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Integral sections of elliptic surfaces and degenerated $(2,3)$ torus decompositions of a 3-cuspidal quartic

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Abstract. In this note, we consider when a plane curve given by a polynomial of the form

$$x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0,$$

where $\deg_t a_i(t) \leq id$ (d: even), has degenerated $(2,3)$ torus decompositions by using arithmetic properties of elliptic surfaces and show that a 3-cuspidal quartic has infinitely many degenerated $(2,3)$ torus decompositions.

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§1. Introduction

In this note, all varieties are defined over the field of complex numbers $\mathbb{C}$. Let d be an even positive integer and let $p(t,x) \in \mathbb{C}[t,x]$ be a polynomial of the form

$$x^3 + a_1(t)x^2 + a_2(t)x + a_3(t) = 0,$$

where $\deg_t a_i(t) \leq id$. Our aim of this note is to consider when $p(t,x)$ has a decomposition of the form

$$(*) \quad p(t,x) = (x - x_o(t))^3 + (c_0(t)x + c_1(t))^2, \quad x_o(t), c_0(t), c_1(t) \in \mathbb{C}[t].$$

The right hand side of $(*)$ is called a $(2,3)$ torus decomposition of the affine curve given by $p(t,x) = 0$. Such decompositions have been considered in, for example, [13, 14, 5, 3] from viewpoint of the topology of the complements to $\{p(t,x) = 0\}$. In this note we add another remark to this problem. In order to state our criterion, we need to introduce some notation.

Let E be an elliptic curve defined over the rational function field of one variable $\mathbb{C}(t)$ given by

$$E : y^2 = p(t, x),$$

and we denote the set of $\mathbb{C}(t)$-rational points and the point at infinity O by $E(\mathbb{C}(t))$. It is well-known that $E(\mathbb{C}(t))$ becomes an abelian group, O being the zero element. Now our first statement is as follows:

Proposition 1. *Assume that both of plane curves given by*

$$p(t, x) = 0 \quad \textit{and} \quad s^{3d}p(1/s, x'/s^d) = 0$$

have at worst simple singularities (see [2] for simple singularities) in both of (t, x) and (s, x') planes. Then $p(t, x)$ has a decomposition as in $(*)$ if and only if $E(\mathbb{C}(t))$ has a point P of order 3*. The polynomial $x_o(t)$ is given by the x-coordinate of P.*

As an application of Proposition 1, we have the following theorem:

Theorem 1. *Let $\mathcal{Q}$ be a quartic with* 3 *cusps and choose a smooth point z_o on $\mathcal{Q}$. There exists a unique irreducible conic $\mathcal{C}$ as follows:*

(i) $\mathcal{C}$ is tangent to $\mathcal{Q}$ at z_o and passes through three cusps of $\mathcal{Q}$.

(ii) Let $F_{\mathcal{Q}}$, $F_{\mathcal{C}}$, and L_{z_o} be defining equations of $\mathcal{Q}$, $\mathcal{C}$ and the tangent line $\mathcal{L}_{z_o}$ of $\mathcal{Q}$ at z_o, respectively. Then there exists a homogeneous polynomial G of degree 3 *such that*

$$(**) \quad L_{z_o}^2 F_{\mathcal{Q}} = F_{\mathcal{C}}^3 + G^2.$$

Remark 1.
- Following [10], we call the decomposition of $F_{\mathcal{Q}}$ as in $(**)$ a *degenerated* $(2, 3)$ *torus decomposition* of projective plane curves. The statement of Theorem 1 can be found in [10, 5.3.2]. We, however, consider that our point of view explains geometry behind the statement, and hope that it is worthwhile mentioning.

- The 5 $(2, 3)$ torus decompositions given in [5] can also be found by Proposition 1. In the terminology of [5], our statement can be rephrased:

  $\mathcal{Q}$ has infinitely many invisible $(2, 3)$ torus decompositions.

- Let z_o be one of 3 cusps of $\mathcal{Q}$ and $\mathcal{L}_{\max,z_o}$ is the tangent line at z_o. Then we also have a degenerated $(2, 3)$ decomposition by using $\mathcal{L}_{\max,z_o}$. This is informed the authors by M. Kawashima. In fact, it is enough to check the statement for one explicit example as any 3-cuspidal quartic is projectively equivalent to each other. For example, we have:

$$Z^2(3T^4 - 2T^3X - 3T^2Z^2 + X^2Z^2 + Z^4) = (XZ^2 - T^3)^2 - (T^2 - Z^2)^3,$$

where $[T, X, Z]$ denote homogeneous coordinates. Note that $[0, 1, 0]$ is a cusp and $Z = 0$ is the maximal tangent line, $\mathcal{L}_{\max,z_o}$. This statement can also be found in [10, 5.3.2]

§2. Preliminaries

2.1. Existence of $\mathcal{C}$

We first show that the conic $\mathcal{C}$ in Theorem 1 exists. Let $[T, X, Z]$ be homogeneous coordinates of $\mathbb{P}^2$.

Lemma 2.1. *(i) Let C be a conic tangent to $\{T = 0\}, \{X = 0\}$ and $\{Z = 0\}$ in $\mathbb{P}^2$. Let Q be the standard quadratic transformation (or the standard Cremona transformation) with respect to $\{T = 0\}, \{X = 0\}$ and $\{Z = 0\}$. Then $Q(C)$ is a quartic whose singularities are only 3 cusps at $[0, 0, 1], [0, 1, 0]$ and $[1, 0, 0]$.*

(ii) Let L be the line tangent to C at a point $P = [T_0, X_0, Z_0] \in C$. If L is different from $\{T = 0\}, \{X = 0\}$ and $\{Z = 0\}$, then $Q(L)$ is a conic tangent to $Q(C)$ at $Q(P) = [X_0Z_0, T_0Z_0, T_0X_0]$ and passes through $[0, 0, 1], [0, 1, 0]$ and $[1, 0, 0]$.

(iii) Conversely any conic such that it is tangent to a smooth point of a 3-cuspidal quartic $\mathcal{Q}$ and passes through the 3 cusps of $\mathcal{Q}$ can be obtained as above.

Since both of these statements are well-known, we omit their proofs. Let $\mathcal{L}_{Q(P)}$ be the tangent line to $Q(C)$ at $Q(P)$ and let Φ be a coordinate change such that $\mathcal{L}_{Q(P)}$ is transformed into the line $Z = 0$ and $Q(P)$ is mapped to $[0, 1, 0]$.

Then $\Phi(Q(C))$ has an affine equation of the form $x^3 + b_1(t)x^2 + b_2(t)x + b_3(t) = 0$, where $t = T/Z, x = X/Z$, $b_i(t) \in \mathbb{C}[t]$ and $\deg_t b_i(t) \le i + 1$. Also $\Phi(Q(L))$ is given by an equation of the form $x - x_o(t) = 0$, where $x_o(t) \in \mathbb{C}[t]$ and $\deg x_o(t) = 2$.

2.2. Elliptic Surfaces

As for details on the results in this subsection, we refer to [6], [7], [8], [12], [16] and [1].

2.2.1. Some terminologies

Throughout this article, an elliptic surface always means a smooth projective surface S with a fibration $\varphi : S \to C$ over a smooth projective curve, C, as follows:

(i) There exists non empty finite subset $\mathrm{Sing}(\varphi) \subset C$ such that $\varphi^{-1}(v)$ is a smooth curve of genus 1 for $v \in C \setminus \mathrm{Sing}(\varphi)$, while $\varphi^{-1}(v)$ is not a smooth curve of genus 1 for $v \in \mathrm{Sing}(\varphi)$.

(ii) There exists a section $O : C \to S$ (we identify O with its image in S).

(iii) there is no exceptional curve of the first kind in any fiber.

In this note, we only consider an elliptic surface over $\mathbb{P}^1$, $\varphi : S \to \mathbb{P}^1$.

We call $F_v = \varphi^{-1}(v)(v \in \mathrm{Sing}(\varphi))$ a singular fiber over v. In order to describe the type of singular fibers, we use notation given in Kodaira ([6]). We denote the irreducible decomposition of F_v by

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i}\Theta_{v,i},$$

where m_v is the number of irreducible components of F_v and $\Theta_{v,0}$ is the irreducible component with $\Theta_{v,0}O = 1$. We call $\Theta_{v,0}$ *the identity component.* We also define a subset $\mathrm{Red}(\varphi)$ of $\mathrm{Sing}(\varphi)$ to be $\mathrm{Red}(\varphi) := \{v \in \mathrm{Sing}(\varphi) \mid F_v \text{ is reducible}\}$. For $s \in \mathrm{MW}(S)$, s is said to be *integral* if $sO = 0$. It is known that any torsion element in $\mathrm{MW}(S)$ is integral (*cf.*[7]).

Let $\mathrm{MW}(S)$ be the set of sections of $\varphi : S \to \mathbb{P}^1$. By our assumption, $\mathrm{MW}(S) \neq \emptyset$. On a smooth fiber F of φ, by regarding $F \cap O$ as the zero element, we can consider the abelian group structure on F. Hence for $s_1, s_2 \in \mathrm{MW}(S)$, one can define the addition $s_1 \dot{+} s_2$ or the multiplication-by-m map $[m]s_1$ on $\mathbb{P}^1 \setminus \mathrm{Sing}(\varphi)$. By [6, Theorem 9.1], $s_1 \dot{+} s_2$ and $[m]s_1$ can be extended over $\mathbb{P}^1$, and we can consider $\mathrm{MW}(S)$ as an abelian group. On the other hand, we can regard the generic fiber $E := S_\eta$ of S as a curve of genus 1 over $\mathbb{C}(\mathbb{P}^1)$, the rational function field of $\mathbb{P}^1$. The restriction of O to E gives rise to a $\mathbb{C}(\mathbb{P}^1)$-rational point of E, and one can regard E as an elliptic curve over $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(t)$, O being the zero element. By considering the restriction to the generic fiber for each section, $\mathrm{MW}(S)$ can be identified with the set of $\mathbb{C}(t)$-rational points $E(\mathbb{C}(t))$. Conversely, any element $P \in E(\mathbb{C}(t))$ gives rise to a section determined by P, which we denote by s_P. We also denote the addition and the multiplication-by-m map on $E(\mathbb{C}(t))$ by $\dot{+}$ and $[m]$, respectively.

In [12], Shioda introduced a $\mathbb{Q}$-valued bilinear form on $E(\mathbb{C}(t))$ called the height pairing. We denote it by $\langle\, ,\, \rangle$. For our later use, we give two basic properties of $\langle\, ,\, \rangle$:

- $\langle P, P\rangle \geq 0$ for $\forall P \in E(\mathbb{C}(t))$ and the equality holds if and only if P is an element of finite order in $E(\mathbb{C}(t))$.
- An explicit formula for $\langle P_1, P_2\rangle$ $(P_1, P_2 \in E(\mathbb{C}(t)))$ is given as follows:

$$\langle P_1, P_2\rangle = \chi(\mathcal{O}_S) + s_{P_1}O + s_{P_2}O - s_{P_1}s_{P_2} - \sum_{v\in \mathrm{Red}(\varphi)} \mathrm{Contr}_v(s_{P_1}, s_{P_2}),$$

where s_{P_i} $(i = 1, 2)$ denote the sections in $\mathrm{MW}(S)$ determined by P_i $(i = 1, 2)$, and $\mathrm{Contr}_v(s_{P_1}, s_{P_2})$ is determined at which component s_{P_1} and s_{P_2} meet at F_v. As for explicit values of $\mathrm{Contr}_v(s_{P_1}, s_{P_2})$, we refer to [12, (8.16)]. Note that since $s_{P_i}^2 = -\chi(\mathcal{O}_S)$, we have

$$\langle P_1, P_1\rangle = 2\chi(\mathcal{O}_S) + 2s_{P_1}O - \sum_{v\in \mathrm{Red}(\varphi)} \mathrm{Contr}_v(s_{P_1}, s_{P_1}),$$

2.2.2. Double cover construction of elliptic surfaces and their Weierstrass equations

Let Σ_d (d: even) be the Hirzebruch surface of degree d. We first give a method in constructing elliptic surfaces over $\mathbb{P}^1$ as double covers of Σ_d as follows:

Let Δ_0 and Δ denotes sections of Σ_d with $\Delta_0^2 = -d, \Delta^2 = d$ and $\Delta_0 \cap \Delta = \emptyset$. Note that $\Delta \sim \Delta_0 + d\mathfrak{f}$, where $\mathfrak{f}$ denotes a fiber of $\Sigma_d \to \mathbb{P}^1$ and $\sim$ means the linear equivalence of divisors. Let $\mathcal{T}$ be a reduced divisor on Σ_d such that

(i) $\mathcal{T} \sim 3\Delta$ $(\sim 3(\Delta_0 + d\mathfrak{f}))$, and

(ii) $\mathcal{T}$ has at worst simple singularities (see [2] for simple singularities).

Let $f' : S' \to \Sigma_d$ be the double cover with branch locus $\Delta_{f'} = \Delta_0 + \mathcal{T}$ (cf. [2, III, §7]). We denote the diagram of the canonical resolution by

$$\begin{array}{ccc} S' & \xleftarrow{\ \mu\ } & S \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Sigma_d & \xleftarrow[\ q\ ]{} & \widehat{\Sigma}_d. \end{array}$$

(see [4]). Namely, μ is the minimal resolution of singularities and q is a composition of blowing-ups so that the branch locus of f becomes smooth. Then the induced morphism $\varphi : S \to \Sigma_d \to \mathbb{P}^1$ gives rise to an elliptic fibration over $\mathbb{P}^1$.

Conversely it is known that any elliptic surface $\varphi : S \to \mathbb{P}^1$ is obtained in this way.

We next consider a Weierstrass equation of the generic fiber of S. Choose affine open sets U_1 and U_2 of Σ_d as in [1, 2.2.3]. Namely $U_i \cong \mathbb{C}^2$ $(i = 1, 2)$ with coordinates (t, x) (resp. (s, x')) on U_1 (resp. U_2) with relations $t = 1/s, x = x'/s^d$. With these coordinates, $\mathcal{T}$ is given by equations of the form

$$p_{\mathcal{T}}(t, x) = x^3 + a_1(t)x^2 + a_2(t)x + a_3(t), \quad a_i \in \mathbb{C}[t], \deg a_i \leq id.$$

on U_1 and $s^{3d}p_{\mathcal{T}}(1/s, x'/s^d) = 0$ on U_2. Over U_1, $S'|_{f'^{-1}(U_1)}$ is given by

$$y^2 - p_{\mathcal{T}}(t, x) = 0 \subset \mathbb{C}^3,$$

and the covering morphism f' is given by the restriction of the projection $(t, x, y) \mapsto (t, x)$. The covering transformation $\sigma_{f'}$ is given by $(t, x, y) \mapsto (t, x, -y)$. Thus we infer that the generic fiber of $\varphi : S \to \mathbb{P}^1$ is an elliptic curve E over $\mathbb{C}(t)$ given by the above Weierstrass equation. Note that if $s \in \mathrm{MW}(S)$ is integral, then the corresponding point $P_s \in E(\mathbb{C}(t))$ has polynomial coordinate components whose degrees are at most d (resp. $3d/2$) for the x-coordinate (resp. the y-coordinate). In what follows, we say $P = (x(t), y(t))$ is integral if $x(t), y(t) \in \mathbb{C}[t], \deg x(t) \leq d$, $\deg y(t) \leq 3d/2$, .

Let $P_o = (x_o(t), y_o(t)) \in E(\mathbb{C}(t))$ be an integral point of the elliptic curve E as in Introduction. Assume $y_o(t) \neq 0$ and let

$$y = l(t, x), \quad l(t, x) = m(t)(x - x_o(t)) + y_o(t)$$

be the tangent line at P_o and put $[2]P_o = (x_1(t), y_1(t))$.

Lemma 2.2. *If $[2]P_o$ is also an integral point, then $m(t) \in \mathbb{C}[t]$.*

Proof. From the definition of addition, we have

$$p_{\mathcal{T}}(t, x) - \{l(t, x)\}^2 = (x - x_o(t))^2(x - x_1(t)).$$

By comparing the coefficients of x^2 of the above equality, we have

$$a_1 - \{m(t)\}^2 = -2x_o(t) - x_1(t).$$

This implies $m(t) \in \mathbb{C}[t]$ □

Corollary 2.1. *Under the assumption of Lemma 2.2, $p(t, x)$ has a decomposition*

$$p_{\mathcal{T}}(t, x) = (x - x_o(t))^2(x - x_1(t)) + \{l(t, x)\}^2.$$

Since any element of finite order in $E(\mathbb{C}(t))$ is always integral under our assumption, we have

Corollary 2.2. *If P is an element of finite order in $E(\mathbb{C}(t))$, $p(t,x)$ has a decomposition*

$$p_{\mathcal{T}}(t,x) = (x - x_o(t))^2(x - x_1(t)) + \{l(t,x)\}^2.$$

In particular, if P is an element of order three, as the x-coordinates of $[2]P$ and $-P$ are the same, we have

$$p_{\mathcal{T}}(t,x) = (x - x_o(t))^3 + \{l(t,x)\}^2.$$

Proof of Proposition 1. The half of Proposition 1 follows form Corollary 2.2, as the degree of $l(t,x)$ with respect to x is equal to 1. Conversely, if $p_{\mathcal{T}}(t,x)$ has the decomposition described in Proposition 1, $(x_o(t), \pm(c_0(t)x_o(t)+c_1(t)))$ are 3-torsions of $E(\mathbb{C}(t))$. Thus we have Proposition 1. □

§3. Rational elliptic surface $S_{\mathcal{Q},z_o}$

An elliptic surface is said to be rational if it is a rational surface. Any rational elliptic surface obtained as a double cover of Σ_2 described in §1. Let $\mathcal{Q}$ be a 3-cuspidal quartic as before and let z_o be a smooth point on $\mathcal{Q}$. Likewise in the second author's article (e.g., [15, 1.3]), we associate a rational elliptic surface with $\mathcal{Q}$ and z_o, which we denote by $\varphi : S_{\mathcal{Q},z_o} \to \mathbb{P}^1$. The tangent line l_{z_o} gives rise to a singular fiber of φ whose type is determined by how l_{z_o} intersects with $\mathcal{Q}$ as follows:

Table 1: l_{z_o} and the corresponding singular fiber

(i)	I_2	l_{z_o} meets $\mathcal{Q}$ with two other distinct points.
(ii)	III	l_{z_o} is a 3-fold tangent point.
(iii)	I_3	l_{z_o} is a bitangent line.
(iv)	IV	l_{z_o} is a 4-fold tangent point.
(v)	I_5	l_{z_o} passes through a cusp of $\mathcal{Q}$

By [8, Table 6.2] and Table 1 as above, possible configurations of singular fibers of $S_{\mathcal{Q},z_o}$ are as follows:

Table 2: Possible configurations of singular fibers of $S_{\mathcal{Q},z_o}$

	Singular fibers	the position of l_{z_o}
Case 1	$3\,\mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1$	(i)
Case 2	$\mathrm{IV}, 2\,\mathrm{I}_3, \mathrm{I}_2$	(ii)
Case 3	$3\,\mathrm{I}_3, \mathrm{III}$	(ii)
Case 4	$4\,\mathrm{I}_3$	(iii)
Case 5	$3\,\mathrm{I}_3, \mathrm{IV}$	(iv)
Case 6	$\mathrm{I}_5, 2\,\mathrm{I}_3, \mathrm{I}_1$	(v)

The Table 2 give us possible cases, but by [11], the Cases 3, 5 and 6 in Table 2 do not occur. Let $\mathcal{C}$ be the conic described in Theorem 1. Note that $\mathcal{C}$ exists by Lemma 2.1. Then by our construction of $S_{\mathcal{Q},z_o}$, $\mathcal{C}$ gives rise to two sections, $s_{\mathcal{C}}^{\pm}$, which meets singular fibers as in the following figures if we label irreducible components of singular fibers suitably. Let $P_{\mathcal{C}^+}$ and $P_{\mathcal{C}^-}$ be the corresponding rational points to $s_{\mathcal{C}^+}$ and $s_{\mathcal{C}^-}$, respectively. Then we have $\langle P_{\mathcal{C}^\pm}, P_{\mathcal{C}^\pm}\rangle = 0$ and $P_{\mathcal{C}^\pm}$ are torsions and their orders are 3 by [11] or [9].

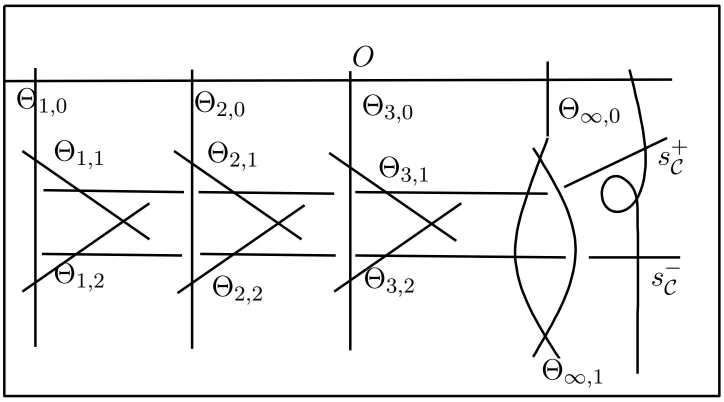

Figure 1: Case 1

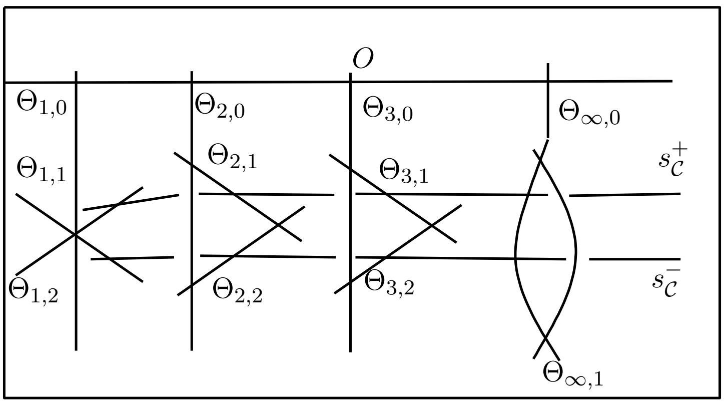

Figure 2: Case 2

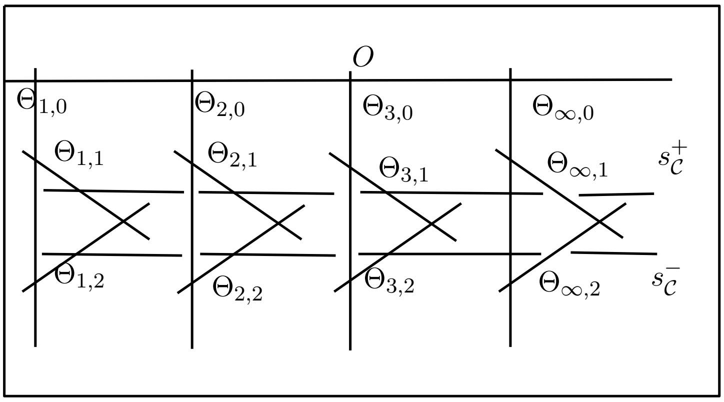

Figure 3: Case 4

§4. Proof of Theorem 1

Choose homogeneous coordinates $[T, X, Z]$ of $\mathbb{P}^2$ such that $l_{z_o} : Z = 0$ and $z_o = [0, 1, 0]$. Then $F_{\mathcal{Q}}$ and $F_{\mathcal{C}}$ are of the form

$$\begin{aligned} F_{\mathcal{Q}}(T, X, Z) &= X^3Z + b_2(T, Z)X^2 + b_3(T, Z)X + b_4(T, Z), \\ F_{\mathcal{C}}(T, X, Z) &= XZ - c_0T^2 - c_1TZ - c_2Z^2, \ c_i \in \mathbb{C}(i = 0, 1, 2), c_0 \neq 0 \end{aligned}$$

where b_i $(i = 2, 3, 4)$ are homogeneous polynomial of degree $\leq i$. Put $p_{\mathcal{Q}}(t, x) = F_{\mathcal{Q}}(t, x, 1)$ and $x_o(t) = c_0t^2 + c_1t + c_2$. Then the elliptic curve $E_{\mathcal{Q}}$ given by $y^2 = p_{\mathcal{Q}}(t, x)$ has a 3 torsion point $P_{\mathcal{C}^+}$ in $E_{\mathcal{Q}}(\mathbb{C}(t))$ and $x_o(t)$ is its x-coordinate. Hence by Proposition 1, we have

$$F_{\mathcal{Q}}(t, x, 1) = (x - c_0t^2 - c_1t - c_2)^3 + \{m(t)(x - c_0t^2 - c_1t - c_2) + y_o(t)\}^2,$$

where $y_o(t)$ is the y-coordinate of $P_{\mathcal{C}^+}$ and $y = m(t)(x - c_0t^2 - c_1t - c_2) + y_o(t)$ is the tangent line at $P_{\mathcal{C}^+}$. By comparing the coefficients of both hand side with respect to x, we have

$$\begin{aligned} b_2(t, 1) &= \{m(t)\}^2 - 3(c_0t^2 + c_1t + c_2), \\ b_4(t, 1) &= \{-m(t)(c_0t^2 + c_1t + c_2) + y_o(t)\}^2 - (c_0t^2 + c_1t + c_2)^3. \end{aligned}$$

Hence, we infer that $\deg m(t) \leq 1, \deg y_o(t) \leq 3$, and we have

$$Z^2F_{\mathcal{Q}}(T, X, Z) = F_{\mathcal{C}}(T, X, Z)^3 + \{Zm(T/Z)F_{\mathcal{C}}(T, X, Z) + Z^3y_o(T/Z)\}^2.$$

This implies Theorem 1. □

Remark 4.1. (i) Note that we also obtain a rational elliptic surface $S_{\mathcal{Q}_1,z_o}$ from a reduced quartic $\mathcal{Q}_1$, which is not concurrent 4 lines, and a distinguished smooth point. A 3-cuspidal quartic and a quartic consisting of a cuspidal cubic and its unique inflectional tangent line are the only ones so that $\mathrm{MW}(S_{\mathcal{Q}_1,z_o})$ has a 3-torsion point for a general z_o. This explains why a 3-cuspidal quartic is so special and we have Theorem 1. We hope this point of view is new.

(ii) As for the case of a cuspidal cubic and its unique inflectional tangent line, the configurations of singular fibers of $S_{\mathcal{Q}_1,z_o}$ is either $\mathrm{I}_6, \mathrm{I}_3, \mathrm{I}_2, \mathrm{I}_1$, $\mathrm{IV}^*, \mathrm{I}_3, \mathrm{I}_1$, or $\mathrm{IV}^*, \mathrm{IV}$.

§5. Example

Now let us consider an explicit example. Let $C : T^2 - XZ = 0$ and Q is the standard quadratic transformation with respect to $\{-2T + X + Z = 0\}, \{2T + X + Z = 0\}$ and $\{Z = 0\}$.

If $P = [a, a^2, 1], a \in \mathbb{C}, a \neq \pm 1$, then tangent line at P is $-2aT+x+a^2Z = 0$. Hence $Q(C), Q(L)$ and $Q(P)$ are given as follows:

$$\begin{aligned} F_{Q(C)} &= 16T^2X^2 - 8T^2XZ + T^2Z^2 - 8TX^2Z - 2TXZ^2 + X^2Z^2, \\ F_{Q(L)} &= 2a^2TX + (1+a)XZ + (1-a)ZT - 2TX, \\ Q(P) &= [(a+1)^2, (a-1)^2, (a+1)^2(a-1)^2]. \end{aligned}$$

The tangent line, $L_{Q(P)}$, to $Q(C)$ at $Q(P)$ has the following equation:

$$(a-1)^3T - (a+1)^3X + 2Z = 0.$$

Let Φ be a coordinate change such that $L_{Q(P)}$ is transformed into the line $Z = 0$ and $Q(P)$ is mapped to $[0, 1, 0]$. Then $\Phi(Q(C))$ and $\Phi(Q(L))$ are given as follows in the affine equations:

$$\begin{aligned} F_{\Phi(Q(C))} &= x^3 + \left(\frac{3(a+1)}{2(a-1)}t^2 + \frac{3}{2}t - \frac{(a+3)^2}{8(a^2-1)}\right)x^2 + \\ &+ \left(\frac{2a(a+1)}{(a-1)^2}t^3 - \frac{3(a+1)}{(a-1)^2}t^2 + \frac{a+3}{(a-1)^2(a+1)}t\right)x \\ &- \frac{2(a+1)}{(a-1)^3}t^4 + \frac{4}{(a-1)^3}t^3 - \frac{2}{(a-1)^3(a+1)}t^2 = 0, \\ F_{\Phi(Q(L))} &= x + \frac{2(a+1)}{a-1}t^2 - \frac{2}{a-1}t = 0, \end{aligned}$$

where $t = T/Z$ and $x = X/Z$.

Then we have

$$\begin{aligned} F_{\Phi(Q(C))} &= F^3_{\Phi(Q(L))} + l_a(t,x)^2, \\ l_a(t,x) &= \frac{6(a+1)t-(a+3)}{\sqrt{-8(a-1)(a+1)}}x + \frac{4(a+1)^2t^3-6(a+1)t^2+2t}{\sqrt{-2(a-1)^3(a+1)}}. \end{aligned}$$

If we first homogenize these equations, then apply Φ^{-1}, we have the following degenerated $(2,3)$ torus decomposition:

$$\begin{aligned} L_a^2 F_{Q(C)} &= -8F^3_{Q(L)} + G^2, \\ L_a &= -(a-1)^3T + (a+1)^3X - 2Z, \\ G &= 4(a-1)^3T^2X - (a-1)^3T^2Z + 4(a+1)^3TX^2 - (a+1)^3X^2Z + \\ &\quad + 2a(a^2-9)TXZ + 2TZ^2 - 2XZ^2. \end{aligned}$$

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