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On presentations of Hochschild extension algebras for a class of self-injective Nakayama algebras

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Abstract. For a bound quiver algebra satisfying the condition that the every oriented cycles in the quiver are vanished in the algebra, Fernández and Platzeck determined the bound quiver algebra which is isomorphic to the trivial extension algebra. In this paper, we consider a Hochschild extension algebra which is a generalization of a trivial extension algebra. The purpose of this paper is to determine the bound quiver algebras which are isomorphic to Hochschild extension algebras of some finite dimensional self-injective Nakayama algebras.

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§1. Introduction

This paper is a sequel to [7] and the notation in this paper is almost same as in [7]. Let K be an algebraically closed field and A a finite dimensional, basic and connected algebra over K. Then A is isomorphic to $K\Delta/I$, where Δ is a finite connected quiver and I is an admissible ideal of $K\Delta$. We denote by $D(A)$ the standard duality module $\mathrm{Hom}_K(A, K)$. We recall the definitions of a Hochschild extension and a Hochschild extension algebra from [6] and [13]. By a Hochschild extension over A by $D(A)$, we mean an exact sequence

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$$

such that T is a K-algebra, ρ is an algebra epimorphism and κ is a T-bimodule monomorphism. The algebra T is called a Hochschild extension algebra. In the above, $D(A)$ is an A-bimodule, so $D(A)$ is regarded as a T-bimodule by means of ρ. It is well known that the ring structure of T is determined by a 2-cocycle $\alpha : A \times A \to D(A)$. Moreover, Hochschild [6] proved that the set of equivalent classes of Hochschild extensions over A by $D(A)$ is in one-to-one correspondence with the second Hochschild cohomology group $H^2(A, D(A))$. We denote by $T_\alpha(A)$ the Hochschild extension algebra corresponding to a 2-cocycle α. Then, $T_0(A)$ is just the trivial extension algebra $A \ltimes D(A)$. Hochschild extension algebras and trivial extension algebras play an important role in the representation theory of self-injective algebras (e.g. [8, 9, 12, 13]).

In [5], Fernández and Platzeck determined the ordinary quiver for any trivial extension algebras. Moreover, they described the relations for the trivial extension algebra $T_0(A)$ under the assumption that any oriented cycle in the ordinary quiver of A is zero in A. We are interested in the ordinary quiver $\Delta_{T_\alpha(A)}$ for a general Hochschild extension algebra $T_\alpha(A)$ and the ideal $I_{T_\alpha(A)}$ of relations for $T_\alpha(A)$, that is, the quiver and the ideal such that $T_\alpha(A) \cong K\Delta_{T_\alpha(A)}/I_{T_\alpha(A)}$ holds. In [7], we determined the ordinary quiver for the Hochschild extension algebras of self-injective Nakayama algebras. In that paper, we referred to the Sköldberg's results in [10], that is, for a truncated quiver algebra A, the Hochschild homology group $HH_p(A)$ is $\mathbb{N}$-graded by the length of cycles in the quiver, and the degree q part $HH_{p,q}(A)$ is explicitly computed. Consequently, in [7], for a 2-cocycle $\alpha : A \times A \to D(A)$ obtained through the isomorphism $\Theta : \bigoplus_q D(HH_{2,q}(A)) \xrightarrow{\sim} H^2(A, D(A))$, we showed that the ordinary quiver $\Delta_{T_\alpha(A)}$ is either Δ or $\Delta_{T_0(A)}$ (see Theorem 2.1 and Corollary 2.2). Moreover, we also described the relations for $T_\alpha(A)$ such that $\Delta_{T_\alpha(A)} = \Delta$ holds (see Corollary 2.3). However, it seems that there is little more information about the relations for general Hochschild extension algebras.

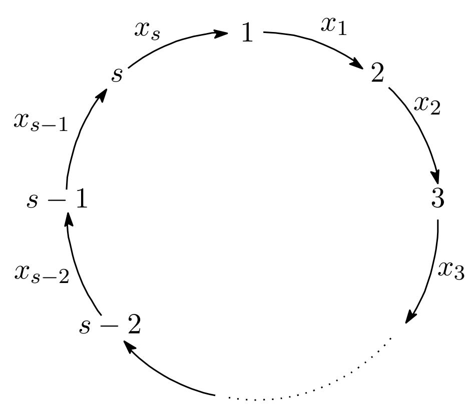
The aim of the present paper is to determine the relations for Hochschild extension algebras of a self-injective Nakayama algebra A under the assumption that $HH_2(A) = HH_{2,q}(A)$ for some $q \in \mathbb{N}$. For the Hochschild extension algebra $T_\alpha(A)$ such that the ordinary quiver coincides with $\Delta_{T_0(A)}$, our main theorems in Section 3 are generalizations of the results of [5] for self-injective Nakayama algebras.

This paper is organized as follows: In Section 2, we recall some facts from [7] about the ordinary quiver for Hochschild extension algebras of self-injective Nakayama algebras. Moreover, we mention the aim of this paper. In Section 3, we will give the main theorems. That is, we determine the relations for Hochschild extension algebras of self-injective Nakayama algebras under the assumption that $HH_2(A) = HH_{2,q}(A)$ for some $q \in \mathbb{N}$. Note that this assumption is weaker than the one of Fernández and Platzeck [5]. In order to prove the main theorems, we refer to the way used in the proof of [5, Theorem 3.9]. Finally, in Section 4, we exhibit four examples to clarify the main theorems.

For general facts on quivers and bound quiver algebras, we refer to [2] and [11]. Also for Hochschild homology and cohomology for bound quiver algebras, we refer to [1], [4], [7] and [10]. Moreover, the notation including Δ_0, Δ_1, Δ_+ and the isomorphism $\Theta : \bigoplus_q D(HH_{2,q}(A)) \xrightarrow{\sim} H^2(A, D(A))$ are same as in [7].

§2. Preliminaries and aims

In this section, we will recall some results from [7] and [10]. From now on, we deal with self-injective Nakayama algebras. We fix an integer $s \geq 1$. Let Δ be the following cyclic quiver with s vertices and s arrows:



Suppose that $n \geq 2$ and $A = K\Delta/R_\Delta^n$, which is called a truncated cycle algebra in [3], where R_Δ^n is the two-sided ideal of $K\Delta$ generated by the paths of length n. We regard the subscripts i of e_i and x_i as modulo $s\,(1 \leq i \leq s)$. By [10], the 2nd Hochschild homology $HH_2(A) = \bigoplus_{q=0}^{\infty} HH_{2,q}(A)$ is given by

(2.1) $HH_{2,q}(A)$

$$= \begin{cases} K & \text{if } s|q \text{ and } n+1 \leq q \leq 2n-1, \\ K^{s-1} \oplus \mathrm{Ker}(\cdot \frac{n}{s} : K \to K) & \text{if } s|q \text{ and } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

In [7], we had the following isomorphisms:

$$\Theta : \bigoplus_{q \geq 0} D(HH_{2,q}(A)) \xrightarrow{\sim} D(\bigoplus_{q \geq 0} HH_{2,q}(A)) = D(HH_2(A)) \xrightarrow{\sim} H^2(A, D(A)).$$

Moreover, we had the following theorem and the corollary about the ordinary quiver of Hochschild extension algebras.

Theorem 2.1 ([7, Theorem 4.3]). *Suppose that $n \geq 2$, $A = K\Delta/R_\Delta^n$ and $n \leq q \leq 2n-1$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle such that the cohomology class $[\alpha]$ of α belongs to $\Theta(D(HH_{2,q}(A)))$, and let $T_\alpha(A)$ be the Hochschild extension algebra of A defined by α. Then the ordinary quiver $\Delta_{T_\alpha(A)}$ is given by*

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \text{if } n \leq q \leq 2n-2, \\ \Delta & \text{if } q = 2n-1. \end{cases}$$

Corollary 2.2 ([7, Corollary 4.4]). *Suppose that $n \geq 2$ and $A = K\Delta/R_\Delta^n$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle and $[\alpha] = \sum_{q=n}^{2n-1}[\beta_q]$, where $\beta_q : A \times A \to D(A)$ is a 2-cocycle such that the cohomology class $[\beta_q]$ of β_q belongs to $\Theta(D(HH_{2,q}(A)))$. Then the following equation holds*:

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \text{if } [\beta_{2n-1}] = 0, \\ \Delta & \text{if } [\beta_{2n-1}] \neq 0. \end{cases}$$

We note that, by [5, Proposition 2.2], the ordinary quiver $\Delta_{T_0(A)}$ for the trivial extension algebra $T_0(A)$ is given by $(\Delta_{T_0(A)})_0 = \Delta_0$ and $(\Delta_{T_0(A)})_1 = \Delta_1 \cup \{y_1, y_2, \dots, y_s\}$, where y_i is an arrow from $t(p_i)$ to $s(p_i)$ corresponding to $p_i := x_{i-n+1}x_{i-n+2}\cdots x_{i-1}$ for each i $(1 \leq i \leq s)$.

Corollary 2.3 ([7, Corollary 4.5]). *Suppose that $n \geq 2$ and $A = K\Delta/R_\Delta^n$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle. If $\Delta_{T_\alpha(A)} = \Delta$, then $T_\alpha(A)$ is isomorphic to $K\Delta/R_\Delta^{2n}$ and $T_\alpha(A)$ is symmetric.*

From now on, we will assume that n satisfies the inequalities

$$(t-1)s < n \leq ts \leq 2n-1 < (t+1)s \tag{2.2}$$

for some $t \geq 1$. Noting that, by (2.1),

$$HH_2(A) = \bigoplus_{q=0}^{\infty} HH_{2,q}(A) = \bigoplus_{\substack{n \leq q \leq 2n-1 \\ \text{with } s|q}} HH_{2,q}(A) = \bigoplus_{\substack{n \leq ms \leq 2n-1 \\ (m \geq 1)}} HH_{2,ms}(A)$$

holds, we see that n satisfies (2.2) if and only if $HH_2(A)$ has the only degree ts part $HH_{2,ts}(A)$. Since (2.2) yields

$$2(t-1)s - 1 < 2n-1 < (t+1)s,$$

$s(3-t) \geq 1$ holds. So $t = 1, 2$. Then, we have

$$\begin{aligned} HH_2(A) &= \bigoplus_{\substack{n \leq ms \leq 2n-1 \\ (m \geq 1)}} HH_{2,ms}(A) \\ &= \begin{cases} HH_{2,s}(A) & \text{if } t = 1, \text{ i.e. if } n \leq s \leq 2n-1, \\ HH_{2,2s}(A) & \text{if } t = 2, \text{ i.e. if } (n/2 \leq)(2n-1)/3 < s \leq n - 1/2. \end{cases} \end{aligned}$$

Our aim is to describe the ideal $I_{T_\alpha(A)}$ of relations for $T_\alpha(A)$ defined by α such that the cohomology class $[\alpha]$ belongs to $\Theta(D(HH_2(A)))$. We remark that if $s = 2n-1$ or $2s = 2n-1$, we already determined $I_{T_\alpha(A)}$ by Theorem 2.1 and Corollary 2.3. So, in the next section, we will investigate the relations for every n with $n \leq s \leq 2n-2$ or $(2n-1)/3 < s < n - 1/2$. In these cases, we note that $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$ holds by Theorem 2.1.

§3. Main theorems

Let Δ be the cyclic quiver same as in Section 2 and $A = K\Delta/R^n_\Delta$ an algebra such that any oriented cycle in Δ is zero in A for n with $n \leq s \leq 2n-2$ or $(2n-1)/3 < s \leq n - 1/2$. From now on, $\Delta_{T_\alpha(A)} (= \Delta_{T_0(A)})$ is denoted by Δ_T. In this section, we will determine the ideal $I_{T_\alpha(A)}$ of relations for the Hochschild extension algebra $T_\alpha(A)$ of A by a 2-cocycle α, that is, the ideal $I_{T_\alpha(A)}$ such that $K\Delta_T/I_{T_\alpha(A)}$ is isomorphic to $T_\alpha(A)$. We will investigate the relations dividing into the following two cases: Case 1: $n+1 \leq s \leq 2n-2$ or $(2n-1)/3 < s < n - 1/2$, and Case 2: $s = n$. In order to prove the main theorems, we refer to the way used in the proof of [5, Theorem 3.9].

3.1. Case 1: $n+1 \leq s \leq 2n-2$ or $(2n-1)/3 < s < n - 1/2$

Let

$$q := \begin{cases} s & \text{if } n+1 \leq s \leq 2n-2, \\ 2s & \text{if } (2n-1)/3 < s < n - 1/2. \end{cases}$$

In this case, note that $\dim_K HH_{2,q}(A) = 1$ by (2.1). So we have the following proposition.

Proposition 3.1. *We define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, 2, \ldots, s)$ by*

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} (\overline{x_{i+m} \cdots x_{i+q-1}})^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A,\ n \leq m \leq q \\ & \quad \text{and } ab = x_i \cdots x_{i+m-1}, \\ 0 & \text{otherwise}, \end{cases}$$

where a, b are paths in Δ, and m denotes the length of ab. Then $\sum_{i=1}^s \alpha_i$ is a 2-cocycle, and the cohomology class $[\sum_{i=1}^s \alpha_i]$ is a K-basis of $H^2(A, D(A))$.

Proof. In the proof of [7, Theorem 4.3], it is proved that $[\sum_{i=1}^s \alpha_i]$ is a K-basis of $H^2(A, D(A))$. □

It is easy to see that any Hochschild extension algebra in Case 1 is isomorphic to $T_\alpha(A)$ with a 2-cocycle $\alpha = k \sum_{i=1}^s \alpha_i$ for some $k \in K$. We describe the relations for $T_\alpha(A)$ above.

Theorem 3.2. *Let* $\alpha = k\sum_{i=1}^{s}\alpha_i$ *for* $k \in K$, *where* α_i*'s are the maps in* Proposition 3.1. *Let* I' *be the ideal in* $K\Delta_T$ *generated by*

$$x_iy_{i+1} - y_ix_{i-n+1}, \quad y_iy_{i-n+1},$$
$$x_ix_{i+1}\cdots x_{i+n-1} - ky_ix_{i-n+1}x_{i-n+2}\cdots x_{i-n+(2n-q-1)}$$

for $i = 1, 2, \ldots, s$. *Then* I' *is admissible and* $I' = I_{T_\alpha(A)}$. *So* $T_\alpha(A)$ *is isomorphic to* $K\Delta_T/I'$.

Proof. Consider the homomorphism of K-algebras $\Phi : K\Delta_T \to T_\alpha(A)$ defined on the trivial paths and the arrows as follows:

$$\Phi(e_i) = (\overline{e_i}, 0), \quad \Phi(x_i) = (\overline{x_i}, 0), \quad \Phi(y_i) = (0, \overline{p_i}^*)$$

for $i = 1, \ldots, s$, where $p_i := x_{i-n+1}x_{i-n+2}\cdots x_{i-1}$. Then Φ is surjective. We define homomorphisms

$$\varphi_1 := \pi_1\Phi : K\Delta_T \to A \quad \text{and} \quad \varphi_2 := \pi_2\Phi : K\Delta_T \to D(A),$$

where π_1 and π_2 denote the projections induced by the decomposition $T_\alpha(A) = A \oplus D(A)$. It is easy to see that

$$I' \subseteq \operatorname{Ker}\Phi, \ R_\Delta^{q+1} \subseteq I', \ R_{\Delta_T}^{q+1} \subseteq I'.$$

Since Φ is surjective and $I' \subseteq \operatorname{Ker}\Phi$, we have the canonical epimorphism

$$K\Delta_T/I' \to K\Delta_T/\operatorname{Ker}\Phi (\cong T_\alpha(A)).$$

Hence, it suffices to show that

$$\dim_K \Delta_T/I' = \dim_K T_\alpha(A) (= 2\dim_K A).$$

We denote $K\Delta/R_\Delta^{q+1}$ by A'. The image of an element $a \in K\Delta$ under the canonical epimorphism $K\Delta \to A'$ is denoted by $\hat{a}$. The inclusion of A' in $K\Delta_T/\operatorname{Ker}\Phi$ factors through $K\Delta_T/I'$ by $R_\Delta^{q+1} \subseteq I'$. Thus the map $\iota : A' \to K\Delta_T/I'$ induced by the embedding of $K\Delta$ in $K\Delta_T$ is a monomorphism. We have that $e_iK\Delta_Te_j = e_iK\Delta e_j + e_iYe_j$, for each i and j in $(\Delta_T)_0$, where Y denotes the ideal generated by the elements $y_i\,(1 \le i \le s)$ in $K\Delta_T$. Let $\pi : K\Delta_T \to K\Delta_T/I'$ be the canonical epimorphism. We define the subspaces $\mathcal{P}_{ij} := \pi(e_iK\Delta e_j)$ and $\mathcal{F}_{ij} := \pi(e_iYe_j)$ in $K\Delta_T/I'$ for i, j $(1 \le i, j \le s)$. Then $\mathcal{P}_{ij} = \iota(\hat{e}_iA'\hat{e}_j) \cong \hat{e}_iA'\hat{e}_j$, so we have

$$\begin{aligned}
\sum_{i,j=1}^{s} \dim_K \mathcal{P}_{ij} &= \sum_{i,j=1}^{s} \dim_K \hat{e}_iA'\hat{e}_j \\
&= \dim_K A' \\
&= \dim_K K\Delta/R_\Delta^{q+1} \\
&= \dim_K K\Delta/R_\Delta^{n} + s(q-n+1) \\
&= \dim_K A + s(q-n+1).
\end{aligned}$$

Since we have

$$\mathcal{F}_{ij} = \begin{cases} K(e_i y_i e_{i-n+1} + I') & \text{if } j = i-n+1, \\ K(e_i y_i x_{i-n+1} \cdots x_{j-1} + I') & \text{if } j = i-n+2, \dots, i, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\sum_{i,j=1}^{s} \dim_K \mathcal{F}_{ij} = \sum_{i=1}^{s} \sum_{j=i-n+1}^{i} \dim_K \mathcal{F}_{ij} = \sum_{i=1}^{s} n = sn = \dim_K A.$$

Moreover, we have

$$\begin{aligned} &\mathcal{P}_{ij} \cap \mathcal{F}_{ij} \\ &\quad = \begin{cases} K(e_i y_i x_{i-n+1} \cdots x_{i-n+(j-i+n-1)} e_{i-n+(j-i+n)} + I') & \text{if } i+n-q \le j \le i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, it holds that

$$\begin{aligned} \sum_{i,j=1}^{s} \dim_K \mathcal{P}_{ij} \cap \mathcal{F}_{ij} &= \sum_{i=1}^{s} \sum_{j=i+n-q}^{i} \dim_K \mathcal{P}_{ij} \cap \mathcal{F}_{ij} \\ &= \sum_{i=1}^{s} (q-n+1) = s(q-n+1). \end{aligned}$$

Since $K\Delta_T/I'$ maps onto $T_\alpha(A)$, we have

$$\begin{aligned} \dim_K T_\alpha(A) &\le \dim_K K\Delta_T/I' \\ &= \sum_{i,j=1}^{s} (\dim_K \mathcal{P}_{ij} + \dim_K \mathcal{F}_{ij} - \dim_K \mathcal{P}_{ij} \cap \mathcal{F}_{ij}) \\ &= \dim_K A + s(q-n+1) + \dim_K A - s(q-n+1) \\ &= 2\dim_K A \\ &= \dim_K T_\alpha(A). \end{aligned}$$

This completes the proof. □

Remark 1. It is easy to see that if $q = s$ and $\alpha = 0$ (that is, $k = 0$), then the ideal I' of relations for $T_0(A)$ coincides with the ideal given by [5, Theorem 3.9].

3.2. Case 2: $s = n$

In this case, we note that $\dim_K HH_{2,s}(A) = s-1$ by (2.1). We have the following proposition.

Proposition 3.3. *We define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, 2, \dots, s-1)$ by*

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} \overline{e_i}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A, \text{ and } ab = x_i \cdots x_{i+s-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then α_i's are 2-cocycles, and the set of their cohomology classes is a K-basis of $H^2(A, D(A))$.

Proof. In the proof of [7, Theorem 4.3], it is proved that $\{[\alpha_i] \mid i = 1, 2, \dots, s-1\}$ is a K-basis of $H^2(A, D(A))$. □

It is easy to see that any Hochschild extension algebra in Case 2 is isomorphic to $T_\alpha(A)$, where $\alpha = \sum_{i=1}^{s-1} k_i\alpha_i$ for some $k_i \in K$. We describe the relations for $T_\alpha(A)$ above.

Theorem 3.4. *Let $\alpha = \sum_{i=1}^{s-1} k_i\alpha_i$ for $k_i \in K$, where α_i's are the 2-cocycles as in* Proposition 3.3. *Let I' be the ideal in $K\Delta_{T_\alpha(A)}$ generated by*

$$x_jy_{j+1} - y_jx_{j+1}, \quad y_jy_{j+1}, \quad x_sx_1 \cdots x_{s-1},$$
$$x_lx_{l+1} \cdots x_{l+s-1} - k_ly_lx_{l+1} \cdots x_{l+s-1}$$

for $j = 1, 2, \dots, s$ and $l = 1, 2, \dots, s-1$. Then I' is admissible and $I' = I_{T_\alpha(A)}$. So $T_\alpha(A)$ is isomorphic to $K\Delta_T/I'$.

Proof. Consider the homomorphisms $\Phi : K\Delta_T \to T_\alpha(A)$, $\varphi_1 : K\Delta_T \to A$ and $\varphi_2 : K\Delta_T \to D(A)$ defined in the same way as the proof of Theorem 3.2. Then Φ is surjective. It is easy to see that

$$I' \subseteq \operatorname{Ker}\Phi, \quad R_\Delta^{s+1} \subseteq I', \quad R_{\Delta_T}^{s+1} \subseteq I'.$$

Since Φ is surjective and $I' \subseteq \operatorname{Ker}\Phi$, we have the canonical epimorphism

$$K\Delta_T/I' \to K\Delta_T/\operatorname{Ker}\Phi(\cong T_\alpha(A)).$$

Hence, in order to prove the statement, it suffices to show that

$$\dim_K \Delta_T/I' = \dim_K T_\alpha(A)(= 2\dim_K A).$$

Let J be the ideal

$$\langle R_\Delta^{s+1},\ x_sx_1 \cdots x_{s-1},\ x_lx_{l+1} \cdots x_{l+s-1} \mid 1 \le l \le s-1 \text{ and } k_l = 0 \rangle$$

of $K\Delta$. We denote $K\Delta/J$ by A'. The image of an element $a \in K\Delta$ under the canonical epimorphism $K\Delta \to A'$ is denoted by $\hat{a}$. Since $J \subseteq I'$, the inclusion of A' to $K\Delta_T/\mathrm{Ker}\,\Phi$ factors through $K\Delta_T/I'$. Thus the map $\iota : A' \to K\Delta_T/I'$ induced by the embedding of $K\Delta$ in $K\Delta_T$ is a monomorphism. We have $K\Delta_T = K\Delta + Y$, where Y denotes the ideal generated by the elements y_i in $K\Delta_T$. Therefore, $e_iK\Delta_Te_j = e_iK\Delta e_j + e_iYe_j$ for each i and j of $(\Delta_T)_0$. Let $\pi : K\Delta_T \to K\Delta_T/I'$ be the canonical epimorphism.

We define the subspaces $\mathcal{P}_{ij} := \pi(e_iK\Delta e_j)$ and $\mathcal{F}_{ij} := \pi(e_iYe_j)$ in $K\Delta_T/I'$ for $i,\, j$ $(1 \le i,\, j \le s)$. Then $\mathcal{P}_{ij} = \iota(\hat{e}_iA'\hat{e}_j) \cong \hat{e}_iA'\hat{e}_j$, so we have

$$\begin{aligned}\sum_{i,\,j=1}^{s} \dim_K \mathcal{P}_{ij} &= \sum_{i,\,j=1}^{s} \dim_K \hat{e}_iA'\hat{e}_j \\ &= \dim_K A' \\ &= \dim_K A + s - u - 1,\end{aligned}$$

where $u := |\{l \mid 1 \le l \le s-1 \text{ and } k_l = 0\}|$. Since $\mathcal{F}_{ij} = K(y_ix_{i+1}\cdots x_{j-1}+I')$ is a 1-dimensional K-vector space for each $i,\, j$ $(1 \le \imath,\, j \le s)$, we have

$$\sum_{i,\,j=1}^{s} \dim_K \mathcal{F}_{ij} = s^2 = \dim_K A.$$

Moreover, it is easy to see that $\mathcal{P}_{ij} \cap \mathcal{F}_{ij} = 0$ if $i = j$ $(1 \le i,\, j \le s)$. For $1 \le i \le s-1$, we have

$$\mathcal{P}_{ij} \cap \mathcal{F}_{ij} = \begin{cases} K(x_ix_{i+1}\cdots x_{i+s-1} + I') & \text{if } k_i \ne 0, \\ 0 & \text{if } k_i = 0, \end{cases}$$

and $\mathcal{P}_{ij} \cap \mathcal{F}_{ij} = 0$ holds. Hence, it holds that

$$\sum_{i,\,j=1}^{s} \dim_K \mathcal{P}_{ij} \cap \mathcal{F}_{ij} = s - u - 1.$$

Since $K\Delta_T/I'$ maps onto $T_\alpha(A)$, we have

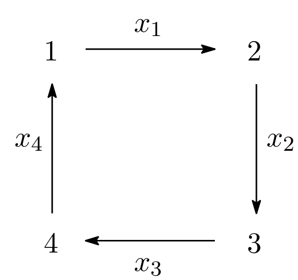
$$\begin{aligned}\dim_K T_\alpha(A) &\le \dim_K K\Delta_T/I' \\ &= \sum_{i,\,j=1}^{s} (\dim_K \mathcal{P}_{ij} + \dim_K \mathcal{F}_{ij} - \dim_K \mathcal{P}_{ij} \cap \mathcal{F}_{ij}) \\ &= (\dim_K A + s - u - 1) + \dim_K A - (s - u - 1) \\ &= 2\dim_K A \\ &= \dim_K T_\alpha(A).\end{aligned}$$

This completes the proof. $\square$

Remark 2. It is easy to see that if $\alpha = 0$, that is, $k_i = 0$ for every i, then the ideal I' above coincides with the ideal given by [5, Theorem 3.9].

§4. Examples

In this section, we consider the case $s = 4$ as examples of the main theorems. That is, let Δ be the following quiver:

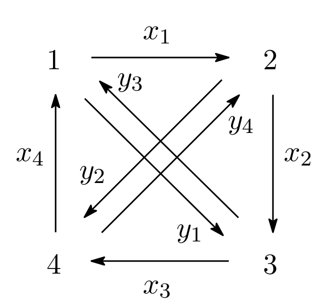


and denote A the bound quiver algebra $K\Delta/R^n_\Delta$ for $n \geq 2$.

Example 1. First, we consider the case $n = 3$, that is, an example of Case 1 ($n + 1 \leq s \leq 2n - 2$). Then $\sum_{i=1}^4 \alpha_i$ is a 2-cocycle, where $\alpha_i : A \times A \to D(A)$ is given by

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} \overline{x_{i+3}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_i x_{i+1} x_{i+2}, \\ \overline{e_{i+4}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_i x_{i+1} x_{i+2} x_{i+3}, \\ 0 & \text{otherwise}, \end{cases}$$

for $i = 1, 2, 3, 4$. For a 2-cocycle $\alpha := k \sum_{i=1}^4 \alpha_i$ $(k \in K)$, by Theorem 2.1, we have $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$. By [5, Proposition 2.2], $\Delta_{T_0(A)}$ is the following quiver:



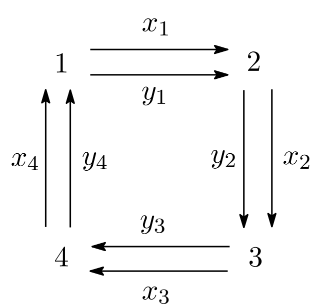
In this case, by Theorem 3.2, we find that $T_\alpha(A)$ is given by the quiver $\Delta_{T_0(A)}$ and the ideal $I_{T_\alpha(A)}$ of relations generated by

$$x_1y_2 - y_1x_3, \quad x_2y_3 - y_2x_4, \quad x_3y_4 - y_3x_1, \quad x_4y_1 - y_4x_2,$$
$$y_1y_3, \quad y_3y_1, \quad y_2y_4, \quad y_4y_2,$$
$$x_1x_2x_3 - ky_1x_3, \quad x_2x_3x_4 - ky_2x_4, \quad x_3x_4x_1 - ky_3x_1, \quad x_4x_1x_2 - ky_4x_2.$$

Example 2. Second, we consider the case $n = 4$, that is, an example of Case 2. Let $\alpha_i : A \times A \to D(A)$ be the 2-cocycle given by

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} \overline{e_i}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_ix_{i+1}x_{i+2}x_{i+3}, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3$. By Theorem 2.1, for a 2-cocycle $\alpha := \sum_{i=1}^{3} k_i\alpha_i \, (k_i \in K)$, we have $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$, and by [5, Proposition 2.2], $\Delta_{T_0(A)}$ is the following quiver:



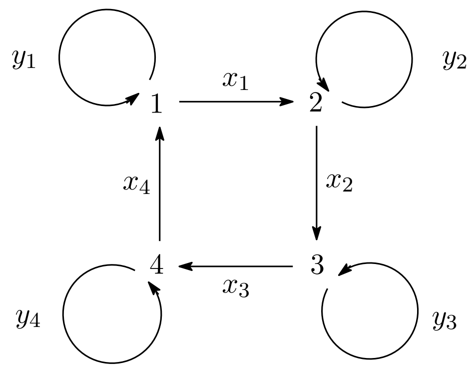
In this case, by Theorem 3.4, we find that $T_\alpha(A)$ is given by $\Delta_{T_0(A)}$ and the ideal $I_{T_\alpha(A)}$ of relations generated by

$$x_1y_2 - y_1x_2, \quad x_2y_3 - y_2x_3, \quad x_3y_4 - y_3x_4, \quad x_4y_1 - y_4x_1,$$
$$y_1y_2, \quad y_2y_3, \quad y_3y_4, \quad y_4y_1, \quad x_4x_1x_2x_3,$$
$$x_1x_2x_3x_4 - k_1y_1x_2x_3x_4, \quad x_2x_3x_4x_1 - k_2y_2x_3x_4x_1, \quad x_3x_4x_1x_2 - k_3y_3x_4x_1x_2.$$

Example 3. Third, we consider the case $n = 5$, that is, an example of Case 1 ($(2n-1)/3 < s < n - 1/2$). Then $\sum_{i=1}^{4} \alpha_i$ is a 2-cocycle, where $\alpha_i : A \times A \to D(A)$ is given by

$$\alpha_i(\overline{a}, \overline{b})$$
$$= \begin{cases} \overline{x_{i+5}x_{i+6}x_{i+7}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}, \\ \overline{x_{i+6}x_{i+7}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}, \\ \overline{x_{i+7}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}x_{i+6}, \\ \overline{e_{i+8}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}x_{i+6}x_{i+7}, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3, 4$. For a 2-cocycle $\alpha := k\sum_{i=1}^{4}\alpha_i$ $(k \in K)$, by Theorem 2.1, we have $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$. By [5, Proposition 2.2], $\Delta_{T_0(A)}$ is the following quiver:



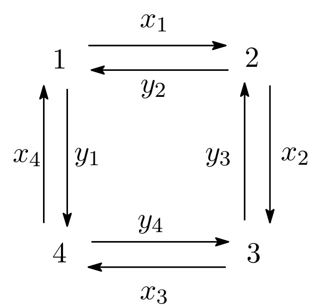
In this case, by Theorem 3.2, we find that $T_\alpha(A)$ is given by the quiver $\Delta_{T_0(A)}$ and the ideal $I_{T_\alpha(A)}$ of relations generated by

$$\begin{aligned}
&x_1y_2 - y_1x_1, \quad x_2y_3 - y_2x_2, \quad x_3y_4 - y_3x_3, \quad x_4y_1 - y_4x_4,\\
&y_1^2, \quad y_2^2, \quad y_3^2, \quad y_4^2,\\
&x_1x_2x_3x_4x_1 - ky_1x_1, \quad x_2x_3x_4x_1x_2 - ky_2x_2,\\
&x_3x_4x_1x_2x_3 - ky_3x_3, \quad x_4x_1x_2x_3x_4 - ky_4x_4.
\end{aligned}$$

Example 4. Finally, we consider the case $n = 6$, that is, an example of Case 1 $((2n-1)/3 < s < n - 1/2)$. Then $\sum_{i=1}^{4}\alpha_i$ is a 2-cocycle, where $\alpha_i : A \times A \to D(A)$ is given by

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} \overline{x_{i+6}x_{i+7}}^* & \text{if } \overline{a}, \overline{b} \neq 0\ ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5},\\ \overline{x_{i+7}}^* & \text{if } \overline{a}, \overline{b} \neq 0\ ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}x_{i+6},\\ \overline{e_{i+8}}^* & \text{if } \overline{a}, \overline{b} \neq 0\ ab = x_ix_{i+1}x_{i+2}x_{i+3}x_{i+4}x_{i+5}x_{i+6}x_{i+7},\\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, 3, 4$. For a 2-cocycle $\alpha := k\sum_{i=1}^{4}\alpha_i$ $(k \in K)$, by Theorem 2.1, we have $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$. By [5, Proposition 2.2], $\Delta_{T_0(A)}$ is the following quiver:



In this case, by Theorem 3.2, we find that $T_\alpha(A)$ is given by the quiver $\Delta_{T_0(A)}$ and the ideal $I_{T_\alpha(A)}$ of relations generated by

$$\begin{aligned}
&x_1y_2 - y_1x_4, \quad x_2y_3 - y_2x_1, \quad x_3y_4 - y_3x_2, \quad x_4y_1 - y_4x_3,\\
&y_1y_4, \quad y_2y_1, \quad y_3y_2, \quad y_4y_3,\\
&x_1x_2x_3x_4x_1x_2 - ky_1x_4x_1x_2, \quad x_2x_3x_4x_1x_2x_3 - ky_2x_1x_2x_3,\\
&x_3x_4x_1x_2x_3x_4 - ky_3x_2x_3x_4, \quad x_4x_1x_2x_3x_4x_1 - ky_4x_3x_4x_1.
\end{aligned}$$

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References

[1] G. Ames, L. Cagliero, P. Tirao, Comparison morphisms and the Hochschild cohomology ring of truncated quiver algebras, J. Algebra 322 (5) (2009) 1466–1497.

[2] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras I*, London Math. Soc. Student Texts, vol. 65. Cambridge: Cambridge Univ. Press, 2006.

[3] M. Bardzell, A. Locateli, E. Marcos, On the Hochschild cohomology of truncated cycle algebras, Comm. Algebra 28(3) (2000) 1615–1639.

[4] C. Cibils, Cohomology of incidence algebras and simplicial complexes, J. Pure Appl. Algebra 56 (1989) 221–232.

[5] E. Fernández, M. Platzeck, Presentations of trivial extensions of finite dimensional algebras and a theorem of Sheila Brenner, J. Algebra 249 (2002) 326–344.

[6] G. Hochschild, On the cohomology groups of an associative algebra, Ann. Math. (2) 46 (1945) 58–67.

[7] H. Koie, T. Itagaki, K. Sanada, The ordinary quivers of Hochschild extension algebras for self-injective Nakayama algebras, arXiv:1707.01618.

[8] Y. Ohnuki, Stable equivalence induced by a socle equivalence, Osaka J. 39(1) (2002) 259–266.

[9] Y. Ohnuki, K. Takeda, K. Yamagata, Symmetric Hochschild extension algebras, Colloq. Math. 80 (1999) 155–174.

[10] E. Sköldberg, The Hochschild homology of truncated and quadratic monomial algebra, J. Lond. Math. Soc. (2) 59 (1999) 76–86.

[11] A. Skowroński, K. Yamagata, *Frobenius Algebras I: Basic Representation Theory.* EMS Textbooks Math. Zürich: Eur. Math. Soc. 2011.

[12] K. Yamagata, Representations of non-splittable extension algebras, J. Algebra 115 (1988) 32–45.

[13] K. Yamagata, Frobenius Algebras, *Handbook of Algebra* 1: 841–887, Elsevier/ North-Holland, Amsterdam, 1996.

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