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On Brauer indecomposability of Scott modules

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Abstract. Let k be an algebraically closed field of prime characteristic p, G a finite group and P a p-subgroup of G. We investigate the relationship between the fusion system $\mathcal{F}_P(G)$ and the Brauer indecomposability of the Scott kG-module. We give a result which shows that there exists some relationship between G and its local subgroups in terms of the Brauer indecomposability of Scott modules.

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§1. Introduction

Let p be a prime number and k an algebraically closed field of characteristic p. For a p-subgroup Q of a finite group G and a kG-module M, the Brauer quotient $M(Q)$ of M with respect to Q is naturally a $kN_G(Q)$-module. A kG-module M is said to be *Brauer indecomposable* if $M(Q)$ is indecomposable or zero as a $kC_G(Q)$-module for any p-subgroup Q of G ([6, p. 90]). Brauer indecomposability of p-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2, 6.3]).

For subgroups Q, R of G, we denote by $\mathrm{Hom}_G(Q, R)$ the set of all group homomorphisms from Q to R which are induced by conjugation in G. For a p-subgroup P of G, the fusion system $\mathcal{F}_P(G)$ of G over P is the category whose objects are the subgroups of P and whose morphism set from Q to R is $\mathrm{Hom}_G(Q, R)$. We refer the reader to [1] for background involving fusion systems.

There is a connection between Brauer indecomposability of p-permutation kG-modules and fusion systems, as shown in [6]. The main theorem in [6] says that, for indecomposable p-permutation kG-module M with vertex P, the

Brauer indecomposability of M implies that $\mathcal{F}_P(G)$ is saturated ([6, Theorem 1.1]).

Moreover, in the case that P is abelian and M is the Scott kG-module $S(G,P)$, the converse of the above theorem holds, that is, if $\mathcal{F}_P(G)$ is saturated, then M is Brauer indecomposable ([6, Theorem 1.2]). In general, the converse does not hold for non-abelian P, as demonstrated in section 3.

However, there are some cases in which the Scott kG-module $S(G,P)$ is Brauer indecomposable for non-abelian P (see [5, 7]). Moreover, it was shown that there are some relationships between Brauer indecomposability of Scott modules and fusion systems ([3, 5]). In particular, we prove the following theorem in [3].

Theorem 1.1 ([3, Theorem 1.3])**.** *Let G be a finite group and P a p-subgroup of G. Suppose that $M = S(G,P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.*

(i) M is Brauer indecomposable.

(ii) $\mathrm{Res}^{N_G(Q)}_{QC_G(Q)} S(N_G(Q), N_P(Q))$ is indecomposable for each fully normalized subgroup Q of P.

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The above theorem gives a criterion to determine whether the Scott module $S(G,P)$ is Brauer indecomposable.

We investigate the possibility of providing applications of the above theorem. In this paper, we will prove the following result.

Theorem 1.2. *Let G be a finite group and P a p-subgroup of G. Suppose that $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated fusion system. Consider the following two conditions:*

(i) $S(N_G(Q), N_P(Q))$ is Brauer indecomposable for each non-trivial fully $\mathcal{F}$-normalized subgroup $Q \leq P$.

(ii) $S(G,P)$ is Brauer indecomposable.

Then (i) implies (ii), and the converse holds if $\mathcal{F} = \mathcal{F}_P(N_G(P))$.

The above theorem shows that there exists some relationship between G and its local subgroups in terms of the Brauer indecomposability of Scott modules, and will be a useful tool for the study of the Brauer indecomposability of Scott modules.

In this paper, we write ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$ for $g \in G$ and a subgroup $H \leq G$. We denote by $K \cap_G H$ the set $\{{}^gK \cap H | g \in G\}$ for subgroups H, K of G.

§2. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2.

For a saturated fusion system $\mathcal{F}$ over p-group P and a subgroup Q of P, the normalizer fusion system $N_{\mathcal{F}}(Q)$ of Q is defined and is a fusion system over $N_P(Q)$ (see [1, II, §2]). We note that if $\mathcal{F} = \mathcal{F}_P(G)$, then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_P(Q)}(N_G(Q))$.

Proof of Theorem 1.2. Suppose that (i) holds. Let Q be a non-trivial fully $\mathcal{F}$-normalized subgroup of P. Then $S(N_G(Q), N_P(Q))(Q)$ is indecomposable as a $kC_G(Q)$-module, and we have that

$$S(N_G(Q), N_P(Q)) \cong S(N_G(Q), N_P(Q))(Q).$$

Therefore, $S(G, P)$ is Brauer indecomposable by Theorem 1.1 (ii) $\Rightarrow$ (i).

Next, suppose that (ii) and $\mathcal{F} = \mathcal{F}_P(N_G(P))$ hold. Then any subgroup Q of P is fully $\mathcal{F}$-normalized. Let Q be any subgroup of P. Then $\mathcal{F}_{N_P(Q)}(N_G(Q)) = N_{\mathcal{F}}(Q)$ is saturated by [1, II, Theorem 2.1]. Let R be a subgroup of $N_P(Q)$. It is sufficient to show that $S(N_{N_G(Q)}(R), N_{N_P(Q)}(R))$ is indecomposable as a $kC_{N_G(Q)}(R)$-module by Theorem 1.1 (ii) $\Rightarrow$ (i).

Since QR is fully $\mathcal{F}$-normalized, $S(N_G(QR), N_P(QR))$ is indecomposable as a $kC_G(QR)$-module by Theorem 1.1 (i) $\Rightarrow$ (ii), and hence is also indecomposable as a $kC_{N_G(Q)}(R)$-module. Therefore, it is sufficient to show that

$$\mathrm{Res}^{N_G(QR)}_{N_{N_G(Q)}(R)} S(N_G(QR), N_P(QR)) \cong S(N_{N_G(Q)}(R), N_{N_P(Q)}(R)),$$

and if we show that $N_{N_P(Q)}(R)$ is a maximal element of $N_P(QR) \cap_{N_G(QR)} N_{N_G(Q)}(R)$, then the isomorphism holds by [4, Theorem 1.7] and the indecomposability of $\mathrm{Res}^{N_G(QR)}_{N_{N_G(Q)}(R)} S(N_G(QR), N_P(QR))$.

Let g be an element of $N_G(QR)$. Then we have $(QR)^g = QR \leq P$ and hence there is $h \in N_G(P)$ such that $c := gh^{-1} \in C_G(QR) \subseteq N_G(Q) \cap N_G(R)$ since $\mathcal{F} = \mathcal{F}_P(N_G(P))$. Then $h = c^{-1}g \in C_G(QR)N_G(QR) = N_G(QR)$ and so $h \in N_G(P) \cap N_G(QR)$. We have that

$$\begin{aligned} {}^{g}N_P(QR) \cap N_{N_G(Q)}(R) &= {}^{ch}N_P(QR) \cap N_{N_G(Q)}(R) \\ &= {}^{c}N_P(QR) \cap N_{N_G(Q)}(R) \\ &= {}^{c}(N_P(QR) \cap N_{N_G(Q)}(R)) \\ &= {}^{c}N_{N_P(Q)}(R) \end{aligned}$$

Hence the order of any subgroup in $N_P(QR) \cap_{N_G(QR)} N_{N_G(Q)}(R)$ is equal to $|N_{N_P(Q)}(R)|$ and $N_{N_P(Q)}(R)$ is a maximal element in $N_P(QR) \cap_{N_G(QR)} N_{N_G(Q)}(R)$, as desired. □

§3. Examples

As mentioned in the introduction, this section is devoted to examples showing that the converse of Theorem 1.1 in [6] does not hold in general. These examples are due to T. Okuyama, who was inspired by private discussions with S. Koshitani. Such examples are notable, but are not known widely, so we include them in this paper.

3.1. Case 1: p is an odd prime

Consider the group $G = M(p) \times D_{2p}$ where

$$M(p) \coloneqq \langle a, y, z \mid a^p = y^p = z^p = 1, [a, z] = [y, z] = 1, [a, y] = z \rangle$$

is the extra-special p-group of order p^3 with exponent p and

$$D_{2p} \coloneqq \langle t, b \mid t^2 = b^p = 1, b^t = b^{-1} \rangle$$

is the dihedral group of order $2p$. We can view $M(p)$ and D_{2p} as subgroups of G by abuse of notation. We set $x \coloneqq ab$, $P_0 \coloneqq M(p) \times \langle b \rangle$, $Q \coloneqq \langle y, z \rangle$, and $P \coloneqq Q \rtimes \langle x \rangle$. Then P_0 is a normal Sylow p-subgroup of G with index 2. Moreover, we have $G = P \ltimes D_{2p}$, $N_G(Q) = G$, and $C_G(Q) = Q \times D_{2p}$.

With the above notation, the following hold.

(1) The fusion system $\mathcal{F}_P(G)$ is saturated. In fact, for two subsets $S, T \subseteq P$ and for element $ug \in G$ ($u \in P$, $g \in D_{2p}$), if ${}^{ug}S \subseteq T$, then $gsg^{-1} \in T^u \subseteq P$ for all $s \in S$. Hence $s^{-1}gsg^{-1} \in P \cap D_{2p} = 1$. Therefore, $g \in C_{D_{2p}}(S)$, and so $\mathcal{F}_P(G)$ is equals to $\mathcal{F}_P(P)$, which is saturated.

(2) $S(G, P)(Q)$ is not indecomposable as a $kC_G(Q)$-module. Indeed, $\mathrm{Ind}_P^G k_P$ is indecomposable since $I_G(\mathrm{Ind}_P^{P_0} k_p) = P_0$, where $I_G(\mathrm{Ind}_P^{P_0} k_p)$ is the inertial subgroup of $\mathrm{Ind}_P^{P_0} k_p$, and so we have $S(G, P) = \mathrm{Ind}_P^G k_P$. Hence, by Mackey decomposition theorem,

$$\begin{aligned}
\mathrm{Res}^{N_G(Q)}_{C_G(Q)}(S(G, P)(Q)) &\cong \mathrm{Res}^G_{C_G(Q)} \mathrm{Ind}_P^G k_P \\
&\cong \bigoplus_{t \in C_G(Q) \backslash G / P} \mathrm{Ind}^{C_G(Q)}_{C_G(Q) \cap {}^tP} \mathrm{Res}^{{}^tP}_{C_G(Q) \cap {}^tP} {}^t k_P \\
&\cong \mathrm{Ind}^{C_G(Q)}_{C_P(Q)} k_{C_P(Q)} \\
&= \mathrm{Ind}^{C_G(Q)}_Q k_Q \\
&\cong kD_{2p}.
\end{aligned}$$

Hence $S(G, P)(Q)$ is isomorphic to kD_{2p} as a $kC_G(Q)$-module, and is not indecomposable.

Therefore, $\mathcal{F}_P(G)$ is saturated, and $S(G, P)$ is not Brauer indecomposable.

3.2. Case 2: $p = 2$

Consider the group $G = D_8 \times A_4$ where

$$D_8 \coloneqq \langle a, y, z \mid a^2 = y^2 = z^2 = 1, [a, z] = [y, z] = 1, [a, y] = z\rangle$$

is the dihedral group of order 2^3 and

$$A_4 \coloneqq \langle t, b, c \mid t^3 = b^2 = c^2 = 1, [b, c] = 1, b^t = c, c^t = bc\rangle$$

is the alternating group of degree 4. We can view D_8 and A_4 as subgroups of G by abuse of notation. We set $x \coloneqq ab$, $P_0 \coloneqq D_8 \times \langle b, c\rangle$, $Q \coloneqq \langle y, z\rangle$, and $P \coloneqq Q \rtimes \langle x\rangle$. Then P_0 is a normal Sylow p-subgroup of G with index 3.

Then a similar argument shows that $\mathcal{F}_P(G)$ is saturated, and that $S(G, P)$ is not Brauer indecomposable.

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