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Strong instability of standing waves for nonlinear Schrödinger equations with double power nonlinearity

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Abstract. We prove strong instability (instability by blowup) of standing waves for some nonlinear Schrödinger equations with double power nonlinearity.

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§1. Introduction

In this paper, we study instability of standing wave solutions $e^{i\omega t}\phi_\omega(x)$ for nonlinear Schrödinger equations with double power nonlinearity:

$$i\partial_t u = -\Delta u - a|u|^{p-1}u - b|u|^{q-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where a and b are positive constants, $1 < p < q < 2^* - 1$, $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.

Moreover, we assume that $\omega > 0$ and $\phi_\omega \in H^1(\mathbb{R}^N)$ is a ground state of

$$-\Delta\phi + \omega\phi - a|\phi|^{p-1}\phi - b|\phi|^{q-1}\phi = 0, \quad x \in \mathbb{R}^N. \tag{1.2}$$

For the definition of ground state, see (1.5) below. It is well known that there exists a ground state ϕ_ω of (1.2) (see, e.g., [2, 15]).

The Cauchy problem for (1.1) is locally well-posed in the energy space $H^1(\mathbb{R}^N)$ (see, e.g., [3, 7, 8]). That is, for any $u_0 \in H^1(\mathbb{R}^N)$ there exist $T^* = T^*(u_0) \in (0, \infty]$ and a unique solution $u \in C([0,T^*), H^1(\mathbb{R}^N))$ of (1.1) with $u(0) = u_0$ such that either $T^* = \infty$ (global existence) or $T^* < \infty$ and $\lim_{t \to T^*} \|\nabla u(t)\|_{L^2} = \infty$ (finite time blowup).

Furthermore, the solution $u(t)$ satisfies

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \tag{1.3}$$

for all $t \in [0, T^*)$, where the energy E is defined by

$$E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

Here we give the definitions of stability and instability of standing waves.

Definition 1. We say that the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0 - \phi_\omega\|_{H^1} < \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists globally and satisfies

$$\sup_{t \ge 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^N} \|u(t) - e^{i\theta}\phi_\omega(\cdot + y)\|_{H^1} < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega$ is said to be *unstable*.

Definition 2. We say that $e^{i\omega t}\phi_\omega$ is *strongly unstable* if for any $\varepsilon > 0$ there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0 - \phi_\omega\|_{H^1} < \varepsilon$ and the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time.

Before we consider the double power case, we recall some well-known results for the single power case:

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.4}$$

When $1 < p < 1 + 4/N$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.4) is stable for all $\omega > 0$ (see [4]). While, if $1 + 4/N \le p < 2^* - 1$, then $e^{i\omega t}\phi_\omega$ is strongly unstable for all $\omega > 0$ (see [1] and also [3]).

Next, we consider the double power case (1.1) with $a > 0$ and $b > 0$. From Berestycki and Cazenave [1], we see that if $1 + 4/N \le p < q < 2^* - 1$, then the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is strongly unstable for all $\omega > 0$ (see [14] for the case $p = 1 + 4/N < q$).

On the other hand, when $1 < p < 1 + 4/N < q < 2^* - 1$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.1) is unstable for sufficiently large ω (see [13]), while $e^{i\omega t}\phi_\omega$ is stable for sufficiently small ω (see [5] and also [12, 11] for more results in one dimensional case). However, it was not known whether $e^{i\omega t}\phi_\omega$ is strongly unstable or not for the case where $1 < p < 1 + 4/N < q < 2^* - 1$ and ω is sufficiently large.

Now we state our main result in this paper.

Theorem 1. *Let $a > 0$, $b > 0$, $1 < p < 1+4/N < q < 2^*-1$, and let $\phi_\omega \in \mathcal{G}_\omega$. Then there exists $\omega_1 > 0$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of* (1.1) *is strongly unstable for all $\omega \in (\omega_1, \infty)$.*

For $\omega > 0$, we define functionals S_ω and K_ω on $H^1(\mathbb{R}^N)$ by

$$S_\omega(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{a}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b}{q+1}\|v\|_{L^{q+1}}^{q+1},$$
$$K_\omega(v) = \|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - a\|v\|_{L^{p+1}}^{p+1} - b\|v\|_{L^{q+1}}^{q+1}.$$

Note that (1.2) is equivalent to $S'_\omega(\phi) = 0$, and

$$K_\omega(v) = \partial_\lambda S_\omega(\lambda v)\big|_{\lambda=1} = \langle S'_\omega(v), v\rangle$$

is the so-called Nehari functional. We denote the set of nontrivial solutions of (1.2) by

$$\mathcal{A}_\omega = \{v \in H^1(\mathbb{R}^N) : S'_\omega(v) = 0,\ v \neq 0\},$$

and define the set of ground states of (1.2) by

$$\mathcal{G}_\omega = \{\phi \in \mathcal{A}_\omega : S_\omega(\phi) \le S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega\}. \tag{1.5}$$

Moreover, consider the minimization problem:

$$d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N),\ K_\omega(v) = 0,\ v \neq 0\}. \tag{1.6}$$

Then, it is well known that $\mathcal{G}_\omega$ is characterized as follows.

$$\mathcal{G}_\omega = \{\phi \in H^1(\mathbb{R}^N) : S_\omega(\phi) = d(\omega),\ K_\omega(\phi) = 0\}. \tag{1.7}$$

The proof of finite time blowup for (1.1) relies on the virial identity (1.8). If $u_0 \in \Sigma := \{v \in H^1(\mathbb{R}^N) : |x|v \in L^2(\mathbb{R}^N)\}$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ belongs to $C([0,T^*),\Sigma)$, and satisfies

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8P(u(t)) \tag{1.8}$$

for all $t \in [0,T^*)$, where

$$P(v) = \|\nabla v\|_{L^2}^2 - \frac{a\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}$$

with $\alpha = \dfrac{N}{2}(p-1)$, $\beta = \dfrac{N}{2}(q-1)$ (see, e.g., [3]).

Note that for the scaling $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$ for $\lambda > 0$, we have

$$\|\nabla v^\lambda\|_{L^2}^2 = \lambda^2\|\nabla v\|_{L^2}^2,\ \|v^\lambda\|_{L^{p+1}}^{p+1} = \lambda^\alpha\|v\|_{L^{p+1}}^{p+1},\ \|v^\lambda\|_{L^{q+1}}^{q+1} = \lambda^\beta\|v\|_{L^{q+1}}^{q+1},$$
$$\|v^\lambda\|_{L^2}^2 = \|v\|_{L^2}^2, \quad P(v) = \partial_\lambda E(v^\lambda)\big|_{\lambda=1}.$$

The method of Berestycki and Cazenave [1] is based on the fact that $d(\omega) = S_\omega(\phi_\omega)$ can be characterized as

$$d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N),\ P(v) = 0,\ v \neq 0\} \tag{1.9}$$

for the case $1 + 4/N \le p < q < 2^* - 1$. Using this fact, it is proved in [1] that if $u_0 \in \Sigma \cap \mathcal{B}_\omega^{BC}$ then the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time, where

$$\mathcal{B}_\omega^{BC} = \{v \in H^1(\mathbb{R}^N) : S_\omega(v) < d(\omega),\ P(v) < 0\}.$$

We remark that (1.9) does not hold for the case $1 < p < 1+4/N < q < 2^* - 1$.

On the other hand, Zhang [16] and Le Coz [9] gave an alternative proof of the result of Berestycki and Cazenave [1]. Instead of (1.9), they proved that

$$d(\omega) \le \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N),\ P(v) = 0,\ K_\omega(v) < 0\} \tag{1.10}$$

holds for all $\omega > 0$ if $1 + 4/N \le p < q < 2^* - 1$ (compare with Lemma 2 below). Using this fact, it is proved in [16, 9] that if $u_0 \in \Sigma \cap \mathcal{B}_\omega^{ZL}$ then the solution $u(t)$ of (1.1) with $u(0) = u_0$ blows up in finite time, where

$$\mathcal{B}_\omega^{ZL} = \{v \in H^1(\mathbb{R}^N) : S_\omega(v) < d(\omega),\ P(v) < 0,\ K_\omega(v) < 0\}.$$

In this paper, we use and modify the idea of Zhang [16] and Le Coz [9] to prove Theorem 1. For $\omega > 0$ with $E(\phi_\omega) > 0$, we introduce

$$\begin{aligned} \mathcal{B}_\omega = \{v \in H^1(\mathbb{R}^N) : 0 < E(v) < E(\phi_\omega),\ &\|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2, \\ &P(v) < 0,\ K_\omega(v) < 0\}. \end{aligned} \tag{1.11}$$

Then we have the following.

Theorem 2. *Let $a > 0$, $b > 0$, $1 < p < 1 + 4/N < q < 2^* - 1$, and assume that $\phi_\omega \in \mathcal{G}_\omega$ satisfies $E(\phi_\omega) > 0$. If $u_0 \in \Sigma \cap \mathcal{B}_\omega$, then the solution $u(t)$ of* (1.1) *with $u(0) = u_0$ blows up in finite time.*

Remark. Our method is not restricted to the double power case (1.1), but is also applicable to other type of nonlinear Schrödinger equations. For example, we consider nonlinear Schrödinger equation with a delta function potential:

$$i\partial_t u = -\partial_x^2 u - \gamma\delta(x)u - |u|^{q-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.12}$$

where $\delta(x)$ is the Dirac measure at the origin, $\gamma > 0$ and $1 < q < \infty$. The energy of (1.12) is given by

$$E(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{q+1}\|v\|_{L^{q+1}}^{q+1}.$$

The standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.12) exists for $\omega \in (\gamma^2/4, \infty)$.

For the case $q > 5$, it is proved in [6] that there exists $\omega_2 \in (\gamma^2/4, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.12) is stable for $\omega \in (\gamma^2/4, \omega_2)$, and it is unstable for $\omega \in (\omega_2, \infty)$. Since the graph of the function

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\partial_x v\|_{L^2}^2 - \frac{\gamma\lambda}{2}|v(0)|^2 - \frac{\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}$$

with $\beta = \dfrac{q-1}{2} > 2$ has the same properties as in Lemma 1 for (1.1), we can prove that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.12) is strongly unstable for ω satisfying $E(\phi_\omega) > 0$ (see also Theorem 5 of [10] for the case $\gamma < 0$).

The rest of the paper is organized as follows. In Section 2, we give the proof of Theorem 2. In Section 3, we show that $E(\phi_\omega) > 0$ for sufficiently large ω, and prove Theorem 1 using Theorem 2.

§2. Proof of Theorem 2

Throughout this section, we assume that

$$a > 0, \quad b > 0, \quad 1 < p < 1 + 4/N < q < 2^* - 1, \quad E(\phi_\omega) > 0.$$

Recall that $0 < \alpha = \dfrac{N}{2}(p-1) < 2 < \beta = \dfrac{N}{2}(q-1)$, and

$$E(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 - \frac{a\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1}, \tag{2.1}$$

$$P(v^\lambda) = \lambda^2\|\nabla v\|_{L^2}^2 - \frac{a\alpha\lambda^\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1} - \frac{b\beta\lambda^\beta}{q+1}\|v\|_{L^{q+1}}^{q+1} = \lambda\partial_\lambda E(v^\lambda), \tag{2.2}$$

$$K_\omega(v^\lambda) = \lambda^2\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \lambda^\alpha a\|v\|_{L^{p+1}}^{p+1} - \lambda^\beta b\|v\|_{L^{q+1}}^{q+1}. \tag{2.3}$$

Lemma 1. *If $v \in H^1(\mathbb{R}^N)$ satisfies $E(v) > 0$, then there exist $\lambda_k = \lambda_k(v)$ $(k = 1, 2, 3, 4)$ such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and*

- *$E(v^\lambda)$ is decreasing in $(0, \lambda_1) \cup (\lambda_3, \infty)$, and increasing in (λ_1, λ_3).*
- *$E(v^\lambda)$ is negative in $(0, \lambda_2) \cup (\lambda_4, \infty)$, and positive in (λ_2, λ_4).*
- *$E(v^\lambda) < E(v^{\lambda_3})$ for all $\lambda \in (0, \lambda_3) \cup (\lambda_3, \infty)$.*

Proof. Since $a > 0$, $b > 0$, $0 < \alpha < 2 < \beta$ and $E(v) > 0$, the conclusion is easily verified by drawing the graph of (2.1) (see Figure 1 below). □

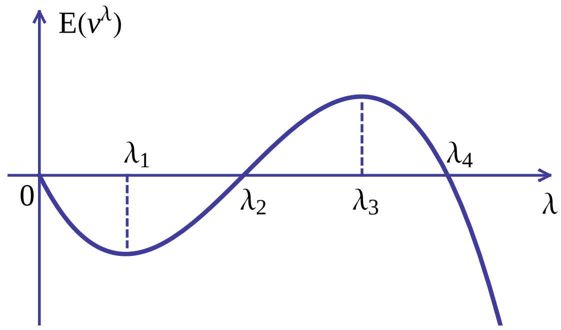

Figure 1: The graph of $\lambda \mapsto E(v^\lambda)$ for the case $E(v) > 0$.

Lemma 2. *If $v \in H^1(\mathbb{R}^N)$ satisfies $E(v) > 0$, $K_\omega(v) < 0$ and $P(v) = 0$, then $d(\omega) < S_\omega(v)$.*

Proof. We consider two functions $f(\lambda) = K_\omega(v^\lambda)$ and $g(\lambda) = E(v^\lambda)$.

Since $f(0) = \omega\|v\|_{L^2}^2 > 0$ and $f(1) = K_\omega(v) < 0$, there exists $\lambda_0 \in (0,1)$ such that $K_\omega(v^{\lambda_0}) = 0$. Moreover, since $v^{\lambda_0} \neq 0$, it follows from (1.6) that

$$d(\omega) \le S_\omega(v^{\lambda_0}).$$

On the other hand, since $g'(1) = P(v) = 0$ and $g(1) = E(v) > 0$, it follows from Lemma 1 that $\lambda_3 = 1$ and $g(\lambda) < g(1)$ for all $\lambda \in (0,1)$.

Thus, we have $E(v^{\lambda_0}) < E(v)$, and

$$d(\omega) \le S_\omega(v^{\lambda_0}) = E(v^{\lambda_0}) + \frac{\omega}{2}\|v^{\lambda_0}\|_{L^2}^2 < E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 = S_\omega(v).$$

This completes the proof. □

Lemma 3. *The set $\mathcal{B}_\omega$ is invariant under the flow of (1.1). That is, if $u_0 \in \mathcal{B}_\omega$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{B}_\omega$ for all $t \in [0, T^*)$.*

Proof. Let $u_0 \in \mathcal{B}_\omega$ and let $u(t)$ be the solution of (1.1) with $u(0) = u_0$. Then, by the conservation laws (1.3), we have

$$0 < E(u(t)) = E(u_0) < E(\phi_\omega), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$$

for all $t \in [0, T^*)$.

Next, we prove that $K_\omega(u(t)) < 0$ for all $t \in [0, T^*)$. Suppose that this were not true. Then, since $K_\omega(u_0) < 0$ and $t \mapsto K_\omega(u(t))$ is continuous on $[0, T^*)$, there exists $t_1 \in (0, T^*)$ such that $K_\omega(u(t_1)) = 0$. Moreover, since $u(t_1) \neq 0$, by (1.6), we have $d(\omega) \le S_\omega(u(t_1))$. Thus, we have

$$d(\omega) \le S_\omega(u(t_1)) = E(u_0) + \frac{\omega}{2}\|u_0\|_{L^2}^2 < E(\phi_\omega) + \frac{\omega}{2}\|\phi_\omega\|_{L^2}^2 = d(\omega).$$

This is a contradiction. Therefore, $K_\omega(u(t)) < 0$ for all $t \in [0, T^*)$.

Finally, we prove that $P(u(t)) < 0$ for all $t \in [0, T^*)$. Suppose that this were not true. Then, there exists $t_2 \in (0, T^*)$ such that $P(u(t_2)) = 0$. Since $E(u(t_2)) > 0$ and $K_\omega(u(t_2)) < 0$, it follows from Lemma 2 that $d(\omega) < S_\omega(u(t_2))$. Thus, we have

$$d(\omega) < S_\omega(u(t_2)) = E(u_0) + \frac{\omega}{2}\|u_0\|_{L^2}^2 < E(\phi_\omega) + \frac{\omega}{2}\|\phi_\omega\|_{L^2}^2 = d(\omega).$$

This is a contradiction. Therefore, $P(u(t)) < 0$ for all $t \in [0, T^*)$. □

Lemma 4. *For any $v \in \mathcal{B}_\omega$,*

$$E(\phi_\omega) \le E(v) - P(v).$$

Proof. Since $K_\omega(v) < 0$, as in the proof of Lemma 2, there exists $\lambda_0 \in (0,1)$ such that $S_\omega(\phi_\omega) = d(\omega) \le S_\omega(v^{\lambda_0})$. Moreover, since $\|v^{\lambda_0}\|_{L^2}^2 = \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$, we have

$$E(\phi_\omega) \le E(v^{\lambda_0}). \tag{2.4}$$

On the other hand, since $P(v^\lambda) = \lambda \partial_\lambda E(v^\lambda)$, $P(v) < 0$ and $E(v) > 0$, it follows from Lemma 1 that $\lambda_3 < 1 < \lambda_4$. Moreover, since $\partial_\lambda^2 E(v^\lambda) < 0$ for $\lambda \in [\lambda_3, \infty)$, by a Taylor expansion, we have

$$E(v^{\lambda_3}) \le E(v) + (\lambda_3 - 1)P(v) \le E(v) - P(v). \tag{2.5}$$

Finally, by (2.4), (2.5) and the third property of Lemma 1, we have

$$E(\phi_\omega) \le E(v^{\lambda_0}) \le E(v^{\lambda_3}) \le E(v) - P(v).$$

This completes the proof. □

Now we give the proof of Theorem 2.

Proof of Theorem 2. Let $u_0 \in \Sigma \cap \mathcal{B}_\omega$ and let $u(t)$ be the solution of (1.1) with $u(0) = u_0$. Then, by Lemma 3, $u(t) \in \mathcal{B}_\omega$ for all $t \in [0, T^*)$.

Moreover, by the virial identity (1.8) and Lemma 4, we have

$$\frac{1}{8}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = P(u(t)) \le E(u(t)) - E(\phi_\omega) = E(u_0) - E(\phi_\omega) < 0$$

for all $t \in [0, T^*)$, which implies $T^* < \infty$. This completes the proof. □

§3. Proof of Theorem 1

First, we prove the following lemma.

Lemma 5. *Let $a > 0$, $b > 0$, $1 < p < 1 + 4/N < q < 2^* - 1$, and let $\phi_\omega \in \mathcal{G}_\omega$. Then there exists $\omega_1 > 0$ such that $E(\phi_\omega) > 0$ for all $\omega \in (\omega_1, \infty)$.*

Proof. Since $P(\phi_\omega) = 0$, we see that $E(\phi_\omega) > 0$ if and only if

$$\frac{(2-\alpha)a}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} < \frac{(\beta-2)b}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1}. \tag{3.1}$$

Moreover, in the same way as the proof of Theorem 2 in [13], we can prove that

$$\lim_{\omega\to\infty} \frac{\|\phi_\omega\|_{L^{p+1}}^{p+1}}{\|\phi_\omega\|_{L^{q+1}}^{q+1}} = 0.$$

Thus, there exists $\omega_1 > 0$ such that (3.1) holds for all $\omega \in (\omega_1, \infty)$. □

Proof of Theorem 1. Let $\omega \in (\omega_1, \infty)$. Then, by Lemma 5, $E(\phi_\omega) > 0$.

For $\lambda > 0$, we consider the scaling $\phi_\omega^\lambda(x) = \lambda^{N/2}\phi_\omega(\lambda x)$, and prove that there exists $\lambda_0 \in (1, \infty)$ such that $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda \in (1, \lambda_0)$.

First, we have $\|\phi_\omega^\lambda\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$ for all $\lambda > 0$. Next, since $P(\phi_\omega) = 0$ and $E(\phi_\omega) > 0$, by Lemma 1 and (2.2), there exists $\lambda_4 > 1$ such that

$$0 < E(\phi_\omega^\lambda) < E(\phi_\omega), \quad P(\phi_\omega^\lambda) < 0$$

for all $\lambda \in (1, \lambda_4)$. Finally, since $P(\phi_\omega) = 0$, we have

$$\partial_\lambda K_\omega(\phi_\omega^\lambda)\big|_{\lambda=1} = -\frac{(p-1)a\alpha}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} - \frac{(q-1)b\beta}{q+1}\|\phi_\omega\|_{L^{q+1}}^{q+1} < 0.$$

Since $K_\omega(\phi_\omega) = 0$, there exists $\lambda_0 \in (1, \lambda_4)$ such that $K_\omega(\phi_\omega^\lambda) < 0$ for all $\lambda \in (1, \lambda_0)$.

Therefore, $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda \in (1, \lambda_0)$. Moreover, since $\phi_\omega^\lambda \in \Sigma$ for $\lambda > 0$, it follows from Theorem 2 that for any $\lambda \in (1, \lambda_0)$, the solution $u(t)$ of (1.1) with $u(0) = \phi_\omega^\lambda$ blows up in finite time.

Finally, since $\lim_{\lambda\to 1}\|\phi_\omega^\lambda - \phi_\omega\|_{H^1} = 0$, the proof is completed. □

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