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The existence of universal flat topological connections with discrete structure group

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Abstract. In a topological connection theory, we give a general way of constructing flat slicing functions in locally trivial principal bundles with a discrete group as structure group. Slicing functions play a role of connections in smooth category. By applying this construction to the universal principal bundle over a classifying space which comes from the Milnor construction, we obtain an explicit description of the universal flat slicing function. Using this explicit nature, we show that flat slicing functions given to respective contexts are pulled back from the universal one.

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§1. Introduction and the main results

In various fields of topology and geometry, slicing functions have been studied or used in various contexts. Slicing function is a continuous map of a bundle which maps each fiber to a fiber (Definition 3.1). In the theory of fibrations, slicing functions have been studied as one of the structures of bundles ([1, 6]). Then Milnor [12] generalized the definition of slicing functions to show the existence of a slicing function in a locally trivial principal G-bundle over a polyhedron of a countable simplicial complex in the weak topology. On the other hand, A. Asada [2, 3, 4] generalized connection theory to topological fiber bundles using a germ of certainly a slicing function. Recently, J. Kubarski and N. Teleman [11, 16] showed that the infinitesimal part of a smooth slicing function, which they call a linear direct connection, yields a linear connection in smooth vector bundles. Originally, so-called direct connections have been used without systematic study for several constructions in K-theory and cyclic homology ([5, 8]).

In previous papers [9, 10], we showed that slicing functions in topological principal G-bundles are a generalization of connections in the smooth category [9], and introduced the notion of flatness for slicing functions as a generalization of that for connections [10]. Moreover, we defined parallel displacements along sequences, which are closely related to slicing functions, and showed holonomy reduction theorems, the discreteness of strong holonomy groups of flat ones, and classification theorems of topological principal G-bundles as topological counterparts of those in the smooth category.

In the smooth category, Narasimhan, M. S. and Ramanan, S [14, 15] showed the existence of universal connections for connections in bundles with a Lie group with a finite number of connected components. It is natural to examine whether or not topological counterparts of universal connections exist. In [10], we showed the existence of a flat slicing function in a locally trivial principal bundle with discrete group under a special condition for a bundle atlas. In this paper, we can get rid of the condition for a bundle atlas to construct a flat slicing function. By applying the construction to the universal topological principal bundle $\pi_G : E_G \to B_G$ with a discrete group G, we obtain in flat case a topological counterpart of Narasimhan and Ramanan's universal connection as in Theorems 1.2, 1.3, and 1.4 below.

The purpose of this paper is to construct a flat slicing function in arbitrary locally trivial principal bundles with a discrete group, and give proofs of Theorems 1.1, 1.2, 1.3, and 1.4 below. In Subsection 4.1, we will give a map shown to be a slicing function in a subsequent subsection. In Theorem 1.1, we will show that under the special condition for a bundle atlas, the domain of flat slicing functions constructed in this paper coincide with that in [10]. Next, in Theorem 1.2 we will show the map given in Subsection 4.1 is in fact a flat slicing function, and the uniqueness of a flat slicing function ω_G in the universal principal bundle $\pi_G : E_G \to B_G$ which comes from the Milnor construction [7, 12]. Then, geometrical properties of ω_G will be described in Theorems 1.3 and 1.4: flat slicing functions given to each context are those which are pulled back from ω_G. Finally, compared to the above theorems dealing with discrete groups, we will show in Proposition 1.5 that even if G is not discrete, local trivializations are induced from a specific G-morphism consisting of a locally finite countable partition of unity Λ and a (not necessarily flat) slicing function related to Λ.

Now, let us state the main theorems. For notations, see Sections 2, 3, and 4. The following theorem shows that the flat slicing function constructed in this paper is a generalization of that in [10].

Theorem 1.1. Let G be a discrete group and $\pi : E \to X$ a principal G-bundle with a bundle atlas A. If $U_\alpha \cap U_\beta$ is connected for any $\alpha, \beta \in A$, then $E^2|_{U_A} = (E)^\flat_A$.

The following theorem shows the existence of flat slicing functions in any locally trivial principal bundle and the uniqueness of a flat slicing function in π_G over $(B_G)^\flat_{A_G}$, where A_G is a bundle atlas of π_G.

Theorem 1.2. Let G be a discrete group.

(i) Let $\pi : E \to X$ be a locally trivial principal G-bundle. For any bundle atlas A, there exists a subset $(X)^\flat_A \subset X^2$ with $\Delta_X \subset (X)^\flat_A$ and a $\mathcal{C}_A$-flat G-compatible slicing function ω_A in π over $(X)^\flat_A$.

(ii) Let $\pi_G : E_G \to B_G$ be the universal principal G-bundle which comes from the Milnor construction. Then, $\omega_G := \omega_{A_G}$ is unique for $(B_G)^\flat_{A_G}$.

The following two theorems express the geometrical properties of ω_G.

Theorem 1.3. Let G be a discrete group and π a principal G-bundle with a bundle atlas A. If A is numerable, then there exists a G-morphism $(h_A, f_A) : \pi \to \pi_G$ preserving $(\omega_A, \mathcal{C}_A)$ and $(\omega_G, \mathcal{C}_G)$. In other words, ω_A is induced from ω_G by f_A.

Theorem 1.4. Let G be a discrete group, π a principal G-bundle over X, and Λ a locally finite countable partition of unity on X. Then for any $\mathcal{C}_\Lambda$-flat G-compatible slicing function ω in π over U_Λ, there exists a G-morphism $(h_{\Lambda,\omega}, f_{\Lambda,\omega}) : \pi \to \pi_G$ preserving $(\omega, \mathcal{C}_\Lambda)$ and $(\omega_G, \mathcal{C}_G)$, where $\mathcal{C}_\Lambda$ and U_Λ are sets given by $\{\lambda^{-1}((0,1]) \mid \lambda \in \Lambda\}$ and $\bigcup_{\lambda\in\Lambda} \lambda^{-1}((0,1]) \times \lambda^{-1}((0,1])$ respectively. In other words, ω is induced from ω_G by $f_{\Lambda,\omega}$.

Let G be a (not necessarily discrete) topological group. We do not know whether there exists a universal (not necessarily flat) slicing function for G. However, for any principal G-bundle π with a locally finite countable partition of unity Λ and a G-compatible slicing function ω in π over U_Λ, we can construct a G-morphism $(h_{\Lambda,\omega}, f_{\Lambda,\omega}) : \pi \to \pi_G$ by a similar way of Theorem 1.4. Thus, we have the following proposition:

Proposition 1.5. Let π be a principal G-bundle over X, where G is not necessarily discrete. If there exists a locally finite countable partition of unity Λ on X and a G-compatible slicing function ω in π over U_Λ, then π is locally trivial.

§2. Preliminaries

First, let us prepare notation and some topological facts.

2.1. Notation for bundles

We mostly follow the terminology of Husemoller [7] with slight changes in notation. Thus, we are going to set up notation for bundles. For a continuous map $\pi : E \to X$, we call the map $\pi : E \to X$ itself a *bundle* while usually the triple $\xi = (E, \pi, X)$ (c.f. [7]) or the total space E is referred to as a bundle. Let $\pi : E \to X$ and $\pi' : E' \to X'$ be two bundles. For continuous maps $h : E \to E'$ and $f : X \to X'$, we call $(h, f) : \pi \to \pi'$ a *bundle morphism* if $\pi' \circ h = f \circ \pi$. If $X = X'$, we call $(h, id_X) : \pi \to \pi'$ an X-*morphism* and denote it simply by h. For $Y \subset X$, put

$$E|_Y := \pi^{-1}(Y),\ \pi|_Y := \pi|_{\pi^{-1}(Y)}.$$

We call $\pi|_Y : E|_Y \to Y$ the *restricted bundle* of π to Y. For a continuous map $f : Z \to X$, the *induced bundle* or *pull-back* of π is denoted by $f^*\pi : f^*E \to Z$, where

$$f^*E := Z \times_X E := \{(z, u) \in Z \times E \mid f(z) = \pi(u)\}$$

is a fiber product of $Z \xrightarrow{f} X \xleftarrow{\pi} E$. Put $\overline{f} := \mathrm{pr}_2|_{f^*E} : f^*E \to E$. Then, a bundle morphism $(\overline{f}, f) : f^*\pi \to \pi$ is called the *canonical bundle map*. For topological spaces X and F, a bundle $\mathrm{pr}_1 : X \times F \to X$ is called a *product bundle*. If π is X-isomorphic to a product bundle, we say that π is *trivial*. We say that $\pi : E \to X$ is *locally trivial* if for any $x \in X$, there exists an open neighborhood V of x in X such that $\pi|_V$ is V-isomorphic to a product bundle $\mathrm{pr}_1 : V \times F \to V$. A V-isomorphism $\pi|_V \to \mathrm{pr}_1$ is called a *local trivialization*.

2.2. G-spaces

Let us recall the notion of G-space. Let G be a topological group. A *right G-space* is a topological space E equipped with a continuous right action $\mu : E \times G \to E$. We often denote $\mu(u, a)$ simply by ua. A *left G-space* is defined in a similar way. Remark that by a G-space we mean a right G-space, unless otherwise mentioned. Now, let E be a G-space. We call E a *free G-space* if the right action is free. Denote by E/G the orbit space with the quotient topology, and by $q_G^E : E \to E/G$ the natural projection. Put

$$E^* := \{(u, ua) \in E^2 \mid a \in G\}.$$

A map $T : E^* \to G$ (not necessarily continuous) is called a *translation function* when T satisfies $uT(u, v) = v$ for any $(u, v) \in E^*$. When E is a free G-space, we have a translation function $T : E^* \to G$ by setting

$$T(u, v) := a$$

because for any $(u, v) \in E^*$ there exists a unique $a \in G$ satisfying $v = ua$. Then, this T satisfies

(1) $T(u, u) = 1_G$ for any $u \in E$;

(2) $(ua, vb) \in E^*$ and $T(ua, vb) = a^{-1}T(u, v)b$ for any $(u, v) \in E^*$, $(a, b) \in G^2$;

(3) $T(u, v)T(v, w) = T(u, w)$ for any $(u, v, w) \in E^3$ with $(u, v), (v, w) \in E^*$.

We call a free G-space E a *principal G-space* if T is continuous.

2.3. Principal G-bundles

Let $\pi : E \to X$ be a bundle such that E is a G-space. We call π a *G-bundle* if q^E_G and π are isomorphic by (id_E, f), where f is a unique continuous map such that $f \circ q^E_G = \pi \circ id_E$. Let $\pi : E \to X$ and $\pi' : E' \to X'$ be G-bundles. We call a bundle morphism $(h, f) : \pi \to \pi'$ a *G-morphism* if $h(ua) = h(u)a$ for any $(u, a) \in E \times G$. If $X = X'$, we call $h : \pi \to \pi'$ an *(X, G)-morphism* if it is an X-morphism and a G-morphism. We call a G-bundle $\pi : E \to X$ a *principal G-bundle* if E is a principal G-space. Every morphism in the category of principal G-bundles over X is an (X, G)-isomorphism ([7, Theorem 3.2, Chap. 4]). Let $\pi : E \to X$ be a principal G-bundle. The restricted bundle $\pi|_Y$ and the induced bundle $f^*\pi$ are principal G-bundles in the natural way.

Let $\pi : E \to X$ be a G-bundle. We say that π is *locally G-trivial* or simply *locally trivial* if for any $x \in X$, there exists an open neighborhood V of x in X such that $\pi|_V$ is (V, G)-isomorphic to a product G-bundle $\mathrm{pr}_1 : V \times G \to V$. A (V, G)-isomorphism $\pi|_V \to \mathrm{pr}_1$ is called a *local trivialization*. For a local trivialization $\alpha : \pi|_V \to \mathrm{pr}_1$, put $U_\alpha := V$. For local trivializations α and β, the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is given by

$$g_{\alpha\beta}(x) := (\mathrm{pr}_2 \circ \alpha \circ \beta^{-1})(x, 1_G).$$

Note that a locally trivial G-bundle is a principal G-bundle. For a local trivialization α, let $s_\alpha : U_\alpha \to E|_{U_\alpha}$ be the local section given by $s_\alpha(x) := \alpha^{-1}(x, 1_G)$. Then $T \circ (s_\alpha \hat{\times} s_\beta) = g_{\alpha\beta}$ holds. If π is a locally trivial G-bundle, then both $\pi|_Y$ and $f^*\pi$ are locally trivial.

2.4. The Milnor construction

Following [7], we will recall the universal principal bundle which comes from the Milnor construction [13]. Let G be a topological group and $I = [0, 1]$ the unit interval. Put

$$(I \times G)^\infty := \big\{(t_i, g_i)_{i\in\mathbb{N}} \in (I \times G)^{\mathbb{N}} \mid \#\{i \in \mathbb{N} \mid t_i \neq 0\} < \infty, \sum_{i\in\mathbb{N}} t_i = 1\big\}.$$

For $k \in \mathbb{N}$, let $\bar{t}_k : (I \times G)^\infty \to I,\ \overline{g}_k : (I \times G)^\infty \to G$ be the projections such that $\bar{t}_k((t_j, g_j)_{j\in\mathbb{N}}) := t_k$ and $\overline{g}_k((t_j, g_j)_{j\in\mathbb{N}}) := g_k$ respectively. An equivalence relation on $(I\times G)^\infty$ is defined as follows. For $(t_i, g_i)_{i\in\mathbb{N}}, (s_i, h_i)_{i\in\mathbb{N}} \in (I\times G)^\infty$, $(t_i, g_i)_{i\in\mathbb{N}} \sim (s_i, h_i)_{i\in\mathbb{N}}$ if $t_i = s_i$ for any $i \in \mathbb{N}$, and $g_i = h_i$ for any $i \in \mathbb{N}$ with $t_i = s_i > 0$. Put

$$E_G := (I \times G)^\infty / \sim$$

and denote by q the natural projection. We denote by $\oplus_{i\in\mathbb{N}} t_i g_i$ or $t_1 g_1 \oplus t_2 g_2 \oplus \cdots$ the image $q((t_i, g_i)_{i\in\mathbb{N}})$ of $(t_i, g_i)_{i\in\mathbb{N}}$. For $k \in \mathbb{N}$, let $\tilde{t}_k : E_G \to I$ be the map induced from $\bar{t}_k$. For $k \in \mathbb{N}$, put $V_k := (\tilde{t}_k)^{-1}((0, 1]) = \{\oplus_{i\in\mathbb{N}} t_i g_i \mid t_k > 0\}$. Let $\tilde{g}_k : V_k \to G$ the map induced from $\overline{g}_k|_{(\bar{t}_k)^{-1}((0,1])}$. Let the topology of E_G be the weakest topology so that $\tilde{t}_k$ and $\tilde{g}_k$ are continuous for any $k \in \mathbb{N}$. A free right action $\mu_G : E_G \times G \to E_G$ is defined by

$$\mu_G(\oplus_{i\in\mathbb{N}} t_i g_i,\ g) := \oplus_{i\in\mathbb{N}} t_i g_i g.$$

From the fact that equalities $\tilde{t}_i(\oplus_{i\in\mathbb{N}} t_i g_i g) = \tilde{t}_i(\oplus_{i\in\mathbb{N}} t_i g_i)$ and $\tilde{g}_i(\oplus_{i\in\mathbb{N}} t_i g_i g) = \tilde{g}_i(\oplus_{i\in\mathbb{N}} t_i g_i) g$ hold for any $\oplus_{i\in\mathbb{N}} t_i g_i \in E_G$ and $g \in G$, we can see that μ_G is continuous. Put

$$B_G := E_G/G$$

and denote by π_G the natural projection . We denote by $[\oplus_{i\in\mathbb{N}} t_i g_i]$ the image $\pi_G(\oplus_{i\in\mathbb{N}} t_i g_i)$ of $\oplus_{i\in\mathbb{N}} t_i g_i$. Let $k \in \mathbb{N}$ and put $U_k := \pi_G(V_k) = \{[\oplus_{i\in\mathbb{N}} t_i g_i] \mid t_i > 0\}$. A local trivialization $\varphi_k : E_G|_{U_k} = V_k \to U_k \times G$ is defined by

$$\varphi_k(\oplus_{i\in\mathbb{N}} t_i g_i) := ([\oplus_{i\in\mathbb{N}} t_i g_i], g_k)$$

for $\oplus_{i\in\mathbb{N}} t_i g_i \in E_G|_{U_i}$. The inverse map $\varphi_k^{-1} : U_k \times G \to E_G|_{U_i}$ is given by

$$\varphi_k^{-1}([\oplus_{i\in\mathbb{N}} t_i g_i], g) = \oplus_{i\in\mathbb{N}} t_i g_i g_k^{-1} g.$$

Then, for $k, l \in \mathbb{N}$, transition function $g_{kl} : U_k \cap U_l \to G$ is given by

$$g_{kl}([\oplus_{i\in\mathbb{N}} t_i g_i]) = g_k g_l^{-1}.$$

Thus, $\pi_G : E_G \to B_G$ is a principal G-bundle. We can see that π_G is a universal bundle, that is, E_G is ∞-connected.

2.5. Numerable bundles

Next, we recall numerable bundles (cf. [7]). Let Y be a topological space. An open covering $(U_i)_{i\in S}$ of Y is said to be *numerable* if there exists a locally finite partition of unity $(\lambda_i)_{i\in S}$ such that $\overline{\lambda_i^{-1}((0, 1])} \subset U_i$ for each $i \in S$. A principal G-bundle π over X is *numerable* if there exists a numerable covering $(U_i)_{i\in S}$ of X such that $\pi|_{U_i}$ is trivial for each $i \in S$.

Lemma 2.1. ([7, Proposition 12.1, Chap. 4]) Let π be a numerable principal G-bundle over a space X. Then there exists a locally finite countable partition of unity $(\lambda_i)_{i\in\mathbb{N}}$ such that $\pi|_{\lambda_i^{-1}((0,1])}$ is trivial for each $i \in \mathbb{N}$.

Let $\pi : E \to X$ be a numerable principal G-bundle. Then, from Lemma 2.1 there exists a bundle atlas (a system of local trivializations) $A = (\alpha_i)_{i\in\mathbb{N}}$ such that there exists a locally finite countable partition of unity $(\lambda_i)_{i\in\mathbb{N}}$ such that $\lambda_i^{-1}((0,1]) = U_{\alpha_i}$ for each $i \in \mathbb{N}$. Now, we define numerable bundle atlas as following:

Definition 2.2. Let π be a principal G-bundle and $A = (\alpha_i)_{i\in\mathbb{N}}$ a bundle atlas. We say that A is *numerable* if there exists a locally finite countable partition of unity $(\lambda_i)_{i\in\mathbb{N}}$ such that $\lambda_i^{-1}((0,1]) = U_{\alpha_i}$ for each $i \in \mathbb{N}$.

Let $\pi : E \to X$ be principal G-bundle and $A = (\alpha_i)_{i\in\mathbb{N}}$ a numerable bundle atlas with a locally finite countable partition of unity $(\lambda_i)_{i\in\mathbb{N}}$. Then, a G-morphism $(h_A, f_A) : \pi \to \pi_G$ is given by

$$h_A(u) := \oplus_{i\in\mathbb{N}}\lambda_i(\pi(u))\mathrm{pr}_2(\alpha_i(u)) \tag{2.1}$$

for $u \in E$, where f_A is the induced map from h_A. Moreover, π and $f_A^*\pi_G$ are (X, G)-isomorphic ([7, Theorem 12.2, Chap. 4]).

§3. Slicing functions and morphisms

In previous papers [9, 10], we studied slicing functions and demonstrated that slicing functions in topological bundles are a generalization of connections in the smooth category [9]. Moreover, we introduced the notion of flatness for slicing functions as a generalization of that for connections [10]. In this section, we recall the definition of slicing functions (cf. [9, 10], [12]) and the Asada's connections ([2, 3, 4]). We note that an Asada's connection is the germ of a slicing function at the diagonal set of the base space of a G-bundle. Next, we introduce morphisms preserving slicing functions given for each bundles. Restricted or induced slicing functions are also introduced and some fundamental properties of them are studied.

3.1. Definition of slicing functions

Let $\pi : E \to X$ be a bundle, Δ_X the diagonal set of X. We consider a subset $U \subset X^2$ with $\Delta_X \subset U$. We define a map $p_i : X^2 \to X$ by

$$p_i(x_1, x_0) := x_i \text{ for } (x_1, x_0) \in U, i = 1, 2.$$

We consider a continuous map $\omega : (p_0|_U)^*E \to E$. Every element of $(p_0|_U)^*E$ is written as (x, y, u) such that $(x, y) \in U,\ u \in E_y$. Then, for $(x, y) \in U$, we set a map $\omega_{x,y} := \omega(x, y, \cdot) : E_y \to E$.

Definition 3.1. (cf. [9, 10], [12]) We call ω a *slicing function in* π *over* U if it satisfies:

(1) $\omega_{x,y}$ induces a map $E_y \to E_x$ for any $(x, y) \in U$;

(2) $\omega_{x,x} = id_{E_x}$ for any $x \in X$.

For a slicing function ω in π over U, we define that ω is invertible, G-compatible, and $\mathcal{C}$-flat respectively in the following:

Definition 3.2. (cf. [9, 10], [12]) **(I)** Suppose that U is symmetric, that is, $(y, x) \in U$ for all $(x, y) \in U$. A slicing function ω over U is said to be *invertible* if $\omega_{x,y}$ is invertible and satisfies

$$\omega_{y,x} = \omega_{x,y}^{-1} \ \text{ for any } (x, y) \in U$$

(II) In the case where π is a G-bundle, we say that a slicing function ω over U is *G-compatible* if

$$\omega_{x,y}(ua) = \omega_{x,y}(u)a \ \text{ for any } (x, y) \in U \text{ and } (u, a) \in E \times G$$

(III) Let $\mathcal{C}$ be a covering of X and ω a invertible slicing function over symmetric U. We say that ω is *$\mathcal{C}$-flat* if it satisfies

$$\omega_{x,y} \circ \omega_{y,z} = \omega_{x,z}$$

for any $C \in \mathcal{C}$ and any $x, y, z \in X$ with $(x, y), (y, z), (x, z) \in U \cap C^2$.

Henceforth, we denote by $SF(\pi, U)$, $SF_{\text{inv}}(\pi, U)$, and $SF_{\mathcal{C}\text{-flat}}(\pi, U)$ the sets of slicing functions, invertible slicing functions, and $\mathcal{C}$-flat slicing functions in π over U, respectively. For a G-bundle π, we denote by $SF(\pi, U)_G$ the set of G-compatible slicing functions in π over U. In addition, we set

$$SF_{\text{inv}}(\pi, U)_G := SF_{\text{inv}}(\pi, U) \cap SF(\pi, U)_G,$$
$$SF_{\mathcal{C}\text{-flat}}(\pi, U)_G := SF_{\mathcal{C}\text{-flat}}(\pi, U) \cap SF_{\text{inv}}(\pi, U)_G.$$

3.2. Asada's connections

In this subsection, we recall Asada's connections and discuss a relation between Asada's connections and G-compatible slicing functions. Let $\pi : E \to X$ be a G-bundle and $U \subset X^2$ with $\Delta_X \subset U$.

Definition 3.3. (cf. [2, 3, 4], [9, 10]) **(I)** Let $C^1(\pi, U)_G$ denotes the set of continuous maps $s : E^2|_U \to G$ such that

(1) $s(u,u) = 1_G$ for $u \in E$,

(2) $s(ua, vb) = a^{-1}s(u,v)b$ for $(u,v) \in E^2|_U$ and $a, b \in G$.

(II) When U is symmetric, we denote by $C^1_{\mathrm{inv}}(\pi, U)_G$ the set of $s \in C^1(\pi, U)_G$ such that

$$s(u,v) = s(v,u)^{-1} \text{ for } (u,v) \in E^2|_U,$$

where $s(v,u)^{-1}$ is the inverse element of $s(v,u)$ in G.
(III) For a covering $\mathcal{C}$ of X, we denote by $C^1_{\mathcal{C}\text{-flat}}(\pi, U)_G$ the set of $s \in C^1_{\mathrm{inv}}(\pi, U)_G$ such that

$$s(u,v)s(v,w) = s(u,w)$$

for any $C \in \mathcal{C}$ and any $u, v, w \in E$ with $(u,v), (v,w), (w,u) \in E^2|_{U \cap C^2}$.

Consider the inductive limit $\varinjlim_U C^1(\pi, U)_G$ over all neighborhoods U of Δ_X in X^2. Regarding elements of $\varinjlim_U C^1(\pi, U)_G$ as connections in π, Asada [2, 3, 4] has constructed a connection theory in a category of topological fiber bundles. Here we discuss a relation between Asada's connections and G-compatible slicing functions. Suppose that π is a principal G-bundle. For a G-compatible slicing function $\omega \in SF(\pi, U)_G$, we set a map $s^\omega : E^2|_U \to G$ given by

$$s^\omega(u,v) := T(u, \omega(\pi(u), \pi(v), v)).$$

Then, we see $s^\omega \in C^1(\pi, U)_G$. On the contrary, for any $s \in C^1(\pi, U)_G$, we set a map $\omega^s : U \times_X E \to E$ given by

$$\omega^s(x,y,u) := vs(v,u), \tag{3.1}$$

where we can fix $v \in E_x$ arbitrarily. Then we see $\omega^s \in SF(\pi, U)_G$. We can see that maps $\omega \mapsto s^\omega$ and $s \mapsto \omega^s$ are inverse maps of each other. Thus, $SF(\pi, U)_G$ corresponds bijectively to $C^1(\pi, U)_G$. In addition, the map $\omega \mapsto s^\omega$ induces bijections $SF_{\mathrm{inv}}(\pi, U)_G \to C^1_{\mathrm{inv}}(\pi, U)_G$ and $SF_{\mathcal{C}\text{-flat}}(\pi, U)_G \to C^1_{\mathcal{C}\text{-flat}}(\pi, U)_G$.

3.3. Morphisms

Next, we introduce bundle morphisms preserving slicing functions.

Definition 3.4. Let $\pi : E \to X$ (resp. $\pi' : E' \to X'$) be a bundle, $\omega \in SF(\pi, U)$ (resp. $\omega' \in SF(\pi', U')$), and $(h, f) : \pi \to \pi'$ a bundle morphism.
(I) We say that (h, f) *preserves* ω and ω' if $f^2(U) \subset U'$ and

$$h(\omega(x,y,u)) = \omega'(f(x), f(y), h(u)) \text{ for any } (x,y,u) \in (p_0|_U)^*E.$$

(II) The concept of $\mathcal{C}$-flatness depends on coverings. When ω (resp. ω') is $\mathcal{C}$-flat (resp. $\mathcal{C}'$-flat), we say that (h, f) *preserves* $(\omega, \mathcal{C})$ and $(\omega', \mathcal{C}')$ if (h, f) preserves ω and ω', and $f_*\mathcal{C}$ is a refinement of $\mathcal{C}'$, where $f_*\mathcal{C} := \{f(C) \mid C \in \mathcal{C}\}$.

It is obvious that when (h, f) preserves $(\omega, \mathcal{C})$ and $(\omega', \mathcal{C}')$, we have

$$h((\omega_{x,y} \circ \omega_{y,z})(u)) = (\omega'_{f(x),f(y)} \circ \omega'_{f(y),f(z)})(h(u))$$

for any $x, y, z \in X$ with $(x, y), (y, z), (z, x) \in U \cap C^2$ and $u \in E_z$.

We obtain a category of bundles with slicing functions whose morphisms are bundle morphisms preserving slicing functions. By considering the concepts of invertible, G-compatible, and $\mathcal{C}$-flat, we get subcategories.

Let $\pi : E \to X$ be a bundle and $\omega \in SF(\pi, U)$. For a subset $V \subset U$ with $\Delta_X \subset V$, ω induces a slicing function $\omega|_V := \omega|_{(p_0|_V)^*E} : (p_0|_V)^*E \to E$, called a *restricted slicing function*. Let $f : X' \to X$ be a continuous map and put $f^*U := (f^2)^{-1}(U)$. Then we have a slicing function $f^*\omega : (p_0|_{f^*U})^* f^*E \to f^*E$ given by

$$(f^*\omega)(x_1, x_0, (x_0, u)) := (x_1, \omega(f(x_1), f(x_0), \overline{f}(x_0, u))) = (x_1, \omega(f(x_1), f(x_0), u))$$

for $(x_1, x_0, (x_0, u)) \in (p_0|_{f^*U})^* f^*E$. We call $f^*\omega$ an *induced slicing function*. The following properties are fundamental.

Proposition 3.5. (1) The isomorphism (id_X, id_E) preserves $\omega|_V$ and ω (not ω and $\omega|_V$ if $V \neq U$);

(2) If π is a G-bundle and ω is G-compatible, then $\omega|_V$ and $f^*\omega$ are also G-compatible;

(3) If ω is $\mathcal{C}$-flat, then $\omega|_V$ (resp. $f^*\omega$) is $\mathcal{C}$-flat (resp. $f^*\mathcal{C}$-flat), where $f^*\mathcal{C} := \{f^{-1}(C) | C \in \mathcal{C}\}$. Moreover, the canonical bundle map $(\overline{f}, f)$ preserves $(f^*\omega, f^*\mathcal{C})$ and $(\omega, \mathcal{C})$;

(4) Suppose that π and π' are principal G-bundles. Consider two slicing functions $\omega \in SF_{\mathcal{C}\text{-flat}}(\pi, U)_G$, $\omega' \in SF_{\mathcal{C}'\text{-flat}}(\pi', U')_G$, and a G-morphism $(h, f) : \pi \to \pi'$ preserving $(\omega, \mathcal{C})$ and $(\omega', \mathcal{C}')$. Then, the canonical (X, G)-isomorphism $\theta : \pi \to f^*\pi'$ (resp. $\theta^{-1} : f^*\pi' \to \pi$) preserves $(\omega, \mathcal{C})$ and $(f^*\omega', f^*\mathcal{C}')$ (resp. $((f^*\omega')|_U, \mathcal{C})$ and $(\omega, \mathcal{C})$).

§4. A construction and Proofs of Theorems

In this section we construct a flat slicing function and present proofs of Theorems 1.1, 1.2, 1.3, and 1.4 in the introduction.

4.1. Construction of flat slicing function

In this subsection, we construct a map s_A which is shown to be a slicing function of Asada's type. At first, we give a domain $(E)^\flat_A \subset E^2$. Then, a map $s_A : (E)^\flat_A \to G$ is defined as follows. Let G be a discrete group and $\pi : E \to X$ a principal G-bundle with a bundle atlas A. We denote by α the local trivialization $E|_{U_\alpha} \xrightarrow{\alpha} U_\alpha \times G$ and by pr_2 the projection $U_\alpha \times G \xrightarrow{\mathrm{pr}_2} G$. Then, we have a map $\mathrm{pr}_2 \circ \alpha : E|_{U_\alpha} \to G$. We consider a domain of s_A. We set $U_A := \bigcup_{\alpha \in A} U_\alpha \times U_\alpha$ and we put a domain

$$(E)^\flat_A := \left\{ (u,v) \in E^2|_{U_A} \;\middle|\; \begin{array}{c} \text{There exists } g \in G \text{ such that} \\ (\mathrm{pr}_2 \circ \alpha)(v) = (\mathrm{pr}_2 \circ \alpha)(u)g \\ \text{for any } \alpha \in A \text{ with } (u,v) \in (E|_{U_\alpha})^2 \end{array} \right\}.$$

Note that for $(u,v) \in (E)^\flat_A$, if $(u,v) \in (E|_{U_\alpha})^2 \cap (E|_{U_\beta})^2$, then

$$(\mathrm{pr}_2 \circ \alpha)(u)^{-1}(\mathrm{pr}_2 \circ \alpha)(v) = (\mathrm{pr}_2 \circ \beta)(u)^{-1}(\mathrm{pr}_2 \circ \beta)(v).$$

Thus, we have a map $s_A : (E)^\flat_A \to G$ such that

$$s_A(u,v) := (\mathrm{pr}_2 \circ \alpha)(u)^{-1}(\mathrm{pr}_2 \circ \alpha)(v), \ (u,v) \in (E|_{U_\alpha})^2. \tag{4.1}$$

Put

$$(X)^\flat_A := (\pi \times \pi)((E)^\flat_A).$$

Then, we have $\Delta_X \subset (X)^\flat_A$. As we shall see in Subsection 4.3, we have $(\pi\times\pi)^{-1}((X)^\flat_A) = (E)^\flat_A$, hence $(E)^\flat_A = E^2|_{(X)^\flat_A}$. Thus, we can take $(X)^\flat_A$ as U in Definition 3.3. To show that s_A is an element of $C^1_{\mathcal{C}_A\text{-flat}}(\pi, (X)^\flat_A)_G$, we will check in Subsection 4.3 the continuity of s_A and the conditions in Definition 3.3, where $\mathcal{C}_A := \{U_\alpha \mid \alpha \in A\}$. As we have already seen in Subsection 3.2, $C^1_{\mathcal{C}_A\text{-flat}}(\pi, (X)^\flat_A)_G$ corresponds bijectively to $SF_{\mathcal{C}_A\text{-flat}}(\pi, (X)^\flat_A)_G$ by the map $s \mapsto \omega^s$ given by (3.1). Thus, once we can see $s_A \in C^1_{\mathcal{C}_A\text{-flat}}(\pi, (X)^\flat_A)_G$, we get a $\mathcal{C}_A$-flat G-compatible slicing function $\omega_A := \omega^{s_A} \in SF_{\mathcal{C}_A\text{-flat}}(\pi, (X)^\flat_A)_G$.

4.2. Proof of Theorem 1.1

Since $E^2|_{U_A} \supset (E)^\flat_A$, it suffices to show $E^2|_{U_A} \subset (E)^\flat_A$. Let $(u,v) \in E^2|_{U_A}$. Then, there exists $\alpha \in A$ such that $(u,v) \in (E|_{U_\alpha})^2$ and we put

$$g := (\mathrm{pr}_2 \circ \alpha)(u)^{-1}(\mathrm{pr}_2 \circ \alpha)(v).$$

For any $\beta \in A$ such that $(u,v) \in (E|_{U_\beta})^2$, the transition function satisfies

$$(\mathrm{pr}_2 \circ \beta)(u) = g_{\beta\alpha}(\pi(u))(\mathrm{pr}_2 \circ \alpha)(u)$$

and also for v. Then, we have

$$\begin{aligned}(\mathrm{pr}_2\circ\beta)(u)^{-1}(\mathrm{pr}_2\circ\beta)(v) &= (g_{\beta\alpha}(\pi(u))(\mathrm{pr}_2\circ\alpha)(u))^{-1}g_{\beta\alpha}(\pi(v))(\mathrm{pr}_2\circ\alpha)(v)\\ &= (\mathrm{pr}_2\circ\alpha)(u)^{-1}g_{\alpha\beta}(\pi(u))g_{\beta\alpha}(\pi(v))(\mathrm{pr}_2\circ\alpha)(v).\end{aligned}$$

Since $U_\alpha\cap U_\beta$ is connected and G is discrete, we see $g_{\alpha\beta}(\pi(u))=g_{\alpha\beta}(\pi(v))$. Then we have

$$(\mathrm{pr}_2\circ\beta)(u)^{-1}(\mathrm{pr}_2\circ\beta)(v)=(\mathrm{pr}_2\circ\alpha)(u)^{-1}(\mathrm{pr}_2\circ\alpha)(v)=g.$$

Thus, $(u,v)\in(E)^\flat_A$, which gives the desired result. □

4.3. Proof of Theorem 1.2

Proof of (i) To show that $s_A\in C^1_{\mathcal{C}_A\text{-flat}}(\pi,(X)^\flat_A)_G$, we will check the following:

(A) s_A satisfies (I),(II), and (III) in Definition 3.3;

(B) the domain $(E)^\flat_A$ of s_A is written as $(\pi\times\pi)^{-1}((X)^\flat_A)$, hence $(E)^\flat_A=E^2|_{(X)^\flat_A}$;

(C) s_A is continuous.

To show (A), (B) and (C), we consider for $\alpha\in A$ a continuous map $F_\alpha:(E|_{U_\alpha})^2\to G$ given by

$$F_\alpha(u,v):=(\mathrm{pr}_2\circ\alpha)(u)^{-1}(\mathrm{pr}_2\circ\alpha)(v).$$

The map F_α has the following properties:

(a) $F_\alpha(u,u)=1_G$ for $u\in E$;

(b) $F_\alpha(ua,vb)=a^{-1}F_\alpha(u,v)b$ for $a,b\in G$ and $(u,v)\in(E|_{U_\alpha})^2$;

(c) $F_\alpha(v,u)=F_\alpha(u,v)^{-1}$ for $(u,v)\in(E|_{U_\alpha})^2$;

(d) $F_\alpha(u,v)F_\alpha(v,w)=F_\alpha(u,w)$ for $(u,v),(v,w)\in(E|_{U_\alpha})^2$.

It is obvious $s_A|_{(E)^\flat_A\cap(E|_{U_\alpha})^2}=F_\alpha|_{(E)^\flat_A\cap(E|_{U_\alpha})^2}$. Then (A) follows from properties (a),(b),(c), and (d).

Next, we shall check (B). Note that $(\pi\times\pi)^{-1}((X)^\flat_A)=\bigcup_{g\in G}(\pi\times\pi)^{-1}((\pi\times\pi)(s_A^{-1}(\{g\})))$. Suppose that $(u,v)\in(\pi\times\pi)^{-1}((\pi\times\pi)(s_A^{-1}(\{g\})))$. Then, we have $(\pi(u),\pi(v))\in(\pi\times\pi)(s_A^{-1}(\{g\}))$. Thus, there exists $(u',v')\in s_A^{-1}(\{g\})$ such that $(\pi(u),\pi(v))=(\pi(u'),\pi(v'))$. Therefore, there exist $a,b\in G$ such that $u'=ua$ and $v'=vb$, hence $(ua,vb)\in s_A^{-1}(\{g\})$. Then, from the condition

(I) of s_A, we have $(u,v) \in s_A^{-1}(\{agb^{-1}\})$. Thus, we get $(\pi \times \pi)^{-1}((X)^\flat_A) \subset (E)^\flat_A$.

Next, we will show (C). We note here that for any $g \in G$, (u,v) is an element of $s_A^{-1}(\{g\})$ if and only if $(u,v) \in E^2|_{U_A}$ and $(u,v) \in F_\alpha^{-1}(\{g\})$ for any $\alpha \in A$ with $(u,v) \in (E|_{U_\alpha})^2$. To show that s_A is continuous, let us take an open set O in G and $(u,v) \in s_A^{-1}(O)$. Then, there exists $g \in O$ such that $(u,v) \in s_A^{-1}(\{g\})$. Thus, we have $(u,v) \in E^2|_{U_A}$ and $(u,v) \in F_\alpha^{-1}(\{g\})$ for any $\alpha \in A$ with $(u,v) \in (E|_{U_\alpha})^2$. From $(u,v) \in E^2|_{U_A}$, there exists $\beta \in A$ such that $(u,v) \in (E|_{U_\beta})^2$. Together with the latter condition we get $(u,v) \in F_\beta^{-1}(\{g\})$. Note that $F_\beta^{-1}(\{g\})$ is an open set in E^2 since G is discrete. Therefore, $F_\beta^{-1}(\{g\}) \cap (E)^\flat_A$ is an open neighborhood of (u,v) in $(E)^\flat_A$. If an inclusion $F_\beta^{-1}(\{g\}) \cap (E)^\flat_A \subset s_A^{-1}(\{g\})$ holds, we see that $s_A^{-1}(O)$ is an open set in $(E)^\flat_A$, hence s_A is continuous. Thus, we will check the inclusion $F_\beta^{-1}(\{g\}) \cap (E)^\flat_A \subset s_A^{-1}(\{g\})$. Let $(u',v') \in F_\beta^{-1}(\{g\}) \cap (E)^\flat_A$. Then, $(u',v') \in F_\beta^{-1}(\{g\})$ and there exists $g' \in G$ such that $(u',v') \in F_\alpha^{-1}(\{g'\})$ for any $\alpha \in A$ with $(u',v') \in (E|_{U_\alpha})^2$. From these two conditions we get $g' = g$. Obviously, we have $(u',v') \in E^2|_{U_A}$. Thus, we obtain $(u',v') \in s_A^{-1}(\{g\})$. This ends the proof of (i).

Proof of (ii) At first, we set up the notation used in the following. We denote by $\varphi_i : E_G|_{U_i} \to U_i \times G$ $(i \in \mathbb{N})$ a local trivialization of $\pi_G : E_G \to B_G$ (see Subsection 2.4) and put $A_G := \{\varphi_i \mid i \in \mathbb{N}\}$. Recall that $\tilde{g}_k : E_G|_{U_k} \to G$ is given by $\tilde{g}_k(\oplus_{i\in\mathbb{N}} t_i g_i) = g_k$. Thus, we have $\tilde{g}_k = \mathrm{pr}_2 \circ \varphi_k$. We put $U_G := U_{A_G} := \bigcup_{i\in\mathbb{N}} U_i \times U_i$. By taking E_G as E, U_G as U_A, and $\tilde{g}_k$ as $\mathrm{pr}_2 \circ \alpha$ in the definition of $(E)^\flat_A$, we get the following expression:

$$(E_G)^\flat_{A_G} = \left\{ (\oplus_{i\in\mathbb{N}} t_i g_i, \oplus_{j\in\mathbb{N}} s_j h_j) \in (E_G)^2|_{U_G} \;\middle|\; \begin{array}{c} \text{There exists } g \in G \text{ such that} \\ h_k = g_k g \text{ for any } k \in \mathbb{N} \\ \text{with } t_k > 0 \text{ and } s_k > 0 \end{array} \right\}.$$

By taking A_G as A in the definition (4.1) of s_A, we have the expression of $s_G := s_{A_G} : (E_G)^\flat_{A_G} \to G$ as follows:

$$s_G(\oplus_{i\in\mathbb{N}} t_i g_i, \oplus_{j\in\mathbb{N}} s_j h_j) := g_k^{-1} h_k$$

if $t_k > 0$ and $s_k > 0$. Then, by the bijection (3.1), we have the expression of $\omega_G := \omega^{s_G} : (B_G)^\flat_{A_G} \times_{B_G} E_G \to E_G$ as follows:

$$\omega_G([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j], \oplus_{j\in\mathbb{N}} s_j h_j) = \oplus_{i\in\mathbb{N}} t_i g_i g_k^{-1} h_k \tag{4.2}$$

if $t_k > 0$ and $s_k > 0$. Under the above notation, we will start proving (ii) of Theorem 1.2. Let $([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j]) \in (B_G)^\flat_{A_G}$. Then, there exists $g \in G$

such that $h_k = g_k g$ for any $k \in \mathbb{N}$ with $t_k > 0$ and $s_k > 0$. Consider a map $c : I = [0,1] \to E_G$ given by

$$c(r) := \oplus_{i\in\mathbb{N}}(t_i r + s_i(1-r))h_i$$

for $r \in I$. We have $(c(r), \oplus_{j\in\mathbb{N}} s_j h_j) \in (E_G)^\flat_{A_G}$ and $([c(r)], [\oplus_{j\in\mathbb{N}} s_j h_j]) \in (B_G)^\flat_{A_G}$ for all $r \in I$. Note that the topology of E_G is the weakest one such that $\tilde{t}_i : E_G \to I$ and $\tilde{g}_i : E_G|_{U_i} \to G$ are continuous for all $i \in \mathbb{N}$. To show that the map c is continuous, we only have to show that $\tilde{t}_i \circ c$ and $\tilde{g}_i \circ c$ are continuous for all $i \in \mathbb{N}$, and it is obvious. We have $[c(0)] = [\oplus_{j\in\mathbb{N}} s_j h_j]$ and $[c(1)] = [\oplus_{i\in\mathbb{N}} t_i h_i] = [\oplus_{i\in\mathbb{N}} t_i g_i g] = [\oplus_{i\in\mathbb{N}} t_i g_i]$. Thus, $\pi_G \circ c : I \to B_G$ is a curve joining $[\oplus_{j\in\mathbb{N}} s_j h_j]$ to $[\oplus_{i\in\mathbb{N}} t_i g_i]$. Let ω be any slicing function in π_G over $(B_G)^\flat_{A_G}$ and suppose that $([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j]) \in U_k \times U_k$. Since $\tilde{g}_k(\omega([c(\cdot)], [c(0)], \oplus_{j\in\mathbb{N}} s_j h_j)) : I \to G$ is a continuous map from a connected set I to a discrete group G, we have

$$\tilde{g}_k(\omega([c(1)], [c(0)], \oplus_{j\in\mathbb{N}} s_j h_j)) = \tilde{g}_k(\omega([c(0)], [c(0)], \oplus_{j\in\mathbb{N}} s_j h_j)).$$

Then, we get

$$\tilde{g}_k(\omega([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j], \oplus_{j\in\mathbb{N}} s_j h_j)) = \tilde{g}_k(\oplus_{j\in\mathbb{N}} s_j h_j)) = h_k.$$

Here, we note that in general, by using a local trivialization α, any point $u \in E|_{U_\alpha}$ is written as $u = \alpha^{-1}(\pi(u), \mathrm{pr}_2(\alpha(u)))$. Thus, we have

$$\begin{aligned}\omega([\oplus_{i\in\mathbb{N}} t_i g_i],&[\oplus_{j\in\mathbb{N}} s_j h_j], \oplus_{j\in\mathbb{N}} s_j h_j) \\ &= \varphi_k^{-1}([\oplus_{i\in\mathbb{N}} t_i g_i], \tilde{g}_k(\omega([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j], \oplus_{j\in\mathbb{N}} s_j h_j)))\end{aligned}$$

Then, the right hand side is equal to

$$\begin{aligned}\varphi_k^{-1}([\oplus_{i\in\mathbb{N}} t_i g_i], h_k) &= \oplus_{i\in\mathbb{N}} t_i g_i g_k^{-1} h_k \\ &= \omega_G([\oplus_{i\in\mathbb{N}} t_i g_i], [\oplus_{j\in\mathbb{N}} s_j h_j], \oplus_{j\in\mathbb{N}} s_j h_j),\end{aligned}$$

hence $\omega = \omega_G$. This is the required result. □

4.4. Proof of Theorem 1.3

Let $A = (\alpha_i)_{i\in\mathbb{N}}$ be a numerable bundle atlas of $\pi : E \to X$ and $(\lambda_i)_{i\in\mathbb{N}}$ a locally finite countable partition of unity such that $\lambda_i^{-1}((0,1]) = U_{\alpha_i}$ for each $i \in \mathbb{N}$. As a G-morphism $\pi \to \pi_G$, we take (h_A, f_A) given by (2.1):

$$h_A(u) = \oplus_{i\in\mathbb{N}} \lambda_i(\pi(u))\mathrm{pr}_2(\alpha(u)).$$

Firstly, we show that $(f_A)^2((X)^\flat_A) \subset (B_G)^\flat_{A_G}$. Let $(x, y) \in (X)^\flat_A$ and take $(u, v) \in (E)^\flat_A$ such that $\pi(u) = x$ and $\pi(v) = y$. Then, there exists $g \in G$ such that $(u, v) \in s_A^{-1}(\{g\})$. From $(u, v) \in E^2|_{U_A}$, there exists $i \in \mathbb{N}$ such that $\lambda_i(x) > 0$ and $\lambda_i(y) > 0$. Thus, we have $(h_A(u), h_A(v)) \in (E_G)^2|_{U_G}$. On the other hand, from $(u, v) \in s_A^{-1}(\{g\})$, for any $k \in \mathbb{N}$, if $\lambda_k(x) > 0$ and $\lambda_k(y) > 0$, then we have $F_{\alpha_k}(u, v) = g$, that is, $\mathrm{pr}_2(\alpha_k(v)) = \mathrm{pr}_2(\alpha_k(u))g$. Therefore, we have $(h_A(u), h_A(v)) \in (E_G)^\flat_{A_G}$ and $(f_A(x), f_A(y)) \in (B_G)^\flat_{A_G}$.

Secondly, we show that (h_A, f_A) preserves ω_A and ω_G. Note that in general, by using a local trivialization α, ω_A is expressed as

$$\omega_A(x, y, u) = \alpha^{-1}(x, \mathrm{pr}_2(\alpha(u))).$$

In fact, by the definition of bijection (3.1) and s_A, we have

$$\begin{aligned}\omega_A(x, y, u) &= \alpha^{-1}(x, 1_G)s_A(\alpha^{-1}(x, 1_G), u)\\ &= \alpha^{-1}(x, 1_G)\mathrm{pr}_2(\alpha(u)) = \alpha^{-1}(x, \mathrm{pr}_2(\alpha(u))).\end{aligned}$$

Let $(x, y, u) \in (X)^\flat_A \times_X E$. Then, by the expression of ω_A, we have for any $i \in \mathbb{N}$,

$$\mathrm{pr}_2(\alpha_i(\omega_A(x, y, u))) = \mathrm{pr}_2(\alpha_i(u)).$$

Then we have

$$h_A(\omega_A(x, y, u)) = \oplus_{i\in\mathbb{N}}\lambda_i(x)\mathrm{pr}_2(\alpha_i(\omega_A(x, y, u))) = \oplus_{i\in\mathbb{N}}\lambda_i(x)\mathrm{pr}_2(\alpha_i(u)).$$

Here, note that we can take $\omega_A(x, y, u)$ as an element of E_x to express $f_A(x)$ as $[h_A(\omega_A(x, y, u))]$. Thus, we have the following expression

$$\begin{aligned}&\omega_G(f_A(x), f_A(y), h_A(u))\\ &\qquad = \omega_G([h_A(\omega_A(x, y, u))], [\oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u))], \oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u))).\end{aligned}$$

Then, from the expression (4.2) of ω_G, if $\lambda_k(x) > 0$ and $\lambda_k(y) > 0$, the right hand side is equal to

$$\begin{aligned}&\omega_G([\oplus_{i\in\mathbb{N}}\lambda_i(x)\mathrm{pr}_2(\alpha_i(u))], [\oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u))], \oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u)))\\ &\qquad = \oplus_{i\in\mathbb{N}}\lambda_i(x)\mathrm{pr}_2(\alpha_i(u))\mathrm{pr}_2(\alpha_k(u))^{-1}\mathrm{pr}_2(\alpha_k(u)))\\ &\qquad = \oplus_{i\in\mathbb{N}}\lambda_i(x)\mathrm{pr}_2(\alpha_i(u)).\end{aligned}$$

Therefore, we get $h_A(\omega_A(x, y, u)) = \omega_G(f_A(x), f_A(y), h_A(u))$.

Finally, we shall check that $f_{A*}\mathcal{C}_A$ is a refinement of $\mathcal{C}_G := \mathcal{C}_{A_G} = \{U_i \mid i \in \mathbb{N}\}$. Let $[\oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u))] \in f_A(U_{\alpha_k})$ with $y \in U_{\alpha_k}$ and $u \in E_y$. Then, $\lambda_k(y) > 0$ holds. Thus, we get $[\oplus_{i\in\mathbb{N}}\lambda_i(y)\mathrm{pr}_2(\alpha_i(u))] \in U_k$. Therefore, (h_A, f_A) preserves $(\omega_A, \mathcal{C}_A)$ and $(\omega_G, \mathcal{C}_G)$. From Proposition 3.5, it follows that ω_A is induced from ω_G. This ends the proof. □

4.5. Proof of Theorem 1.4

Let G be a discrete group, $\pi : E \to X$ a principal G-bundle (not necessarily locally trivial), $\Lambda = (\lambda_i)_{i\in\mathbb{N}}$ a locally finite countable partition of unity. Put $\mathcal{C}_\Lambda := \{\lambda_i^{-1}((0,1]) \mid i \in \mathbb{N}\}$ and $U_\Lambda := \bigcup_{i\in\mathbb{N}} \lambda_i^{-1}((0,1]) \times \lambda_i^{-1}((0,1])$. We remark here that we do not consider slicing functions over $(X)^\flat_A$ but over U_Λ. Hence, let $\omega \in SF_{\mathcal{C}_\Lambda\text{-flat}}(\pi, U_\Lambda)_G$.

Firstly, we give a G-morphism $(h_{\Lambda,\omega}, f_{\Lambda,\omega}) : \pi \to \pi_G$. To this end, fix $z_i \in \lambda_i^{-1}((0,1])$ and $w_i \in E_{z_i}$ for each $i \in \mathbb{N}$, and let T be the translation function of π. Then, a continuous map $h_{\Lambda,\omega} : E \to E_G$ is given by

$$h_{\Lambda,\omega}(u) := \oplus_{i\in\mathbb{N}}\lambda_i(\pi(u))T(w_i, \omega(z_i, \pi(u), u)) \tag{4.3}$$

for $u \in E$. By the definition, we have $h_{\Lambda,\omega}(ua) = h(u)a$ for $(u, a) \in E\times G$. Let $f_{\Lambda,\omega} : X \to B_G$ be the induced map from $h_{\Lambda,\omega}$ such that $f_{\Lambda,\omega} \circ \pi = \pi_G \circ h_{\Lambda,\omega}$. Then, $(h_{\Lambda,\omega}, f_{\Lambda,\omega})$ is a G-morphism.

Secondly, we show that $(f_{\Lambda,\omega})^2(U_\Lambda) \subset (B_G)^\flat_{A_G}$. Let $(x, y) \in U_\Lambda$ and $(u, v) \in E^2|_{U_\Lambda}$ such that $(\pi(u), \pi(v)) = (x, y)$. Then, we put $g := T(u, \omega(x, y, v))$. From the properties of translation function T, for any $k \in \mathbb{N}$, we get

$$\begin{aligned} T(w_k, \omega(z_k, x, u))g &= T(w_k, \omega(z_k, x, u))T(u, \omega(x, y, v)) \\ &= T(w_k, \omega(z_k, x, uT(u, \omega(x, y, v)))) = T(w_k, \omega(z_k, x, \omega(x, y, v))). \end{aligned}$$

Since ω is $\mathcal{C}_\Lambda$-flat, for any $k \in \mathbb{N}$, we get $\omega(z_k, x, \omega(x, y, v))) = \omega(z_k, y, v)$. Thus, for any $k \in \mathbb{N}$, we get

$$T(w_k, \omega(z_k, x, u))g = T(w_k, \omega(z_k, y, v)).$$

This implies $(h_{\Lambda,\omega}(u), h_{\Lambda,\omega}(v)) \in (E_G)^\flat_{A_G}$, hence $(f_{\Lambda,\omega}(x), f_{\Lambda,\omega}(y)) \in (B_G)^\flat_{A_G}$.

Thirdly, we show that $(h_{\Lambda,\omega}, f_{\Lambda,\omega})$ preserves ω and ω_G. Let $(x, y, v) \in U_\Lambda \times_X E$ and $u \in E$ such that $\pi(u) = x$. Suppose that $\lambda_k(x) > 0$ and $\lambda_k(y) > 0$. Then, from the expressions (4.2) of ω_G and (4.3) of $h_{\Lambda,\omega}$, we have

$$\begin{aligned} &\omega_G(f_{\Lambda,\omega}(x), f_{\Lambda,\omega}(y), h_{\Lambda,\omega}(v)) \\ &= \oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i, \omega(z_i, x, u))T(w_k, \omega(z_k, x, u))^{-1}T(w_k, \omega(z_k, y, v)). \end{aligned} \tag{4.4}$$

From the properties of T and the flatness of ω, we have

$$\begin{aligned} T(w_k,&\omega(z_k, x, u))T(u, \omega(x, y, v)) \\ &= T(w_k, \omega(z_k, x, \omega(x, y, v))) = T(w_k, \omega(z_k, y, v)). \end{aligned}$$

Then, the right hand side of (4.4) is equal to

$$\begin{aligned} &\oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i, \omega(z_i, x, u))T(u, \omega(x, y, v)) \\ &\quad = \oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i, \omega(z_i, x, uT(u, \omega(x, y, v)))) \\ &\quad = \oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i, \omega(z_i, x, \omega(x, y, v))) = h_{\Lambda,\omega}(\omega(x, y, v)). \end{aligned}$$

Finally, to show that $f_{\Lambda,\omega *}\mathcal{C}_\Lambda$ is a refinement of $\mathcal{C}_G$, we take an element $[\oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i,\omega(z_i,x,v))]$ of $f_{\Lambda,\omega}(\lambda_k^{-1}((0,1]))$ with $x \in \lambda_k^{-1}((0,1])$ and $v \in E_x$. Then, $\lambda_k(x) > 0$ holds. Thus, we get $[\oplus_{i\in\mathbb{N}}\lambda_i(x)T(w_i,\omega(z_i,x,v))] \in U_k$. Therefore, $(h_{\Lambda,\omega}, f_{\Lambda,\omega})$ preserves $(\omega, \mathcal{C}_\Lambda)$ and $(\omega_G, \mathcal{C}_G)$. From Proposition 3.5, it follows that ω is induced from ω_G. This completes the proof. □

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