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F-geometric mean labeling of graphs obtained by duplicating any edge of some graphs

A. Durai Baskar and S. Arockiaraj

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Abstract. A function f is called an F-geometric mean labeling of a graph $G = (V, E)$ with p vertices and q edges if $f : V \to \{1, 2, 3, \ldots, q+1\}$ is injective and the induced function $f^* : E \to \{1, 2, 3, \ldots, q\}$ defined as

$$f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor, \text{ for all } uv \in E,$$

is bijective. A graph that admits an F-geometric mean labeling is called an F-geometric mean graph. In this paper we discuss the F-geometric meanness of graphs obtained by duplicating any edge of some graphs.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected simple graph. Let $G = (V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [4] and for a detailed survey on graph labeling, we refer [5].

Path on n vertices is denoted by P_n and a cycle on n vertics is denoted by C_n. The graph $G \circ K_1$ is obtained from the graph G by attaching a new pendant vertex at each vertex of G. A ladder $L_n, n \geq 2$, is the graph $P_2 \times P_n$. Duplicating of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = (N(u) \cup \{v'\}) - \{v\}$ and $N(v') = (N(u) \cup \{u'\}) - \{u\}$ [6].

The concept of F-geometric mean labeling was first introduced by Durai Baskar et al. [1] and they studied the F-geometric mean labeling of some standard graphs [2, 3].

A function f is called an F-geometric mean labeling of a graph $G = (V, E)$ with p vertices and q edges if $f : V \to \{1, 2, 3, \ldots, q + 1\}$ is injective and induced function $f^* : E \to \{1, 2, 3, \ldots, q\}$ defined as $f^*(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$, for all $uv \in E$, is bijective. A graph that admits an F-geometric mean labeling is called an F-geometric mean graph. In this paper we discuss the F-geometric meanness of graphs obtained by duplicating any edge of some graphs.

§2. Main Results

Theorem 1. *Let G be a graph obtained by duplicating an edge e of a path $P_n, n \geq 3$. Then G is an F-geometric mean graph.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n. Let $e' = v'_i v'_{i+1}$ be the duplicating edge of $e = v_i v_{i+1}$ for some $i, 1 \leq i \leq n - 1$.

Case 1. $i = 1$ or $i = n - 1$.

Since the graph G is isomorphic when $i = 1$ or $i = n - 1$, we may take $i = 1$.

Define $f : V(G) \to \{1, 2, 3, \ldots, n + 2\}$ as follows:

$f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq 3 \\ j + 2, & 4 \leq j \leq n \end{cases}$ and $f(v'_j) = 2j$, for $1 \leq j \leq 2$.

Then the induced edge labeling is obtained as follows:

$f^*(v_j v_{j+1}) = \begin{cases} 2j - 1, & 1 \leq j \leq 3 \\ j + 2, & 4 \leq j \leq n - 1, \end{cases}$ $f^*(v'_1 v'_2) = 2$ and $f^*(v'_2 v_3) = 4$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Case 2. $i = 2$ and $n \geq 4$.

Define $f : V(G) \to \{1, 2, 3, \ldots, n + 3\}$ as follows:

$f(v_j) = \begin{cases} 2j - 1, & 1 \leq j \leq 4 \\ j + 3, & 5 \leq j \leq n \end{cases}$ and $f(v'_j) = 2j$, for $2 \leq j \leq 3$.

Then the induced edge labeling is obtained as follows:

$f^*(v_j v_{j+1}) = \begin{cases} 2j - 1, & 1 \leq j \leq 4 \\ j + 3, & 5 \leq j \leq n - 1, \end{cases}$

$f^*(v_1 v'_2) = 2, f^*(v'_2 v'_3) = 4$ and $f^*(v'_3 v_4) = 6$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Case 3. $3 \leq i \leq n - 2$ and $n \geq 5$.

Define $f : V(G) \to \{1, 2, 3, \ldots, n + 3\}$ as follows:

$f(v_j) = \begin{cases} j, & 1 \leq j \leq i - 1 \\ i + 2, & j = i \\ j + 3, & i + 1 \leq j \leq n, \end{cases}$ $f(v'_i) = i + 1$ and $f(v'_{i+1}) = i + 3$.

Then the induced edge labeling is obtained as follows:

$$f^*(v_jv_{j+1}) = \begin{cases} j, & 1 \le j \le i-2 \\ i, & j = i-1 \\ i+2, & j = i \\ j+3, & i+1 \le j \le n-1 \end{cases}$$

$f^*(v_{i-1}v'_i) = i-1, f^*(v'_iv'_{i+1}) = i+1$ and $f^*(v'_{i+1}v_{i+2}) = i+3$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph. □

The F-geometric mean labeling of G in the above cases are shown in Figure 1.

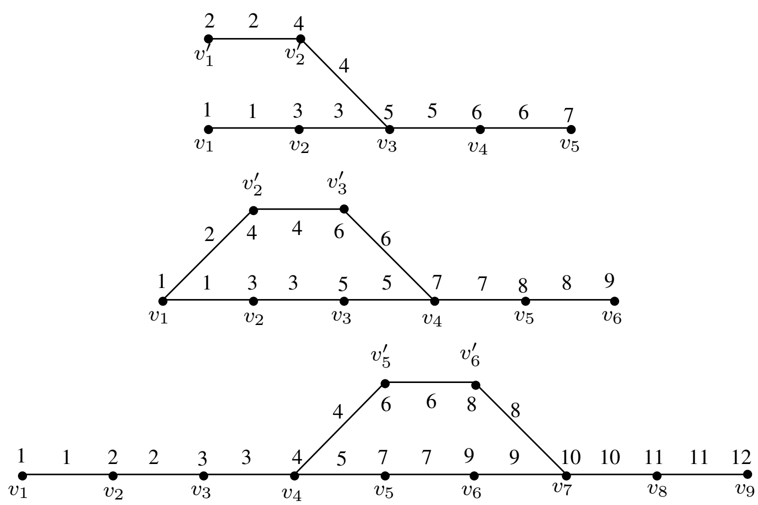


Figure 1.

Theorem 2. *Let G be a graph obtained by duplicating an edge e of a graph $P_n \circ K_1, n \ge 2$. Then G is an F-geometric mean graph.*

Proof. Let $u_1, u_2, \dots, u_n$ be the vertices of the path P_n and v_i be a pendant vertex attached at u_i, for $1 \le i \le n$. When $n = 2$, an F-geometric mean labeling of G is shown in Figures 2 and 3 (Figure 2 is the case $e = u_1v_1$ and Figure 3 is the case $e = u_1u_2$). So we assume $n \ge 3$.

Case 1. $e = u_iv_i$, for $1 \le i \le n$

Let its duplication be $e' = u'_iv'_i$.

Subcase (i). $i = 1$ or $i = n$.

Since the graph G is isomorphic when $i = 1$ or $i = n$, we may take $i = 1$.

Define $f : V(G) \to \{1, 2, 3, \dots, 2n+2\}$ as follows:

$$f(u_j) = \begin{cases} j+3, & 1 \le j \le 2 \\ 2j+2, & 3 \le j \le n, \end{cases} \quad f(v_j) = \begin{cases} 4j-2, & 1 \le j \le 2 \\ 2j+1, & 3 \le j \le n, \end{cases}$$

$f(u'_1) = 3$ and $f(v'_1) = 1$.

Then the induced edge labeling is obtained as follows:

$f^*(u_ju_{j+1}) = 2j+2$, for $1 \le j \le n-1$, $f^*(u_jv_j) = \begin{cases} 2, & j=1 \\ 2j+1, & 2 \le j \le n, \end{cases}$

$f^*(u_1'v_1') = 1$ and $f^*(u_1'u_2) = 3$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$.

Define $f : V(G) \to \{1, 2, 3, \dots, 2n+3\}$ as follows:

$$f(u_j) = \begin{cases} 3, & j=1 \\ 4j-4, & 2 \le j \le 3 \\ 2j+3, & 4 \le j \le n, \end{cases} \quad f(v_j) = \begin{cases} 1, & j=1 \\ 7j-12, & 2 \le j \le 3 \\ 2j+2, & 4 \le j \le n \end{cases}$$

$f(u_2') = 7$ and $f(v_2') = 6$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 2j+1, & 1 \le j \le 2 \\ 2j+3, & 3 \le j \le n-1, \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} j, & 1 \le j \le 2 \\ 2j+2, & 3 \le j \le n, \end{cases}$$

$f^*(u_1u_2') = 4, f^*(u_2'u_3) = 7$ and $f^*(u_2'v_2') = 6$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iii). $3 \le i \le n-1$ and $n \ge 4$.

Define $f : V(G) \to \{1, 2, 3, \dots, 2n+3\}$ as follows.

$$f(u_j) = \begin{cases} 2j, & 1 \le j \le i \\ 2i+4, & j = i+1 \\ 2j+3, & i+2 \le j \le n \end{cases} \quad f(v_j) = \begin{cases} 2j-1, & 1 \le j \le i \\ 2i+5, & j = i+1 \\ 2j+2, & i+2 \le j \le n, \end{cases}$$

$f(u_i') = 2i+3$ and $f(v_i') = 2i+2$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 2j, & 1 \le j \le i-1 \\ 2i+1, & j = i \\ 2j+3, & i+1 \le j \le n-1, \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} 2j-1, & 1 \le j \le i \\ 2j+2, & i+1 \le j \le n, \end{cases}$$

$f^*(u_{i-1}u_i') = 2i, f^*(u_i'u_{i+1}) = 2i+3$ and $f^*(u_i'v_i') = 2i+2$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

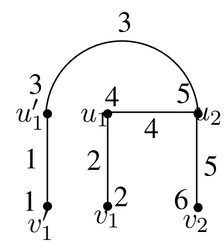


Figure 2

Case 2. $e = u_i u_{i+1}$, for $1 \leq i \leq n-1$

Let its duplication be $e' = u_i' u_{i+1}'$

Subcase (i). $i = 1$ or $i = n-1$.

Since the graph G is isomorphic when $i = 1$ or $i = n-1$, we may take $i = 1$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+4\}$ as follows:

$f(u_j) = \begin{cases} 1, & j = 1 \\ 2j+4, & 2 \leq j \leq n, \end{cases}$ $f(v_j) = \begin{cases} 3j, & 1 \leq j \leq 3 \\ 2j+3, & 4 \leq j \leq n, \end{cases}$

$f(u_1') = 4$ and $f(u_2') = 5$.

Then the induced edge labeling obtained as follows:

$f^*(u_j u_{j+1}) = \begin{cases} 2, & j = 1 \\ 2j+4, & 2 \leq j \leq n-1, \end{cases}$

$f^*(u_j v_j) = \begin{cases} 5j-4, & 1 \leq j \leq 2 \\ 2j+3, & 3 \leq j \leq n, \end{cases}$

$f^*(u_1' v_1) = 3, f^*(u_1' u_2') = 4, f^*(u_2' v_2) = 5$ and $f^*(u_2' u_3) = 7$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$ and $n \geq 4$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+5\}$ as follows:

$f(u_j) = \begin{cases} 5j-3, & 1 \leq j \leq 2 \\ 4j-4, & 3 \leq j \leq 4 \\ 2j+5, & 5 \leq j \leq n, \end{cases}$ $f(v_j) = \begin{cases} 5j-4, & 1 \leq j \leq 2 \\ 3j+1, & 3 \leq j \leq 4 \\ 2j+4, & 5 \leq j \leq n, \end{cases}$

$f(u_2') = 3$ and $f(u_3') = 11$.

Then the induced edge labeling is obtained as follows:

$f^*(u_j u_{j+1}) = \begin{cases} 3, & j = 1 \\ 2j+3, & 2 \leq j \leq 3 \\ 2j+5, & 4 \leq j \leq n-1, \end{cases}$

$f^*(u_j v_j) = \begin{cases} 1, & j = 1 \\ 2j+2, & 2 \leq j \leq 3 \\ 2j+4, & 4 \leq j \leq n, \end{cases}$

$f^*(u_1 u_2') = 2, f^*(u_2' u_3') = 5, f^*(u_3' u_4) = 11, f^*(u_2' v_2) = 4$ and $f^*(u_3' v_3) = 10$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iii). $3 \leq i \leq n-2$ and $n \geq 5$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+5\}$ as follows:

$$f(u_j) = \begin{cases} 2j, & 1 \leq j \leq i-1 \\ 2i+3, & j = i \\ 2i+4, & j = i+1 \\ 2i+8, & j = i+2 \\ 2j+5, & i+3 \leq j \leq n, \end{cases} \qquad f(v_j) = \begin{cases} 2j-1, & 1 \leq j \leq i-1 \\ 2i, & j = i \\ 2i+6, & j = i+1 \\ 2i+9, & j = i+2 \\ 2j+4, & i+3 \leq j \leq n, \end{cases}$$

$f(u'_i) = 2i-1$ and $f(u'_{i+1}) = 2i+7$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 2j, & 1 \leq j \leq i-2 \\ 2i, & j = i-1 \\ 2i+3, & j = i \\ 2i+5, & j = i+1 \\ 2j+5, & i+2 \leq j \leq n-1, \end{cases}$$

$$f^*(u_j v_j) = \begin{cases} 2j-1, & 1 \leq j \leq i-1 \\ 2i+1, & j = i \\ 2i+4, & j = i+1 \\ 2j+4, & i+2 \leq j \leq n, \end{cases}$$

$f^*(u_{i-1}u'_i) = 2i-2, f^*(u'_i u'_{i+1}) = 2i+2, f^*(u'_i u_{i+2}) = 2i+7, f^*(u'_i v_i) = 2i-1$ and $f^*(u'_{i+1} v_{i+1}) = 2i+6$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph. □

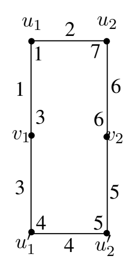
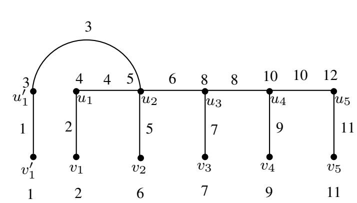


Figure 3

The F-geometric mean labeling of G in the above cases are shown in Figure 4.



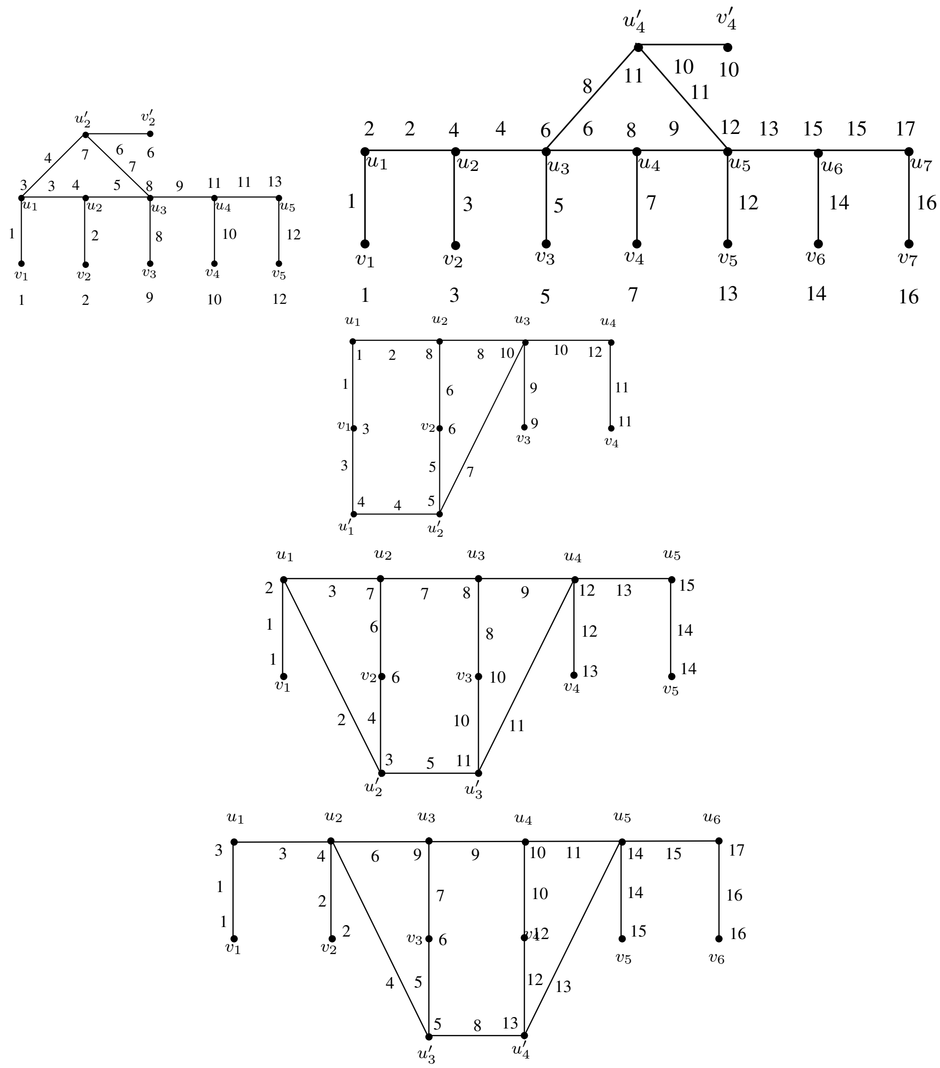


Figure 4

Theorem 3. *Let G be a graph obtained by duplicating an edge e of a graph $C_n \circ K_1, n \geq 3$. Then G is an F-geometric mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n and v_i be a pendant vertex attached at u_i, for $1 \leq i \leq n$. When $n = 3$, an F-geometric mean labeling of G is shown in Figures 5 and 6 (Figure 5 is the case $e = u_1v_1$ and Figure 6 is the case $e = u_1u_2$). So we assume $n \geq 4$.

Case 1. $e = u_iv_i$, for $1 \leq i \leq n$.

Let its duplication be $e' = u_i'v_i'$ and choose arbitrarily $i = 1$.

Subcase (i). n is odd

Define $f : V(G) \to \{1, 2, 3, \ldots, 2n+4\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j-2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is even} \\ 2n+4, & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is odd} \\ 4n+9-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 2j+3, & 1 \le j \le 2 \\ 4j, & 3 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is odd} \\ 4j-2, & 3 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is even} \\ 2n+3, & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ 4n+9-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is odd} \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is even,} \end{cases}$$

$f(u'_1) = 1$ and $f(v'_1) = 2$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+1, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is odd} \\ 2n+2, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is even} \\ 4n+7-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n-1, \end{cases}$$

$$f^*(u_nu_1) = 6, f^*(u_jv_j) = \begin{cases} 3j+1, & 1 \le j \le 2 \\ 4j-2, & 3 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 4n+9-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n, \end{cases}$$

$f^*(u'_1v'_1) = 1, f^*(u_nu'_1) = 3$ and $f^*(u_2u'_1) = 2$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

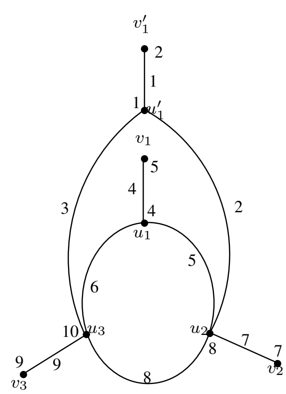


Figure 5

Subcase (ii). n is even.

Define $f : V(G) \to \{1, 2, 3, \dots, 2n + 4\}$ as follows:

$$f(u_j)=\begin{cases}4, & j=1\\ 4j-2, & 2\le j\le \left\lfloor \frac{n}{2}\right\rfloor+1 \text{ and } j \text{ is odd}\\ 4j, & 2\le j\le \left\lfloor \frac{n}{2}\right\rfloor+1 \text{ and } j \text{ is even}\\ 4n+9-4j, & \left\lfloor \frac{n}{2}\right\rfloor+2\le j\le n \text{ and } j \text{ is odd}\\ 4n+11-4j, & \left\lfloor \frac{n}{2}\right\rfloor+2\le j\le n \text{ and } j \text{ is even,}\end{cases}$$

$$f(v_j)=\begin{cases}2j+3, & 1\le j\le 2\\ 4j, & 3\le j\le \left\lfloor \frac{n}{2}\right\rfloor \text{ and } j \text{ is odd}\\ 4j-2, & 3\le j\le \left\lfloor \frac{n}{2}\right\rfloor \text{ and } j \text{ is even}\\ 2n+1, & j=\left\lfloor \frac{n}{2}\right\rfloor+1 \text{ is odd}\\ 2n+3, & j=\left\lfloor \frac{n}{2}\right\rfloor+1 \text{ is even}\\ 2n+2, & j=\left\lfloor \frac{n}{2}\right\rfloor+2 \text{ is odd}\\ 2n+4, & j=\left\lfloor \frac{n}{2}\right\rfloor+2 \text{ is even}\\ 4n+11-4j, & \left\lfloor \frac{n}{2}\right\rfloor+3\le j\le n \text{ and } j \text{ is odd}\\ 4n+9-4j, & \left\lfloor \frac{n}{2}\right\rfloor+3\le j\le n \text{ and } j \text{ is even,}\end{cases}$$

$f(u'_1)=1$ and $f(v'_1)=2$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1})=\begin{cases}5, & j=1\\ 4j, & 2\le j\le \left\lfloor \frac{n}{2}\right\rfloor\\ 2n+2, & j=\left\lfloor \frac{n}{2}\right\rfloor+1\\ 4n+7-4j, & \left\lfloor \frac{n}{2}\right\rfloor+2\le j\le n-1,\end{cases}$$

$$f^*(u_jv_j)=\begin{cases}3j+1, & 1\le j\le 2\\ 4j-2, & 3\le j\le \left\lfloor \frac{n}{2}\right\rfloor\\ 2n+1, & \left\lfloor \frac{n}{2}\right\rfloor+1\le j\le \left\lfloor \frac{n}{2}\right\rfloor+2 \text{ and } j \text{ is odd}\\ 2n+3, & \left\lfloor \frac{n}{2}\right\rfloor+1\le j\le \left\lfloor \frac{n}{2}\right\rfloor+2 \text{ and } j \text{ is even}\\ 4n+9-4j, & \left\lfloor \frac{n}{2}\right\rfloor+3\le j\le n,\end{cases}$$

$f^*(u_nu_1)=6$, $f^*(u'_1v'_1)=1, f^*(u_nu'_1)=3$ and $f^*(u_2u'_1)=2$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Case 2. $e=u_iu_{i+1}$, for $1\le i\le n-1$.

Let its duplication be $e'=u'_iu'_{i+1}$ and choose arbitrarily $i=1$.

Subcase (i). n is odd

Define $f:V(G)\to\{1,2,3,\dots,2n+6\}$ as follows:

$$f(u_j)=\begin{cases}4, & j=1\\ 4j+2, & 2\le j\le \left\lfloor \frac{n}{2}\right\rfloor+1 \text{ and } j \text{ is odd}\\ 4j, & 2\le j\le \left\lfloor \frac{n}{2}\right\rfloor+1 \text{ and } j \text{ is even}\\ 2n+6, & j=\left\lfloor \frac{n}{2}\right\rfloor+2\\ 4n+11-4j, & \left\lfloor \frac{n}{2}\right\rfloor+3\le j\le n \text{ and } j \text{ is odd}\\ 4n+13-4j, & \left\lfloor \frac{n}{2}\right\rfloor+3\le j\le n \text{ and } j \text{ is even,}\end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is odd} \\ 4j+2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is even} \\ 2n+5, & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ 4n+13-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is odd} \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is even,} \end{cases}$$

$f(u_1') = 2$ and $f(u_2') = 6$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j+2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+4, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is odd} \\ 2n+3, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is even} \\ 4n+9-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n-1 \end{cases}$$

$$f^*(u_n u_1) = 6,\ f^*(u_j v_j) = \begin{cases} 2, & j = 1 \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n, \end{cases}$$

$f^*(u_n u_1') = 4, f^*(u_1' u_2') = 3, f^*(u_2' u_3) = 9, f^*(u_1' v_1) = 1$ and $f^*(u_2' v_2) = 7$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

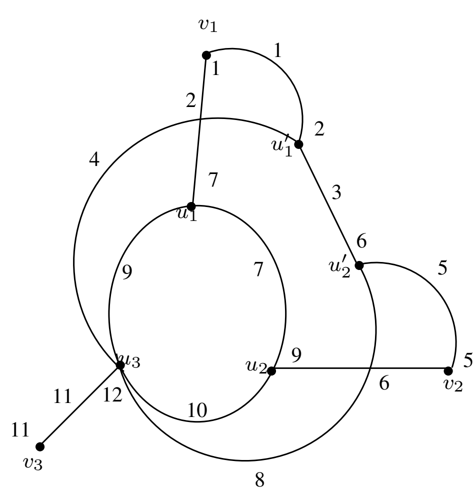


Figure 6

Subcase (ii). n is even.

Define $f : V(G) \to \{1, 2, 3, \dots, 2n+6\}$ as follows:

$$f(u_j) = \begin{cases} 4, & j = 1 \\ 4j+2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is odd} \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } j \text{ is even} \\ 4n+13-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n \text{ and } j \text{ is odd} \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n \text{ and } j \text{ is even,} \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1 \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } j \text{ is odd} \\ 4j+2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \text{ and } j \text{ is even} \\ 2n+5, & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ is odd} \\ 2n+3, & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ is even} \\ 2n+6, & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ is odd} \\ 2n+4, & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ is even} \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is odd} \\ 4n+13-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \text{ and } j \text{ is even,} \end{cases}$$

$f(u_1') = 2$ and $f(u_2') = 6$.

Then the induced edge labeling is obtained as follows:

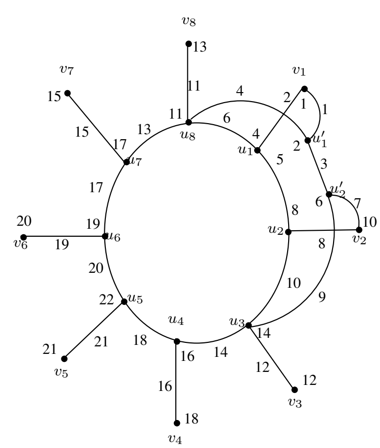
$$f^*(u_j u_{j+1}) = \begin{cases} 5, & j = 1 \\ 4j+2, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor - 2 \\ 2n-2, & j = \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 2n+2, & j = \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+4, & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 4n+9-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n-1, \end{cases}$$

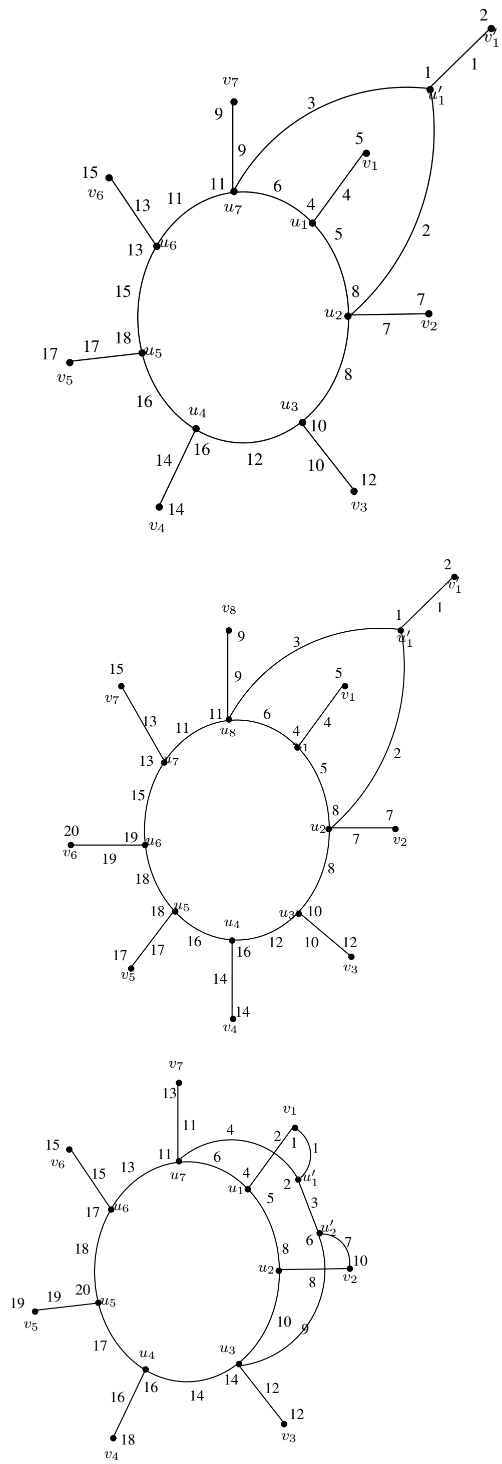
$$f^*(u_j v_j) = \begin{cases} 2, & j = 1 \\ 4j, & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+5, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is odd} \\ 2n+3, & \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } j \text{ is even} \\ 4n+11-4j, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n, \end{cases}$$

$f^*(u_n u_1) = 6$, $f^*(u_n u_1') = 4, f^*(u_1' u_2') = 3, f^*(u_2' u_3) = 9$,
$f^*(u_1' v_1) = 1$ and $f^*(u_2' v_2) = 7$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

The F-geometric mean labeling of G in the above cases are shown in Figure 7.




v_1'
u_1'
v_7
v_1
v_6
u_7
u_1
u_6
u_2
v_2
u_5
v_5
u_4
u_3
v_3
v_4
v_1'
u_1'
v_8
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u_4
v_5
v_3
v_4
v_7
v_1
v_6
u_7
u_1'
u_1
u_6
u_2'
u_2
v_2
v_5
u_5
u_4
u_3
v_3
v_4

Figure 7

□

Theorem 4. *Let G be a graph obtained by duplicating an edge e of a graph $L_n, n \geq 2$, Then $G$is an F-geometric mean graph.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices on the path of length $n-1$ in the ladder L_n. When $n = 2$, an F-geometric mean labeling of G is shown in Figure 8 (Figure 8 is the case $e = u_1v_1$ and $e = u_1u_2$). So we assume $n \geq 3$.

Case 1. $e = u_iv_i$, for $1 \leq i \leq n$

Let its duplication be $e' = u_i'v_i'$.

Subcase (i). $i = 1$ or $i = n$.

Since the graph G is isomorphic when $i = 1$ or $i = n$, we may take $i = 1$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+2\}$ as follows:

$f(u_j) = \begin{cases} 6, & j = 1 \\ 3j+1, & 2 \leq j \leq n, \end{cases}$ $f(v_j) = \begin{cases} 4, & j = 1 \\ 3j+2, & 2 \leq j \leq n, \end{cases}$

$f(u_1') = 2$ and $f(v_1') = 1$.

Then the induced edge labeling is obtained as follows:

$f^*(u_ju_{j+1}) = \begin{cases} 6, & j = 1 \\ 3j+2, & 2 \leq j \leq n-1, \end{cases}$

$f^*(u_jv_j) = 3j+1$, for $1 \leq j \leq n$, $f^*(v_jv_{j+1}) = \begin{cases} 5, & j = 1 \\ 3j+3, & 2 \leq j \leq n-1, \end{cases}$

$f^*(u_1'v_1') = 1, f^*(u_1'u_2) = 3$ and $f^*(v_1'v_2) = 2$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$f(u_j) = \begin{cases} 3, & j = 1 \\ 1, & j = 2 \\ 3j+4, & 3 \leq j \leq n, \end{cases}$ $f(v_j) = \begin{cases} 8, & j = 1 \\ 5, & j = 2 \\ 3j+3, & 3 \leq j \leq n, \end{cases}$

$f(u_2') = 10$ and $f(v_2') = 9$.

Then the induced edge labeling is obtained as follows:

$f^*(u_ju_{j+1}) = \begin{cases} 2j-1, & 1 \leq j \leq 2 \\ 3j+5, & 3 \leq j \leq n-1, \end{cases}$

$f^*(u_jv_j) = \begin{cases} 4, & j = 1 \\ 2, & j = 2 \\ 3j+3, & 3 \leq j \leq n, \end{cases}$

$f^*(v_jv_{j+1}) = \begin{cases} j+5, & 1 \leq j \leq 2 \\ 3j+4, & 3 \leq j \leq n-1, \end{cases}$

$f^*(u_1u_2') = 5, f^*(u_2'u_3) = 11, f^*(v_1v_2') = 8, f^*(v_2'v_3) = 10$ and $f^*(u_2'v_2') = 9$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iii). $i = 3$ and $n \geq 4$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$$f(u_j) = \begin{cases} 3j-2, & 1 \leq j \leq 2 \\ 14, & j = 3 \\ 3j+4, & 4 \leq j \leq n, \end{cases} \qquad f(v_j) = \begin{cases} 2j+1, & 1 \leq j \leq 3 \\ 3j+3, & 4 \leq j \leq n, \end{cases}$$

$f(u'_3) = 10$ and $f(v'_3) = 13$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 5j-3, & 1 \leq j \leq 2 \\ 3j+5, & 3 \leq j \leq n-1, \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} 1, & j = 1 \\ 5j-6, & 2 \leq j \leq 3 \\ 3j+3, & 4 \leq j \leq n, \end{cases}$$

$$f^*(v_jv_{j+1}) = \begin{cases} 3, & j = 1 \\ 5j-5, & 2 \leq j \leq 3 \\ 3j+4, & 4 \leq j \leq n-1, \end{cases}$$

$f^*(u_2u'_3) = 6, f^*(u'_3u_4) = 12, f^*(v_2v'_3) = 8, f^*(v'_3v_4) = 13$ and $f^*(u'_3v'_3) = 11$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iv). $4 \leq i \leq n-1$ and $n \geq 5$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$$f(u_j) = \begin{cases} 3j-2, & 1 \leq j \leq i-1 \\ 3i+3, & j = i \\ 3j+4, & i+1 \leq j \leq n, \end{cases}$$

$$f(v_j) = \begin{cases} 3j-1, & 1 \leq j \leq i-1 \\ 3i-2, & j = i \text{ and } 4 \leq i \leq 6 \\ 3i-3, & j = i \text{ and } 7 \leq i \leq n-1 \\ 3j+3, & i+1 \leq j \leq n, \end{cases}$$

$f(u'_i) = 3i+1$ and $f(v'_i) = 3i+5$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 3j-1, & 1 \leq j \leq i-2 \\ 3i-2, & j = i-1 \\ 3i+4, & j = i \\ 3j+5, & i+1 \leq j \leq n-1, \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} 3j-2, & 1 \leq j \leq i-1 \\ 3i, & j = i \text{ and } 4 \leq i \leq 6 \\ 3i-1, & j = i \text{ and } 7 \leq i \leq n-1 \\ 3j+3, & i+1 \leq j \leq n, \end{cases}$$

$$f^*(v_jv_{j+1}) = \begin{cases} 3j, & 1 \le j \le i-2 \\ 3i-4, & j = i-1 \\ 3i+1, & j = i \\ 3j+4, & i+1 \le j \le n-1, \end{cases}$$

$f^*(u_{i-1}u'_i) = 3i-3, f^*(u'_iu_{i+1}) = 3i+3, f^*(v'_iv_{i+1}) = 3i+5,$

$f^*(u'_iv'_i) = 3i+2$ and $f^*(u_{i-1}v'_i) = \begin{cases} 3i-1, & 4 \le i \le 6 \\ 3i, & 7 \le i \le n-1. \end{cases}$

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Case 2. $e = u_iu_{i+1}$, for $1 \le i \le n-1$

Let its duplication be $e' = u'_iu'_{i+1}$.

Subcase (i). $i = 1$ or $i = n-1$.

Since the graph G is isomorphic when $i = 1$ or $i = n-1$, we may take $i = 1$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+3\}$ as follows:

$f(u_j) = \begin{cases} 5j, & 1 \le j \le 2 \\ 3j+3, & 3 \le j \le n, \end{cases}$ $f(v_j) = \begin{cases} 3, & j = 1 \\ 3j+2, & 2 \le j \le n, \end{cases}$

$f(u'_1) = 1$ and $f(u'_2) = 4$.

Then the induced edge labeling is obtained as follows:

$f^*(u_ju_{j+1}) = 3j+4$, for $1 \le j \le n-1$,

$f^*(u_jv_j) = \begin{cases} 3, & j = 1 \\ 3j+2, & 2 \le j \le n, \end{cases}$

$f^*(v_jv_{j+1}) = \begin{cases} 4, & j = 1 \\ 3j+3, & 2 \le j \le n-1, \end{cases}$

$f^*(u'_1v_1) = 1, f^*(u'_1u'_2) = 2, f^*(u'_2u_3) = 6$ and $f^*(u'_2v_2) = 5$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (ii). $i = 2$ and $n \ge 4$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$f(u_j) = 3j+4$, for $1 \le j \le n, f(v_j) = 3j+3$, for $1 \le j \le n$,

$f(u'_2) = 1$ and $f(u'_3) = 2$.

Then the induced edge labeling is obtained as follows:

$f^*(u_ju_{j+1}) = 3j+5$, for $1 \le j \le n-1$, $f^*(u_jv_j) = 3j+3$, for $1 \le j \le n$,

$f^*(v_jv_{j+1}) = 3j+4$, for $1 \le j \le n-1$,

$f^*(u_1u'_2) = 2, f^*(u'_2u'_3) = 1, f^*(u'_3u_4) = 5, f^*(u'_2v_2) = 3$ and $f^*(u'_3v_3) = 4$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iii). $i = 3$ and $n \ge 5$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$f(u_j) = \begin{cases} j+3, & 1 \le j \le 2 \\ 4j-1, & 3 \le j \le 4 \\ 3j+4, & 5 \le j \le n, \end{cases}$ $f(v_j) = \begin{cases} 2j-1, & 1 \le j \le 2 \\ 4j-3, & 3 \le j \le 4 \\ 3j+3, & 5 \le j \le n, \end{cases}$

$f(u'_3) = 8$ and $f(u'_4) = 16$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 3j+1, & 1 \le j \le 2 \\ 4j, & 3 \le j \le 4 \\ 3j+5, & 5 \le j \le n-1, \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} j+1, & 1 \le j \le 2 \\ 4j-3, & 3 \le j \le 4 \\ 3j+3, & 5 \le j \le n, \end{cases}$$

$$f^*(v_jv_{j+1}) = \begin{cases} 1, & j = 1 \\ 5j-5, & 2 \le j \le 4 \\ 3j+4, & 5 \le j \le n-1, \end{cases}$$

$f^*(u_2u'_3) = 6, f^*(u'_3u'_4) = 11, f^*(u'_4u_5) = 17, f^*(u'_3v_3) = 8$ and $f^*(u'_4v_4) = 14$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph.

Subcase (iv). $4 \le i \le n-2$ and $n \ge 6$.

Define $f : V(G) \to \{1, 2, 3, \ldots, 3n+4\}$ as follows:

$$f(u_j) = \begin{cases} 3j-1, & 1 \le j \le i-2 \\ 3i-3, & j = i-1 \\ 3i+2, & j = i \\ 3i+6, & j = i+1 \\ 3j+4, & i+2 \le j \le n, \end{cases} \qquad f(v_j) = \begin{cases} 3j-2, & 1 \le j \le i-1 \\ 3i-1, & j = i \\ 3i+4, & j = i+1 \\ 3j+3, & i+2 \le j \le n, \end{cases}$$

$f(u'_i) = 3i-2$ and $f(u'_{i+1}) = 3i+8$.

Then the induced edge labeling is obtained as follows:

$$f^*(u_ju_{j+1}) = \begin{cases} 3j, & 1 \le j \le i-2 \\ 3i-1, & j = i-1 \\ 3i+3, & j = i \\ 3i+7, & j = i+1 \\ 3j+5, & i+2 \le j \le n-1 \end{cases}$$

$$f^*(u_jv_j) = \begin{cases} 3j-2, & 1 \le j \le i-1 \\ 3i, & j = i \\ 3i+4, & j = i+1 \\ 3j+3, & i+2 \le j \le n, \end{cases}$$

$$f^*(v_jv_{j+1}) = \begin{cases} 3j-1, & 1 \le j \le i-1 \\ 3i+1, & j = i \\ 3i+6, & j = i+1 \\ 3j+4, & i+2 \le j \le n-1, \end{cases}$$

$f^*(u_{i-1}u'_i) = 3i-3, f^*(u'_iu'_{i+1}) = 3i+2, f^*(u'_iu_{i+2}) = 3i+8$,
$f^*(u'_iv_i) = 3i-2$ and $f^*(u'_{i+1}v_{i+1}) = 3i+5$.

Hence, f is an F-geometric mean labeling of G. Thus the graph G is an F-geometric mean graph. □

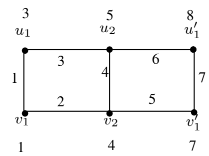
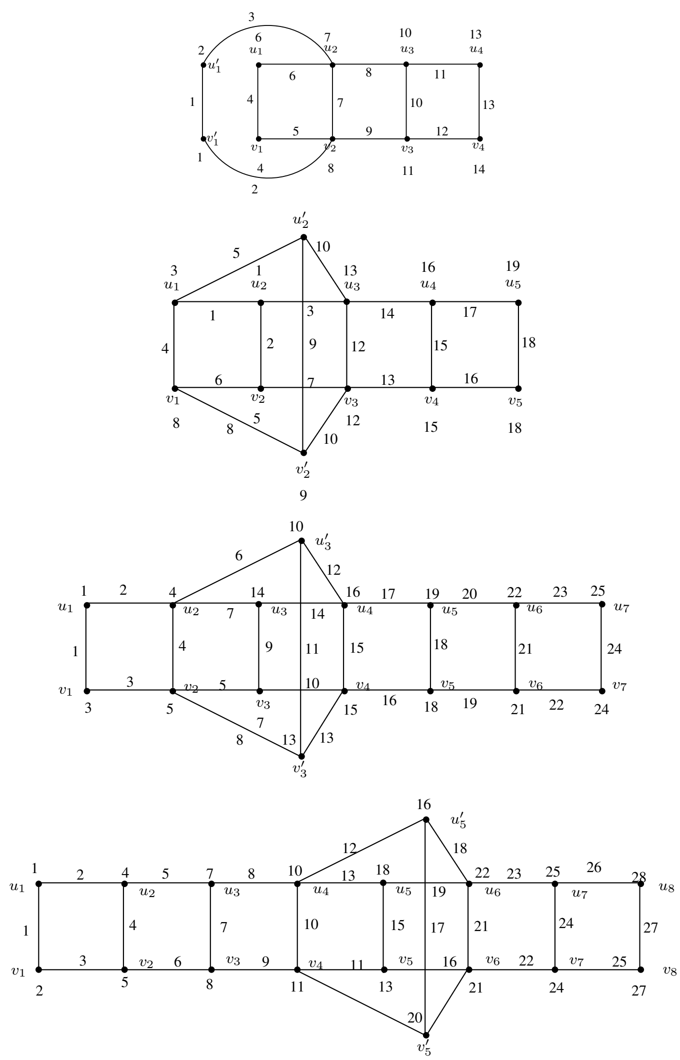


Figure 8

The F-geometric mean labeling of G in the above cases are shown in Figure 9.



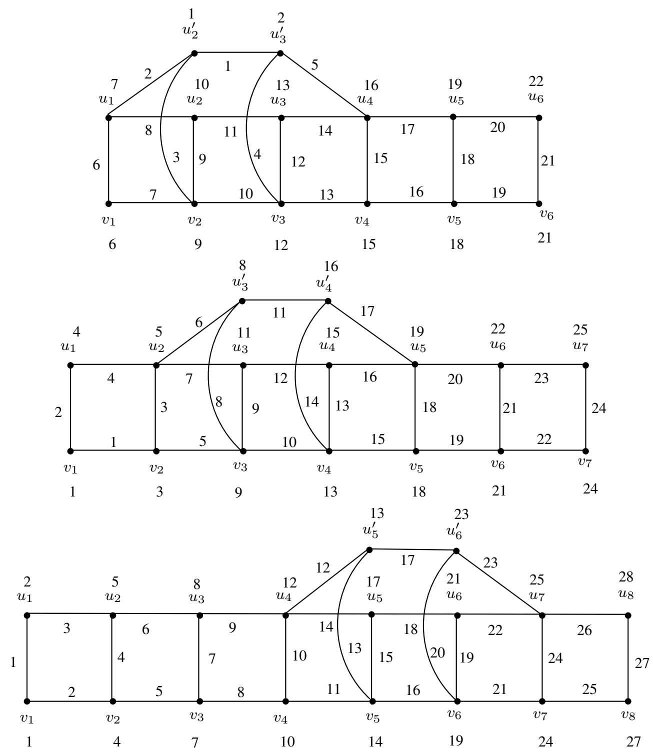


Figure 9

Acknowledgments

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A. Durai Baskar
Department of Mathematics
Mepco Schlenk Engineering College
Mepco Engineering College (PO)
Sivakasi - 626 005
Tamilnadu
India.
E-mail: a.duraibaskr@gmail.com

S. Arockiaraj
Department of Mathematics
Government Arts & Science College
Sivakasi - 626 124
Tamilnadu
India.
E-mail: psarockiaraj@gmail.com