\text{if } \alpha = 2,\ \beta = 1,\ \text{then} \frac{1}{\sqrt{2}} \varepsilon^{231} D_{11} = 0 \Leftrightarrow D_{11} = 0\,;$$

$$\text{if } \alpha = 2,\ \beta = 2,\ \text{then} \frac{1}{\sqrt{2}} \varepsilon^{231} D_{12} = 0 \Leftrightarrow D_{12} = 0\,.$$

So, $D_{11} = D_{22} = D_{33} = D_{12} = D_{13} = D_{23} = 0$, i.e. the matrix (D_{ab}) vanishes:

$$D_{ab} \equiv 0. \tag{2}$$

Remark. We note that it was sufficient to prove only that the components $D_{11},\ D_{22},\ D_{33}$ vanish. The condition to be equal to zero for the rest of the components follows from identities established in [5]:

$$(D_{12})^2 = D_{11}D_{22},\ (D_{13})^2 = D_{11}D_{33},\ (D_{23})^2 = D_{22}D_{33}.$$

As we can see the condition (2) is fulfilled at every point of a 1-type cosymplectic hypersurface of six-dimensional planar Hermitian submanifold of the octave algebra. But this condition is a criterion for the six-dimensional submanifold $M^6 \subset \mathbf{O}$ to be Kählerian [6], [14]. That is why if $M^6 \subset \mathbf{O}$ satisfies the 1-cosymplectic hypersurfaces axiom, then it is a Kählerian manifold. So, the Theorem A is completely proved.

As it was mentioned above, the paper [14] by V.F. Kirichenko contains a complete classification of six-dimensional Kählerian submanifolds of the Octonian. Now, we can state that this paper contains a complete classification of six-dimensional planar Hermitian submanifolds of the octave algebra satisfying the 1-cosymplectic hypersurfaces axiom. We remark also that the property to satisfy the 1-cosymplectic hypersurfaces axiom essentially simplify the structure of the six-dimensional planar Hermitian submanifold of the Octonian.

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