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Large time asymptotics of solutions for the modified KdV equation with a fifth order dispersive term

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Abstract. We study the large time asymptotics of small amplitude solutions to the Cauchy problem for the modified KdV equation with a fifth order dispersive term.

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§1. Introduction

We study the large time asymptotics of small amplitude solutions to the Cauchy problem for the modified Korteweg-de Vries equation with a fifth order dispersive term

$$\begin{cases} \partial_t v - \frac{a}{3}\partial_x^3 v + \frac{b}{5}\partial_x^5 v = \partial_x v^3, \ t > 0, \ x \in \mathbf{R}, \\ v(0,x) = v_0(x), \ x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $a, b > 0$ or $a, b < 0$. The case of $a, b > 0$ is called the positive dispersion, whereas the case of $a, b < 0$ is called the negative dispersion in the study of the solitary waves (see [14]). In this paper we assume that $a, b > 0$ since the case of $a, b < 0$ is considered in the same way. In the case of $a, b > 0$ it is expected that the main term of the large time asymptotics of solutions to (1.1) is located in the positive half-line. We also assume that the initial data are real-valued functions. The problem of large time asymptotics of solutions to (1.1) is open for the complex-valued initial data.

Korteweg-de Vries (KdV) equation

$$\partial_t v - \frac{a}{3}\partial_x^3 v = \partial_x v^2 \tag{1.2}$$

was introduced in [17] to describe a model for unidirectional propagation of nonlinear dispersive long waves. In [13], it was shown that the system of equations for magneto-acoustic waves of small finite amplitude propagating in a cold collision-free plasma can be reduced to KdV equation. It was also found that the coefficient a depends on masses of ion and electron and has a possibility to be zero. Therefore the higher order dispersive terms should be taken into account. Indeed in [12] it was shown that, when $a = 0$, then the system of equations can be reduced to a simple non linear dispersive equation of the form

$$\partial_t v + \frac{b}{5}\partial_x^5 v = \partial_x v^2.$$

Thus, when a is close to 0, we obtain a generalized KdV equation

$$\partial_t v - \frac{a}{3}\partial_x^3 v + \frac{b}{5}\partial_x^5 v = \partial_x v^2. \tag{1.3}$$

In [14], the steady travelling wave solutions of (1.3) were investigated numerically to reveal how the solitary wave solutions of (1.2) are modified by the fifth order dispersive term. In [12], it was also pointed out that the Alfvén waves are described by the modified KdV equation

$$\partial_t v - \frac{a}{3}\partial_x^3 v = \partial_x v^3. \tag{1.4}$$

Thus we arrive to equation (1.1). Concerning the solitary wave solutions of higher order dispersive equations, there are a lot of works (see, e.g., [11], [15]). However the large time asymptotic behavior of small solutions was not studied well. On the other hand, the Cauchy problem (1.3) was considered in papers [2], [3], [20] and problem (1.1) was studied in [19]. In [2] and [19], the local existence in $\mathbf{H}^r$ with $r > 1/4$ and global existence in $\mathbf{H}^2$ were obtained for (1.3) and (1.1), respectively by using the Strichartz estimates. In [3], it was shown via the bilinear estimates that problem (1.3) has local solutions for $\mathbf{H}^r$ data with $r > -1$. This result combined with the $\mathbf{L}^2$- conservation law yields global existence of solutions to (1.3) in $\mathbf{L}^2$. In [20] the order of Sobolev space r was reduced to $r \geq -7/5$ by improving the bilinear estimates. And also the global existence in $\mathbf{H}^r$ with $r > -1/2$ was obtained by using the high-low frequency method from paper [1] developed for (1.2). As far as we know the large time asymptotics of solutions to (1.3) is still an open problem.

For the modified KdV equation (1.4) we studied the large time asymptotics of small solutions and showed that the small amplitude solutions are stable in the neighborhood of the self-similar solutions (see [9]).

In the case of the fourth order nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - \frac{1}{4}\partial_x^4 u = \lambda \left|u\right|^2 u, \ t > 0, \ x \in \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{1.5}$$

we developed the factorization technique used in [7] for the nonlinear Klein-Gordon equation to obtain the large time asymptotics of small solutions. Since the symbol of equation (1.5) $K(\xi) = -\frac{i}{2}\xi^2 - \frac{i}{4}\xi^4$ is inhomogeneous, it seems difficult to apply the operator

$$\mathcal{J}_1 = e^{tK(-i\partial_x)} x e^{-tK(-i\partial_x)} = x + it\left(\partial_x - \partial_x^3\right)$$

to (1.5) directly, so we defined the new operator

$$\mathcal{A}_1 = \frac{\overline{M_1}}{\sqrt{K''(\xi)}} \partial_\xi \frac{M_1}{\sqrt{K''(\xi)}}, \ M_1 = e^{-t(\xi K'(\xi) - K(\xi))}.$$

In the case of the nonlinear Schrödinger equation $i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda |u|^2 u$, the symbol is $-\frac{i}{2}\xi^2$, and we have

$$\begin{aligned} \mathcal{A}_2 &= e^{-\frac{it}{2}\xi^2} \partial_\xi e^{\frac{it}{2}\xi^2} = \partial_\xi + it\xi \\ &= -i\mathcal{F}(x + it\partial_x)\mathcal{F}^{-1} = -i\mathcal{F}\mathcal{J}_2\mathcal{F}^{-1}. \end{aligned}$$

Hence the operator $\mathcal{A}_1$ is a generalization of $\mathcal{A}_2$, and $\mathcal{J}_2 = e^{\frac{it}{2}\partial_x^2} x e^{-\frac{it}{2}\partial_x^2} = x + it\partial_x$ was widely used in the study of the large time behavior of solutions to the nonlinear Schrödinger equations (see [10]). Here we denote by $\mathcal{F}\phi$ or $\hat{\phi}$ the Fourier transform of the function ϕ

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x) dx,$$

then the inverse Fourier transformation $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi) d\xi.$$

Multiplying both sides of (1.5) by $e^{-itK(\xi)}$ and using the factorization technique in [8], we obtained for the gauge invariant nonlinearity

$$\sqrt{K''(\xi)} \left| \frac{\widehat{u}(\xi)}{\sqrt{K''(\xi)}} \right|^2 \frac{\widehat{u}(\xi)}{\sqrt{K''(\xi)}}.$$

Then we can see that the operator $\mathcal{A}_1$ works well for the gauge invariant nonlinearity in (1.5). In the case of equations (1.1) or (1.4) we encounter another difficulty. By the factorization technique the nonlinearity can be decomposed as a combination of the following four terms $\widehat{u}^3$, $|\widehat{u}|^2 \widehat{u}$, $\overline{\widehat{u}}^3$, $|\widehat{u}|^2 \overline{\widehat{u}}$ with some additional oscillating factors, except the term $|\widehat{u}|^2 \widehat{u}$. (Note that for the case of the complex-valued solution, we do not have the same representation for the nonlinearity). Due to these oscillation factors the operator $\mathcal{J}$ does not work on the terms $\widehat{u}^3$, $\overline{\widehat{u}}^3$ and $|\widehat{u}|^2\,\overline{\widehat{u}}$. To avoid this difficulty in [9] we applied for (1.4) the dilation operator $x\partial_x + 3t\partial_t$. However the dilation operators only work well for dispersive equations with homogeneous symbols such as (1.4). So we can not overcome this difficulty by introducing the modified dilation operator due to the inhomogeneous symbol in equation (1.3).

In our previous paper [9] we also used the property that the difference between the total mass of the solution and the self-similar solution is equal to zero. However we do not know the existence of the self-similar solution for (1.1). So we assume in this paper that the total mass of the initial data is zero, i.e. $\int v_0(x)\,dx = 0$. Then by the equation we get $\int v(t,x)\,dx = 0$ for all $t > 0$. Under this condition, taking $u(t,x) = \int_{-\infty}^{x} v(t,x)\,dx$ we can rewrite (1.1) in the potential form

$$\begin{cases} \partial_t u - \frac{a}{3}\partial_x^3 u + \frac{b}{5}\partial_x^5 u = (u_x)^3\,,\ t>0,\ x\in\mathbf{R}, \\ u(0,x) = u_0(x)\,,\ x\in\mathbf{R}. \end{cases} \tag{1.6}$$

This is our equation which we study in this paper.

To state our results precisely we introduce *Notation and Function Spaces.* We denote the Lebesgue space by $\mathbf{L}^p = \{\phi\in\mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$, where the norm $\|\phi\|_{\mathbf{L}^p} = \left(\int |\phi(x)|^p\,dx\right)^{\frac{1}{p}}$ for $1\le p<\infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbf{R}}|\phi(x)|$ for $p=\infty$. The weighted Sobolev space is

$$\mathbf{H}_p^{k,s} = \left\{\varphi\in\mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{k,s}} = \left\|\langle x\rangle^s\langle i\partial_x\rangle^k\phi\right\|_{\mathbf{L}^p} < \infty\right\},$$

$k,s\in\mathbf{R}$, $1\le p\le\infty$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{k,s} = \mathbf{H}_2^{k,s}$, $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly, if it does not cause any confusion. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from an interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants might be denoted by the same letter C. We define the free evolution group $\mathcal{U}(t) = e^{-it\Lambda(-i\partial_x)} = \mathcal{F}^{-1}E\mathcal{F}$, where the multiplication factor $E(t,\xi) = e^{-it\Lambda(\xi)}$, and $\Lambda(\xi) = \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5$ is the dispersion relation for equation (1.6).

We are now in a position to state our main result. Define the Heaviside function $\theta(x) = 1$ for $x\ge 0$ and $\theta(x) = 0$ for $x<0$.

Theorem 1.1. *Assume that the initial data $u_0\in\mathbf{H}^3\cap\mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^3\cap\mathbf{H}^{1,1}}\le\varepsilon$. Then there exists a unique global solution $e^{it\Lambda(-i\partial_x)}u\in\mathbf{C}\left([0,\infty);\mathbf{H}^3\cap\mathbf{H}^{1,1}\right)$ of the Cauchy problem (1.6) satisfying the time decay estimates*

$$\|\partial_x u(t)\|_{\mathbf{L}^\infty} + \left\|\partial_x^2 u(t)\right\|_{\mathbf{L}^\infty} \le C\varepsilon t^{-\frac{1}{2}}.$$

Moreover there exists a unique modified final state $W_+\in\mathbf{L}^\infty$ such that the *asymptotics*

$$\begin{aligned}\partial_x u(t) &= 2Re t^{-\frac{1}{2}} \left|\Lambda''\left(\frac{x}{t}\right)\right|^{-\frac{1}{2}} e^{it\left(\frac{x}{t}\Lambda'\left(\frac{x}{t}\right)-\Lambda\left(\frac{x}{t}\right)\right)} \\ \times\theta\left(\frac{x}{t}\right) & \frac{ix}{t}W_+\left(\frac{x}{t}\right)\exp\left(3i\left|\Lambda''\left(\frac{x}{t}\right)\right|^{-1}\left(\frac{x}{t}\right)^3\left|W_+\left(\frac{x}{t}\right)\right|^2\log t-\frac{\pi}{4}i\right) \\ & +O\left(\varepsilon t^{-\frac{1}{2}-\delta}\right) \qquad (1.7)\end{aligned}$$

is valid for $t\to\infty$ *uniformly with respect to* $x\in\mathbf{R}$, *where* $\delta\in\left(0,\frac{1}{6}\right)$ *is a small constant.*

We introduce the factorization formula for equation (1.6). We have

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{F}^{-1}E\phi=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{it\left(\frac{x}{t}\xi-\Lambda(\xi)\right)}\phi(\xi)\,d\xi \\ &= \mathcal{D}_t\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}}e^{it(x\xi-\Lambda(\xi))}\phi(\xi)\,d\xi,\end{aligned}$$

where $\mathcal{D}_t\phi=(it)^{-\frac{1}{2}}\phi\left(xt^{-1}\right)$. Consider the stationary point defined by the equation $\Lambda'(\xi)=x$. Since $\Lambda'(\xi)=a\xi^2+b\xi^4$ is monotonous in the domain $\xi\in\mathbf{R}_+$, there exists a stationary point $\xi=\eta(x)$, where $\eta(x)=\frac{1}{\sqrt{2b}}\sqrt{\sqrt{4bx+a^2}-a}$, such that $\Lambda'(\eta(x))=x$ for $x>0$. This fact means that the main term of the large time asymptotics of the solutions lies in the positive half-line. We extend $\eta(x)$ for all $x\in\mathbf{R}$ by

$$\eta(x)=\frac{x}{\sqrt{2b}\,|x|}\sqrt{\sqrt{4b\,|x|+a^2}-a}.$$

Define the operator $(\mathcal{B}\phi)(x)=|\Lambda''(\eta(x))|^{-\frac{1}{2}}\phi(\eta(x))$. Since $x=\frac{\eta}{|\eta|}\Lambda'(\eta)$, we get

$$\begin{aligned}\mathcal{U}(t)\mathcal{F}^{-1}\phi &= \mathcal{D}_t\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}}e^{-it(\Lambda(\xi)-x\xi)}\phi(\xi)\,d\xi \\ &= \mathcal{D}_t\sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}}e^{-it\left(\Lambda(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)\xi\right)}\phi(\xi)\,d\xi \\ &= \mathcal{D}_t\mathcal{B}\sqrt{\frac{it\,|\Lambda''(\eta)|}{2\pi}}\int_{\mathbf{R}}e^{-it\left(\Lambda(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)\xi\right)}\phi(\xi)\,d\xi,\end{aligned}$$

Since u is a real-valued function, then if we take $u=\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}$ we have

$\widehat{\varphi}(-\xi) = \overline{\widehat{\varphi}(\xi)}$. Hence

$$\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}\sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}}\int_0^\infty e^{-it\left(\Lambda(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)\xi\right)}\widehat{\varphi}(\xi)\,d\xi$$

$$+\mathcal{D}_t\mathcal{B}\overline{\sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}}\int_0^\infty e^{-it\left(\Lambda(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)\xi\right)}\widehat{\varphi}(\xi)\,d\xi}$$

$$= 2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\widehat{\varphi}(\xi)\,d\xi = 2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}\widehat{\varphi} \tag{1.8}$$

for $\eta \in \mathbf{R}$, where the multiplication factor $M(t,\eta) = e^{it(\eta\Lambda'(\eta)-\Lambda(\eta))\theta(\eta)} = e^{itH(\eta)}$, $H(\eta) = (\eta\Lambda'(\eta) - \Lambda(\eta))\theta(\eta)$, the phase function

$$S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta)\theta(\eta) - \frac{\eta}{|\eta|}\Lambda'(\eta)(\xi - \eta\theta(\eta))$$

and the operator

$$\mathcal{V}\phi = \sqrt{\frac{it|\Lambda''(\eta)|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi.$$

Also we need the representation for the inverse evolution group $\mathcal{F}\mathcal{U}(-t)$ for all $\xi \geq 0$

$$\mathcal{F}\mathcal{U}(-t)\phi = \overline{E}\mathcal{F}\phi = \sqrt{\frac{1}{2\pi}}\int_{\mathbf{R}} e^{it\Lambda(\xi)-ix\xi}\phi(x)\,dx$$

$$= \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{it(\Lambda(\xi)-x\xi)}\mathcal{D}_t^{-1}\phi(x)\,dx$$

$$= \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{it\left(\Lambda(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)\xi\right)}\left(\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi\right)(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta$$

$$= \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{itS(\xi,\eta)}\overline{M}\left(\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi\right)(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta = \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi, \tag{1.9}$$

where

$$\mathcal{D}_t^{-1}\phi = (it)^{\frac{1}{2}}\phi(xt),\ \left(\mathcal{B}^{-1}\phi\right)(\eta) = \left|\Lambda''(\eta)\right|^{\frac{1}{2}}\phi\left(\frac{\eta}{|\eta|}\Lambda'(\eta)\right),$$

and the operator

$$\mathcal{Q}\phi = \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta$$

for $\xi \geq 0$, is considered as the inverse operator to $\mathcal{V}$.

Since $\mathcal{FU}(-t)\mathcal{L} = \partial_t \mathcal{FU}(-t)$ with $\mathcal{L} = \partial_t - \frac{a}{3}\partial_x^3 + \frac{b}{5}\partial_x^5$, applying the operator $\mathcal{FU}(-t)$ to equation (1.6) we get for the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t)\,u(t)$

$$\partial_t \widehat{\varphi} = \partial_t \mathcal{FU}(-t)\,u = \mathcal{FU}(-t)\,\mathcal{L}u = \mathcal{FU}(-t)\,u_x^3.$$

Then by (1.8) and (1.9) we find the following factorization property

$$\begin{aligned} & \mathcal{FU}(-t)\,u_x^3 = \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}u_x^3 \\ = \; & \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(\mathcal{D}_t\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi} + \overline{\mathcal{D}_t\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi}}\right)^3 \\ = \; & \mathcal{Q}\overline{M}\mathcal{B}^{-1}\frac{1}{(it)}\left(\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi} + \frac{(it)^{\frac{1}{2}}}{(-it)^{\frac{1}{2}}}\overline{\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi}}\right)^3 \\ = \; & (it)^{-1}\mathcal{Q}\overline{M}\mathcal{B}^{-1}\left(\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi} + i\overline{\mathcal{B}M\mathcal{V}(i\xi)\,\widehat{\varphi}}\right)^3 \\ = \; & (it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\,\widehat{\varphi} + i\overline{M\mathcal{V}(i\xi)\,\widehat{\varphi}}\right)^3. \end{aligned}$$

Hence

$$\begin{aligned} & \mathcal{FU}(-t)\,u_x^3 \\ = \; & (it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}M^2(\mathcal{V}i\xi\widehat{\varphi})^3 + 3t^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}|\mathcal{V}i\xi\widehat{\varphi}|^2\,\mathcal{V}i\xi\widehat{\varphi} \\ & -3(it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}^2|\mathcal{V}i\xi\widehat{\varphi}|^2\,\overline{\mathcal{V}i\xi\widehat{\varphi}} - t^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}^4\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)^3. \end{aligned}$$

We need to calculate the commutator of $\mathcal{Q}$ and M since $\partial_\xi \mathcal{Q}M$ yields undesirable time growth, when we wish to estimate the derivative $\partial_\xi \widehat{\varphi}$. Note that in the case of the usual nonlinear Schrödinger equation with cubic nonlinearity u^3, we have

$$\mathcal{FU}(-t)\,u^3 = (it)^{-1}\mathcal{Q}M^2(\mathcal{V}\widehat{\varphi})^3$$

with

$$\mathcal{Q}\phi = \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{\frac{it}{2}(\xi-\eta)^2}\phi(\eta)\,d\eta = \mathcal{F}^{-1}e^{-\frac{i}{2t}|x|^2}\mathcal{F}\phi,$$

$$\mathcal{V}\phi = \sqrt{\frac{it}{2\pi}}\int_{\mathbf{R}} e^{-\frac{it}{2}(\xi-\eta)^2}\phi(\xi)\,d\xi = \mathcal{F}e^{\frac{i}{2t}|\xi|^2}\mathcal{F}^{-1}\phi.$$

We can see that $M = e^{\frac{it}{2}\eta^2}$ yields an additional time growth since $\partial_\xi \mathcal{Q}M^2$ has a time growing summand like $-it\xi\mathcal{Q}M^{2\cdot}$. We encounter a similar difficulty in our case.

We have for $\alpha \neq -1$

$$\begin{aligned}
& it\left(S\left(\xi,\eta\right)+\alpha H\left(\eta\right)\right) \\
& = it\Big(\Lambda(\xi)-\Lambda\left(\eta\right)\theta\left(\eta\right)-\frac{\eta}{|\eta|}\Lambda'\left(\eta\right)\left(\xi-\eta\theta\left(\eta\right)\right) \\
& \qquad\qquad + \alpha\left(\eta\Lambda'\left(\eta\right)\theta\left(\eta\right)-\Lambda\left(\eta\right)\theta(\eta)\right)\Big) \\
& = it\left(\Lambda(\xi)-\left(1+\alpha\right)\Lambda\left(\frac{\xi}{1+\alpha}\right)\right) \\
& +i\left(1+\alpha\right)t\left(\Lambda\left(\frac{\xi}{1+\alpha}\right)-\Lambda\left(\eta\right)\theta\left(\eta\right)-\frac{\eta}{|\eta|}\Lambda'\left(\eta\right)\left(\frac{\xi}{1+\alpha}-\eta\theta\left(\eta\right)\right)\right) \\
& = it\left(\Lambda\left(\xi\right)-\left(1+\alpha\right)\Lambda\left(\frac{\xi}{1+\alpha}\right)\right)+i\left(1+\alpha\right)tS\left(\frac{\xi}{1+\alpha},\eta\right).
\end{aligned}$$

Hence

$$\mathcal{Q}\left(t\right)M^{\alpha}\phi=e^{it\left(\Lambda(\xi)-(1+\alpha)\Lambda\left(\frac{\xi}{1+\alpha}\right)\right)}i^{\frac{1}{2}}\mathcal{D}_{1+\alpha}\mathcal{Q}\left(t\left(1+\alpha\right)\right)\phi.$$

This fact implies that our method does not work for the nonlinearity $|u|^2$, namely for the case of equations (1.2) or (1.3) with a quadratic nonlinearity. Thus we obtain from (1.6) the following equation for the new dependent variable $\widehat{\varphi}=\mathcal{FU}\left(-t\right)u\left(t\right)$

$$\begin{aligned}
\partial_t\widehat{\varphi} &= -i^{\frac{3}{2}}t^{-1}e^{it\Omega}\mathcal{D}_3\mathcal{Q}\left(3t\right)\frac{1}{|\Lambda''|}\left(\mathcal{V}i\xi\widehat{\varphi}\right)^3 \\
&+3t^{-1}\mathcal{Q}\left(t\right)\frac{1}{|\Lambda''|}\left|\mathcal{V}i\xi\widehat{\varphi}\right|^2\mathcal{V}i\xi\widehat{\varphi} \\
&+3i^{\frac{3}{2}}t^{-1}\mathcal{D}_{-1}\mathcal{Q}\left(-t\right)\frac{1}{|\Lambda''|}\left|\mathcal{V}i\xi\widehat{\varphi}\right|^2\overline{\mathcal{V}i\xi\widehat{\varphi}} \\
&-i^{\frac{1}{2}}t^{-1}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}\left(-3t\right)\frac{1}{|\Lambda''|}\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)^3, \qquad (1.10)
\end{aligned}$$

where $\Omega=\Lambda\left(\xi\right)-3\Lambda\left(\frac{\xi}{3}\right)$.

It is well known that the operator $\mathcal{J}=\mathcal{U}\left(t\right)x\mathcal{U}\left(-t\right)$ is a useful tool for obtaining the $\mathbf{L}^{\infty}$ - time decay estimates of solutions and has been used widely for the studying the asymptotic behavior of solutions to various nonlinear dispersive equations. We have

$$\begin{aligned}
\mathcal{J} &= \mathcal{U}\left(t\right)x\mathcal{U}\left(-t\right)=\mathcal{F}^{-1}e^{-it\Lambda(\xi)}i\partial_{\xi}e^{it\Lambda(\xi)}\mathcal{F} \\
&= \mathcal{F}^{-1}\left(i\partial_{\xi}-t\Lambda'\left(\xi\right)\right)\mathcal{F}=x-t\Lambda'\left(-i\partial_x\right),
\end{aligned}$$

where $\Lambda'\left(-i\partial_x\right)=-a\partial_x^2+b\partial_x^4$. Note that the commutators are true $[\mathcal{J},\mathcal{L}]=0$, $[\mathcal{J},\partial_x]=-1$, where $\mathcal{L}=\partial_t+i\Lambda\left(-i\partial_x\right)=\partial_t-\frac{a}{3}\partial_x^3+\frac{b}{5}\partial_x^5$. However, it seems that $\mathcal{J}$ does not work well on the nonlinear terms. Then, instead of the operators

$\mathcal{J}$ and the dilation operator used in [5] we apply the modified dilation operator defined by

$$\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a.$$

This is our idea in this paper. Note that $\mathcal{P}$ acts well on the nonlinear terms as the first order differential operator. Also $\mathcal{J}$ and $\mathcal{P}$ are related via the identity $\mathcal{P} = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I}$, where $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$. In order to get the estimate of $\mathcal{J}\partial_x u$, we will show the a-priori estimates of $\mathcal{P}u$, $t\mathcal{L}u$ and $\mathcal{I}u$. Different point compared to the previous works is to consider the estimate of $\mathcal{I}u$ since $\mathcal{I}u$ contains the third order derivatives $t\partial_x^3 u$ with the additional time growth (if $a \neq 0$). Note that the commutators $[\mathcal{P}, \mathcal{L}] = -\mathcal{L}$ and $[\mathcal{P}, \partial_x] = -\frac{1}{5}\partial_x$ are true. Also we have $[\mathcal{I}, \mathcal{L}] = 0$.

Our main task is to estimate each term of the right-hand side of equation (1.10) in $\mathbf{L}^\infty$ and $\mathbf{H}^1$ norms. We organize the rest of our paper as follows. In Section 2, we state the uniform estimates for the decomposition operator $\mathcal{V}$ (Lemma 2.2). The uniform estimates of $\mathcal{Q}(t)$ and $\mathcal{Q}(-t)$ are also obtained in Lemma 2.3. We prove $\mathbf{L}^2$ - estimates of the derivatives of $\mathcal{Q}$ (Lemma 3.1) and $\mathcal{V}$ (Lemma 3.5) in Section 3. Lemma 3.2 and Lemma 3.3 are prepared for proving Lemma 3.5. In Section 4 we divide the nonlinear term into the main term and the remainder terms (Lemma 4.1). In Section 5 we obtain the estimates of $\|\mathcal{F}\mathcal{U}(-t)u\|_{\mathbf{L}^\infty}$ and $\|\mathcal{J}u(t)\|_{\mathbf{H}^1}$. Section 6 is devoted to the proof of Theorem 1.1.

§2. Estimates in the uniform norm

2.1. Estimates for two kernels

Define two kernels, which describe the main term of the large time asymptotics of the operators $\mathcal{V}$ and $\mathcal{Q}$, respectively,

$$A(t,\eta) = \sqrt{\frac{it}{2\pi}} \left|\Lambda''(\eta)\right|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \chi\left(\xi\eta^{-1}\right) d\xi$$

and

$$A^*(t,\xi) = \sqrt{\frac{t}{2\pi i}} \int_0^\infty e^{itS(\xi,\eta)} \left|\Lambda''(\eta)\right|^{\frac{1}{2}} \chi\left(\xi\eta^{-1}\right) d\eta,$$

for $\xi, \eta > 0$, where $S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi - \eta)$, $\Lambda(\xi) = \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5$, and the cut off function $\chi(z) \in \mathbf{C}^2(\mathbf{R})$ is such that $\chi(z) = 0$ for $z \leq \frac{1}{3}$, $\chi(z) = 1$ for $\frac{2}{3} \leq z \leq \frac{3}{2}$, and $\chi(z) = 0$ for $z \geq 3$.

Lemma 2.1. *The estimate*

$$\|A(t)\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|A^*(t)\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \leq C$$

is true for all $t>0$. *Moreover the asymptotics are valid*

$$A\left(t,\eta\right)=1+O\left(\left(t\eta^{3}\right)^{-1}\right)$$

for $t^{\frac{1}{3}}\eta\to\infty$ *and*

$$A^{*}\left(t,\xi\right)=1+O\left(\left(t\xi^{3}\right)^{-1}\right)$$

for $t^{\frac{1}{3}}\xi\to\infty$.

Proof. We use the identity

$$e^{-itS(\xi,\eta)}=H_{1}\partial_{\xi}\left(\left(\xi-\eta\right)e^{-itS(\xi,\eta)}\right) \tag{2.1}$$

with $H_{1}=\left(1-it\left(\xi-\eta\right)S_{\xi}\left(\xi,\eta\right)\right)^{-1}$ in the domain $\xi,\eta>0$, and integrate by parts to get

$$A\left(t,\eta\right)=-\sqrt{\frac{it}{2\pi}}\left|\Lambda''\left(\eta\right)\right|^{\frac{1}{2}}\int_{0}^{\infty}e^{-itS(\xi,\eta)}\left(\xi-\eta\right)\partial_{\xi}\left(H_{1}\chi\left(\xi\eta^{-1}\right)\right)d\xi.$$

Hence in view of the estimate

$$\left|\left(\xi-\eta\right)\partial_{\xi}\left(H_{1}\chi\left(\xi\eta^{-1}\right)\right)\right|\le C\left(1+t\eta\left\langle\eta\right\rangle^{2}\left(\xi-\eta\right)^{2}\right)^{-1} \tag{2.2}$$

in the domain $\frac{\eta}{3}\le\xi\le3\eta$, we obtain

$$\begin{aligned}\left|A\left(t,\eta\right)\right| &\le Ct^{\frac{1}{2}}\left|\Lambda''\left(\eta\right)\right|^{\frac{1}{2}}\int_{\frac{\eta}{3}}^{3\eta}\frac{d\xi}{1+t\eta\left\langle\eta\right\rangle^{2}\left(\xi-\eta\right)^{2}}\\ &\le Ct^{\frac{1}{2}}\eta^{\frac{3}{2}}\left\langle\eta\right\rangle\int_{\frac{1}{3}}^{3}\frac{dy}{1+t\Lambda\left(\eta\right)\left(y-1\right)^{2}}\le C\left|t\Lambda\left(\eta\right)\right|^{\frac{1}{2}}\left\langle t\Lambda\left(\eta\right)\right\rangle^{-\frac{1}{2}}\le C.\end{aligned}$$

In the same manner we use the identity

$$e^{itS(\xi,\eta)}=H_{2}\partial_{\eta}\left(\left(\eta-\xi\right)e^{itS(\xi,\eta)}\right) \tag{2.3}$$

with $H_{2}=\left(1+it\left(\eta-\xi\right)S_{\eta}\left(\xi,\eta\right)\right)^{-1}$ in the domain $\xi,\eta>0$, and integrate by parts

$$\begin{aligned}A^{*}\left(t,\xi\right) &= \sqrt{\frac{t}{2\pi i}}\int_{0}^{\infty}e^{itS(\xi,\eta)}\left|\Lambda''\left(\eta\right)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)d\eta\\ &= -\sqrt{\frac{t}{2\pi i}}\int_{0}^{\infty}e^{itS(\xi,\eta)}\left(\xi-\eta\right)\partial_{\eta}\left(H_{2}\left|\Lambda''\left(\eta\right)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)\right)d\eta.\end{aligned}$$

Then using the estimate

$$\left|(\xi-\eta)\,\partial_\eta\left(H_2\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)\right)\right| \le C\xi^{\frac{1}{2}}\langle\xi\rangle\left(1+t\xi\langle\xi\rangle^2(\xi-\eta)^2\right)^{-1} \tag{2.4}$$

in the domain $\frac{\xi}{3}\le\eta\le 3\xi$, we get as above

$$\begin{aligned}|A^*(t,\xi)| &\le Ct^{\frac{1}{2}}\xi^{\frac{1}{2}}\langle\xi\rangle\int_{\frac{\xi}{3}}^{3\xi}\frac{d\eta}{1+t\xi\langle\xi\rangle^2(\xi-\eta)^2}\\ \le\ & Ct^{\frac{1}{2}}\xi^{\frac{3}{2}}\langle\xi\rangle\int_{\frac{1}{3}}^{3}\frac{dy}{1+t\Lambda(\xi)(y-1)^2}\le C\left|t\Lambda(\xi)\right|^{\frac{1}{2}}\langle t\Lambda(\xi)\rangle^{-\frac{1}{2}}\le C.\end{aligned}$$

To compute the asymptotics of the functions $A(t,\eta)$ and $A^*(t,\xi)$ for large $\eta t^{\frac{1}{3}}$ and $\xi t^{\frac{1}{3}}$ we apply the stationary phase method (see [4], p. 163). We have the asymptotics

$$\int_{\mathbf{R}} e^{irG(x)}f(x)\,dx = e^{irG(x_0)}f(x_0)\sqrt{\frac{2\pi}{r\left|G''(x_0)\right|}}e^{i\frac{\pi}{4}\operatorname{sgn}G''(x_0)}+O\left(r^{-\frac{3}{2}}\right) \tag{2.5}$$

for $r\to+\infty$, where the stationary point x_0 is defined by $G'(x_0)=0$. We change $\xi=x\eta$, then we get

$$\begin{aligned}A(t,\eta) &= \sqrt{\tfrac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\textstyle\int_0^\infty e^{-itS(\xi,\eta)}\chi\left(\xi\eta^{-1}\right)d\xi\\ =\eta\sqrt{\tfrac{it}{2\pi}}&\left|2\eta\left(a+2b\eta^2\right)\right|^{\frac{1}{2}}\textstyle\int_0^\infty e^{-it\eta^3\left(\frac{a}{3}x^3+\frac{b}{5}x^5\eta^2-\frac{a}{3}-\frac{b}{5}\eta^2-\left(a+b\eta^2\right)(x-1)\right)}\chi(x)\,dx\\ &= \sqrt{\tfrac{ir}{2\pi}}\left|2\left(a+2b\eta^2\right)\right|^{\frac{1}{2}}\textstyle\int_0^\infty e^{irG(x,\eta)}\chi(x)\,dx,\end{aligned}$$

where $r=t\eta^3$. By virtue of formula (2.5) with $r=t\eta^3$,

$$f(x)=\chi(x)\,,\ x_0=1$$

and

$$G(x,\eta)=-\left(\frac{a}{3}x^3+\frac{b}{5}x^5\eta^2-\frac{a}{3}-\frac{b}{5}\eta^2-\left(a+b\eta^2\right)(x-1)\right),$$

we get $A(t,\eta)=1+O\left(\left(t\eta^3\right)^{-1}\right)$ for $t^{\frac{1}{3}}\eta\to+\infty$. Also changing $\eta=x\xi$ we obtain

$$\begin{aligned}A^*(t,\xi) &= \sqrt{\frac{t}{2\pi i}}\int_0^\infty e^{itS(\xi,\eta)}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(x^{-1}\right)d\eta\\ =\ & \xi^{\frac{3}{2}}\sqrt{\frac{t}{2\pi i}}\int_0^\infty e^{it\xi^3\left(\frac{a}{3}+\frac{b}{5}\xi^2-\frac{a}{3}x^3-\frac{b}{5}x^5\xi^2-\left(ax^2+bx^4\xi^2\right)(1-x)\right)}\chi\left(x^{-1}\right)\\ &\qquad\times\left|2ax+4bx^3\xi^2\right|^{\frac{1}{2}}dx\\ =\ & \sqrt{\frac{r}{2\pi i}}\int_0^\infty e^{irG(x,\eta)}\chi\left(x^{-1}\right)\left(2x\left(a+2bx^2\xi^2\right)\right)^{\frac{1}{2}}dx,\end{aligned}$$

where $r = t\xi^3$. Then by virtue of formula (2.5) with $r = t\xi^3$,

$$f(x) = \chi\left(x^{-1}\right)\left(2x\left(a + 2bx^2\xi^2\right)\right)^{\frac{1}{2}}, x_0 = 1$$

and

$$G(x,\xi) = \frac{a}{3} + \frac{b}{5}\xi^2 - \frac{a}{3}x^3 - \frac{b}{5}x^5\xi^2 - \left(ax^2 + bx^4\xi^2\right)(1-x),$$

we get $A^*(t,\xi) = 1 + O\left(\left(t\xi^3\right)^{-1}\right)$ as $t^{\frac{1}{3}}\xi \to +\infty$. Lemma 2.1 is proved. □

2.2. Estimates for the operator $\mathcal{V}$

We estimate the operator

$$\mathcal{V}\xi^j\phi = \sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\xi)\,\xi^j d\xi$$

in the uniform norm. Denote the norm

$$\|\phi\|_{\mathbf{Z}_1} = \|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} + \|\langle\xi\rangle\,\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Note that

$$\begin{aligned} |\mathcal{V}\phi| &\le Ct^{\frac{1}{2}}\left|\Lambda''\right|^{\frac{1}{2}}\left\|\langle\xi\rangle^{-2}\right\|_{\mathbf{L}^1(\mathbf{R}_+)}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &\le Ct^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left\|\langle\xi\rangle^2\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}. \end{aligned}$$

Next lemma says that the main term of $(\mathcal{V}\phi)(\eta)$ is $A(t,\eta)\,\phi(\eta)$.

Lemma 2.2. *Let $j = 0,1,2,3$. Then the estimates are valid for all $t \ge 1$*

$$\left\||\eta|^\alpha\langle\eta\rangle^\beta\left(\mathcal{V}\xi^j\phi - A(t,\eta)\,\eta^j\phi(\eta)\right)\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \le C\max\left(t^{-\frac{1}{4}}, t^{-\frac{j+\alpha}{3}}\right)\|\phi\|_{\mathbf{Z}_1}$$

with $\alpha \ge -\frac{1}{2}$, $\beta \le \frac{7}{4} - j - \alpha$, and

$$\left\||\eta|^\alpha\langle\eta\rangle^\beta\,\mathcal{V}\xi^j\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \le C\max\left(t^{-\frac{1}{2}}\log\langle t\rangle, t^{-\frac{j+\alpha}{3}}\right)\|\phi\|_{\mathbf{Z}_1}$$

with $\alpha \ge -\frac{1}{2}$, $\beta \le \frac{5}{2} - j - \alpha$. When $j = 2,3$, the norm $\|\langle\xi\rangle\,\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ can be replaced by $\|\xi\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$.

Proof. By the definition of $\mathcal{V}$ and $A(t,\eta)$, we can write

$$\begin{aligned}
&\mathcal{V}\xi^j\phi - A(t,\eta)\,\eta^j\phi(\eta)\\
=&\sqrt{\frac{it\,|\Lambda''(\eta)|}{2\pi}}\int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi,\eta)}\left(\phi(\xi)\,\xi^j-\phi(\eta)\,\eta^j\right)\chi\left(\xi\eta^{-1}\right)d\xi\\
&+\sqrt{\frac{it\,|\Lambda''(\eta)|}{2\pi}}\int_0^{\frac{2\eta}{3}} e^{-itS(\xi,\eta)}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi(\xi)\,\xi^j d\xi\\
&+\sqrt{\frac{it\,|\Lambda''(\eta)|}{2\pi}}\int_{\frac{3\eta}{2}}^{\infty} e^{-itS(\xi,\eta)}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi(\xi)\,\xi^j d\xi\\
=&\sqrt{\frac{it\,|\Lambda''(\eta)|}{2\pi}}\left(I_1+I_2+I_3\right)
\end{aligned}$$

for $\eta>0$. We integrate by parts via the identity (2.1)

$$\begin{aligned}
I_1 =& -\int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi,\eta)}(\xi-\eta)\,\chi\left(\xi\eta^{-1}\right)H_1\phi(\xi)\,j\xi^{j-1}d\xi\\
&-\int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi,\eta)}\left(\phi(\xi)\,\xi^j-\phi(\eta)\,\eta^j\right)(\xi-\eta)\,\partial_\xi\left(\chi\left(\xi\eta^{-1}\right)H_1\right)d\xi\\
&-\int_{\frac{\eta}{3}}^{3\eta} e^{-itS(\xi,\eta)}(\xi-\eta)\,\chi\left(\xi\eta^{-1}\right)H_1\xi^j\phi_\xi(\xi)\,d\xi.
\end{aligned}$$

Using the estimates

$$|H_1| = \left|1-it(\xi-\eta)\,S_\xi(\xi,\eta)\right|^{-1}\le C\left(1+t\eta\,\langle\eta\rangle^2(\xi-\eta)^2\right)^{-1} \tag{2.6}$$

and (2.2) in the domain $\frac{\eta}{3}<\xi<3\eta$, we obtain for $j=0,1,2,3$

$$\begin{aligned}
|I_1| \le\;& C\,\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\,j\eta^{j-1}\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|\,d\xi}{1+t\eta\,\langle\eta\rangle^2(\xi-\eta)^2}\\
&+C\eta^j\int_{\frac{\eta}{3}}^{3\eta}\frac{|\phi(\xi)-\phi(\eta)|\,d\xi}{1+t\eta\,\langle\eta\rangle^2(\xi-\eta)^2}\\
&+C\eta^j\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|\,|\phi_\xi(\xi)|\,d\xi}{1+t\eta\,\langle\eta\rangle^2(\xi-\eta)^2}.
\end{aligned}$$

Since

$$\begin{aligned}
&|\phi(\xi)-\phi(\eta)|\le C\int_\xi^\eta\langle z\rangle^{-1}\langle z\rangle\,|\partial_z\phi(z)|\,dz\\
\le\;& C\left(\int_\xi^\eta\langle z\rangle^{-2}dz\right)^{\frac{1}{2}}\|\langle\xi\rangle\,\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\le C\,\langle\eta\rangle^{-1}|\xi-\eta|^{\frac{1}{2}}\,\|\phi\|_{\mathbf{Z}_1}
\end{aligned}$$

in the domain $\frac{\eta}{3}<\xi<3\eta$, we get

$$\begin{aligned}
|I_1| &\leq C\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} j\eta^{j-1}\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|\,d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2}\\
&+C\|\phi\|_{\mathbf{Z}_1}\eta^j\langle\eta\rangle^{-1}\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|^{\frac12}\,d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2}\\
&+C\|\phi\|_{\mathbf{Z}_1}\eta^j\langle\eta\rangle^{-1}\left(\int_{\frac{\eta}{3}}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t\eta\langle\eta\rangle^2(\xi-\eta)^2\right)^2}\right)^{\frac12}
\end{aligned}$$

for $j=0,1,2,3$. Note that

$$\begin{aligned}
&\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|\,d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2}\\
\leq\ & C\eta^2\int_{\frac13}^{3}\frac{|y-1|\,dy}{1+t\Lambda(\eta)(y-1)^2}\\
\leq\ & C\langle t\Lambda(\eta)\rangle^{-1}\eta^2\int_{-\frac23 t\sqrt{\Lambda(\eta)}}^{2t\sqrt{\Lambda(\eta)}}\frac{|y|\,dy}{1+y^2}\leq C\eta^2\langle t\Lambda(\eta)\rangle^{-1}\log\langle t\Lambda(\eta)\rangle,
\end{aligned}$$

$$\int_{\frac{\eta}{3}}^{3\eta}\frac{|\xi-\eta|^{\frac12}\,d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2}\leq C\eta^{\frac32}\int_{\frac13}^{3}\frac{|y-1|^{\frac12}\,dy}{1+t\Lambda(\eta)(y-1)^2}\leq C\eta^{\frac32}\langle t\Lambda(\eta)\rangle^{-\frac34}$$

and

$$\begin{aligned}
&\left(\int_{\frac{\eta}{3}}^{3\eta}\frac{(\xi-\eta)^2\,d\xi}{\left(1+t\eta\langle\eta\rangle^2(\xi-\eta)^2\right)^2}\right)^{\frac12}\\
\leq\ & C\eta^{\frac32}\left(\int_{\frac13}^{3}\frac{(y-1)^2\,dy}{\left(1+t\Lambda(\eta)(y-1)^2\right)^2}\right)^{\frac12}\leq C\eta^{\frac32}\langle t\Lambda(\eta)\rangle^{-\frac34}.
\end{aligned}$$

Hence we get for $j=0,1,2,3$

$$\begin{aligned}
&t^{\frac12}\left|\Lambda''(\eta)\right|^{\frac12}|\eta|^\alpha\langle\eta\rangle^\beta|I_1|\\
\leq\ & Ct^{\frac12}\|\phi\|_{\mathbf{Z}_1}\Big(j\eta^{j+\frac32+\alpha}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1}\log\langle t\Lambda(\eta)\rangle\\
&+\eta^{j+2+\alpha}\langle\eta\rangle^\beta\langle t\Lambda(\eta)\rangle^{-\frac34}\Big)\\
\leq\ & C\left(t^{-\frac14}+t^{-\frac{j+\alpha}{3}}\right)\|\phi\|_{\mathbf{Z}_1} \qquad (2.7)
\end{aligned}$$

with $j+\alpha+\beta\le\frac{7}{4}$, $j+2+\alpha\ge 0$, since

$$\begin{aligned}&\eta^{j+\frac{3}{2}+\alpha}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1}\log\langle t\Lambda(\eta)\rangle\\ \le\;& C\eta^{j+\frac{3}{2}+\alpha}\left\langle t^{\frac{1}{3}}\eta\right\rangle^{-3}\log\left\langle t^{\frac{1}{3}}\eta\right\rangle\\ \le\;& C\begin{cases} t^{-\frac{j+\alpha}{3}-\frac{1}{2}},\ 3>j+\frac{3}{2}+\alpha\ge 0\\ t^{-1}\log\langle t\rangle,\ j+\frac{3}{2}+\alpha\ge 3\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\eta^{j+2+\alpha}\langle\eta\rangle^{\beta}\langle t\Lambda(\eta)\rangle^{-\frac{3}{4}}\le C\eta^{j+2+\alpha}\left\langle t^{\frac{1}{3}}\eta\right\rangle^{-\frac{9}{4}}\\ \le\;& C\begin{cases} t^{-\frac{j+\alpha+2}{3}},\ \frac{9}{4}\ge j+2+\alpha\ge 0\\ t^{-\frac{3}{4}},\ j+2+\alpha\ge\frac{9}{4}\end{cases}\end{aligned}$$

for $\eta<1$, also

$$\begin{aligned}&\eta^{j+\frac{3}{2}+\alpha}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1}\log\langle t\Lambda(\eta)\rangle\\ \le\;& C\eta^{j+\frac{5}{2}+\alpha+\beta}\left\langle t^{\frac{1}{5}}\eta\right\rangle^{-5}\log\left\langle t^{\frac{1}{5}}\eta\right\rangle\le Ct^{-1}\log\langle t\rangle\end{aligned}$$

and

$$\eta^{j+2+\alpha}\langle\eta\rangle^{\beta}\langle t\Lambda(\eta)\rangle^{-\frac{3}{4}}\le C\eta^{j+2+\alpha+\beta}\left\langle t^{\frac{1}{5}}\eta\right\rangle^{-\frac{15}{4}}\le Ct^{-\frac{3}{4}}$$

for $\eta>1$ and $j+\alpha+\beta\le\frac{7}{4}$.

Next we consider

$$I_2=\int_0^{\frac{2\eta}{3}}e^{-itS(\xi,\eta)}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi(\xi)\,\xi^j d\xi$$

for $\eta>0$. We integrate by parts via the identity

$$e^{-itS(\xi,\eta)}=H_3\partial_\xi\left(\xi e^{-itS(\xi,\eta)}\right)\tag{2.8}$$

with $H_3=\left(1-it\xi\left(\Lambda'(\xi)-\Lambda'(\eta)\right)\right)^{-1}$ to get

$$\begin{aligned}I_2\;=\;&-\int_0^{\frac{2\eta}{3}}e^{-itS(\xi,\eta)}\phi(\xi)\,\xi\partial_\xi\left(H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^j\right)d\xi\\ &-\int_0^{\frac{2\eta}{3}}e^{-itS(\xi,\eta)}H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^{j+1}\phi_\xi(\xi)\,d\xi.\end{aligned}$$

Using the estimates $|H_3|\le C\left(1+t\xi\Lambda'(\eta)\right)^{-1}$ and

$$\left|\xi\partial_\xi\left(H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^j\right)\right|\le C\xi^j\left(1+t\xi\Lambda'(\eta)\right)^{-1}$$

in the domain $0<\xi<\frac{2\eta}{3}$, we obtain

$$|I_2| \le C\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\int_0^{\frac{2\eta}{3}}\frac{\xi^j d\xi}{1+t\xi\Lambda'(\eta)}+C\int_0^{\frac{2\eta}{3}}\frac{\xi^{j+\frac{1}{2}}\left|\xi^{\frac{1}{2}}\phi_\xi\right|d\xi}{1+t\xi\Lambda'(\eta)}.$$

We have

$$\int_0^{\frac{2\eta}{3}}\frac{\xi^j d\xi}{1+t\xi\Lambda'(\eta)} \le C\eta^{1+j}\int_0^{\frac{2}{3}}\frac{y^j dy}{1+t\Lambda(\eta)y} \le C\eta^{1+j}\langle t\Lambda(\eta)\rangle^{-1}$$

and

$$\int_0^{\frac{2\eta}{3}}\frac{\xi^{2j+1}d\xi}{(1+t\xi\Lambda'(\eta))^2} \le C\eta^{2j+2}\int_0^{\frac{2}{3}}\frac{y^{2j+1}dy}{(1+t\Lambda(\eta)y)^2} \le C\eta^{2j+2}\langle t\Lambda(\eta)\rangle^{-2}.$$

Hence we get

$$\begin{aligned}
& t^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}|\eta|^\alpha\langle\eta\rangle^\beta|I_2| \\
\le\ & Ct^{\frac{1}{2}}\|\phi\|_{\mathbf{Z}_1}\eta^{\alpha+j+\frac{3}{2}}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1} \\
\le\ & C\left(t^{-\frac{1}{2}}+t^{-\frac{\alpha+j}{3}}\right)\|\phi\|_{\mathbf{Z}_1} \qquad (2.9)
\end{aligned}$$

with $\alpha+j+\frac{3}{2}\ge 0$, $j+\alpha+\beta\le\frac{5}{2}$, since

$$\begin{aligned}
& \eta^{\alpha+j+\frac{3}{2}}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1} \le C\eta^{\alpha+j+\frac{3}{2}}\left\langle t^{\frac{1}{3}}\eta\right\rangle^{-3} \\
\le\ & C\begin{cases} t^{-\frac{\alpha+j}{3}-\frac{1}{2}},\ 3\ge\alpha+j+\frac{3}{2}\ge 0 \\ t^{-1},\ \alpha+j+\frac{3}{2}\ge 3\end{cases}
\end{aligned}$$

for $\eta<1$, and

$$\begin{aligned}
& \eta^{\alpha+j+\frac{3}{2}}\langle\eta\rangle^{\beta+1}\langle t\Lambda(\eta)\rangle^{-1} \\
\le\ & C\eta^{j+\frac{5}{2}+\alpha+\beta}\left\langle t^{\frac{1}{5}}\eta\right\rangle^{-5} \le Ct^{-1},
\end{aligned}$$

for $\eta>1$ and $j+\alpha+\beta\le\frac{5}{2}$.

Finally we estimate

$$I_3=\int_{\frac{3\eta}{2}}^{\infty}e^{-itS(\xi,\eta)}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi(\xi)\xi^j d\xi$$

for $\eta>0$. We integrate by parts via the identity (2.8)

$$\begin{aligned}
I_3 =\ & -\int_{\frac{3\eta}{2}}^{\infty}e^{-itS(\xi,\eta)}\phi(\xi)\xi\partial_\xi\left(H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^j\right)d\xi \\
& -\int_{\frac{3\eta}{2}}^{\infty}e^{-itS(\xi,\eta)}H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^{j+1}\phi_\xi(\xi)d\xi.
\end{aligned}$$

Using the estimates $|H_3| \le C(1+t\Lambda(\xi))^{-1}$ and

$$\left|\xi\partial_\xi\left(H_3\left(1-\chi\left(\xi\eta^{-1}\right)\right)\xi^j\right)\right| \le C\xi^j(1+t\Lambda(\xi))^{-1}$$

in the domain $\xi > \frac{3\eta}{2}$, we obtain

$$|I_3| \le C\|\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} + C\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{j+\frac12}\left|\xi^{\frac12}\phi_\xi\right| d\xi}{1+t\Lambda(\xi)}.$$

Since

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} \le Ct^{-1}\int_{\frac{3\eta}{2}}^\infty \xi^{j-5}d\xi \le Ct^{-1}\eta^{j-4}$$

for $\eta > 1$, $t \ge 1$, and for $0<\eta\le 1$, we find

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} \le \int_{\frac{3\eta}{2}}^{\frac32} \frac{\xi^j d\xi}{1+t\xi^3} + Ct^{-1}\int_{\frac32}^\infty \xi^{j-4}d\xi$$

$$\le Ct^{-\frac{j+1}{3}}\int_{\frac{3\eta}{2}t^{\frac13}}^{\frac32 t^{\frac13}} \frac{\xi^j d\xi}{1+\xi^3} + Ct^{-1} \le Ct^{-1}\log\langle t\rangle + Ct^{-\frac{j+1}{3}}\left\langle t^{\frac13}\eta\right\rangle^{j-2}$$

for $j=0,1,2$ and for $j=3$

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} \le \int_{\frac{3\eta}{2}}^{\frac32} \frac{\xi^j d\xi}{1+t\xi^3} + Ct^{-1}\int_{\frac32}^\infty \xi^{j-5}d\xi$$

$$\le Ct^{-\frac{j+1}{3}}\int_{\frac{3\eta}{2}t^{\frac13}}^{\frac32 t^{\frac13}} \frac{\xi^j d\xi}{1+\xi^3} + Ct^{-1} \le Ct^{-1}.$$

Similarly

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1+t\Lambda(\xi))^2} \le Ct^{-2}\int_{\frac{3\eta}{2}}^\infty \xi^{2j-9}d\xi \le Ct^{-2}\eta^{2j-8}$$

for $\eta > 1$. For $0<\eta\le 1$

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1+t\Lambda(\xi))^2} \le \int_{\frac{3\eta}{2}}^{\frac32} \frac{\xi^{2j+1} d\xi}{(1+t\xi^3)^2} + Ct^{-2}\int_{\frac32}^\infty \xi^{2j-9}d\xi$$

$$\le Ct^{-\frac{2j+2}{3}}\int_{\frac{3\eta}{2}t^{\frac13}}^{\frac32 t^{\frac13}} \frac{\xi^{2j+1} d\xi}{(1+\xi^3)^2} + Ct^{-2} \le Ct^{-2}\log\langle t\rangle + Ct^{-\frac{2j+2}{3}}\left\langle t^{\frac13}\eta\right\rangle^{2j-4}$$

for $j=0,1,2$, and for $j=3$

$$\int_{\frac{3\eta}{2}}^\infty \frac{\xi^{2j+1} d\xi}{(1+t\Lambda(\xi))^2} \le Ct^{-2}\int_{\frac{3\eta}{2}}^{\frac32} d\xi + Ct^{-2}\int_{\frac32}^\infty \xi^{2j-9}d\xi \le Ct^{-2}.$$

Hence we get

$$\begin{aligned} & t^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}|\eta|^{\alpha}\langle\eta\rangle^{\beta}|I_3| \\ \leq\ & C\|\phi\|_{\mathbf{Z}_1}\langle\eta\rangle^{j-3+\beta}|\eta|^{\frac{1}{2}+\alpha}\left(t^{-\frac{1}{2}}\log\langle t\rangle+t^{\frac{1}{2}-\frac{j+1}{3}}\left\langle t^{\frac{1}{3}}\eta\right\rangle^{j-2}\right) \\ \leq\ & C\left(t^{-\frac{1}{2}}\log\langle t\rangle+t^{-\frac{j+\alpha}{3}}\right)\|\phi\|_{\mathbf{Z}_1} \end{aligned}$$

for $j=0,1,2$, and for $j=3$

$$t^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}|\eta|^{\alpha}\langle\eta\rangle^{\beta}|I_3|\leq Ct^{-\frac{1}{2}}\|\phi\|_{\mathbf{Z}_1}\langle\eta\rangle^{\beta}|\eta|^{\frac{1}{2}+\alpha}\leq Ct^{-\frac{1}{2}}\|\phi\|_{\mathbf{Z}_1}$$

with $j+\beta+\alpha\leq\frac{5}{2}$, $\alpha\geq-\frac{1}{2}$.

Finally we estimate

$$\mathcal{V}\xi^j\phi=\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\phi(\xi)\,\xi^j d\xi$$

for $\eta<0$. We integrate by parts via the identity

$$e^{-itS(\xi,\eta)}=H_4\partial_{\xi}\left(\xi e^{-itS(\xi,\eta)}\right) \tag{2.10}$$

with $H_4=\left(1-it\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)\right)^{-1}$ to get

$$\begin{aligned} \mathcal{V}\xi^j\phi\ =\ & -\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\phi(\xi)\,\xi\partial_{\xi}\left(H_4\xi^j\right)d\xi \\ & -\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\xi^{j+1}H_4\phi_{\xi}(\xi)\,d\xi. \end{aligned}$$

Using the estimates $|H_4|\leq C\left(1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)\right)^{-1}$ and

$$\left|\xi\partial_{\xi}\left(H_4\xi^j\right)\right|\leq C\xi^j\left(1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)\right)^{-1}$$

we obtain

$$\begin{aligned} \left|\mathcal{V}\xi^j\phi\right|\ \leq\ & C\|\phi\|_{\mathbf{L}^{\infty}(\mathbf{R}_+)}t^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^{\infty}\frac{\xi^j d\xi}{1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)} \\ & +Ct^{\frac{1}{2}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^{\infty}\frac{\xi^{j+\frac{1}{2}}\left|\xi^{\frac{1}{2}}\phi_{\xi}\right|d\xi}{1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)}. \end{aligned}$$

As above we get

$$\begin{aligned}
& \int_0^\infty \frac{\xi^j d\xi}{1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)}+\left(\int_0^\infty \frac{\xi^{2j+1} d\xi}{\left(1+t\xi\left(\Lambda'(\xi)+\Lambda'(\eta)\right)\right)^2}\right)^{\frac{1}{2}} \\
\leq\ & \int_0^{|\eta|} \frac{\xi^j d\xi}{1+t\xi\Lambda'(\eta)}+\int_{|\eta|}^\infty \frac{\xi^j d\xi}{1+t\Lambda(\xi)} \\
& +\left(\int_0^{|\eta|} \frac{\xi^{2j+1} d\xi}{\left(1+t\xi\Lambda'(\eta)\right)^2}\right)^{\frac{1}{2}}+\left(\int_{|\eta|}^\infty \frac{\xi^{2j+1} d\xi}{\left(1+t\Lambda(\xi)\right)^2}\right)^{\frac{1}{2}} \\
\leq\ & C\eta^{1+j}\left\langle t\Lambda(\eta)\right\rangle^{-1}+C\langle\eta\rangle^{j-4}\begin{cases} t^{-1}\log\langle t\rangle+t^{-\frac{j+1}{3}}\left\langle t^{\frac{1}{3}}\eta\right\rangle^{j-2}, & j=0,1,2, \\ t^{-1}, & j=3. \end{cases}
\end{aligned}$$

Then we find

$$\left\||\eta|^\alpha\langle\eta\rangle^\beta\mathcal{V}\xi^j\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_-)}\leq C\left(t^{-\frac{1}{2}}\log\langle t\rangle+t^{-\frac{\alpha+1}{3}}\right)\|\phi\|_{\mathbf{Z}_1}$$

if $\alpha\geq-\frac{1}{2}$, $\alpha+\beta+j\leq\frac{5}{2}$. Lemma 2.2 is proved. □

2.3. Estimates for the operator $\mathcal{Q}$

In this subsection we consider the estimate of the operator

$$\mathcal{Q}\phi=\mathcal{Q}(t)\phi=\sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}}e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta$$

in the uniform norm. Note that

$$\|\mathcal{Q}\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\leq Ct^{\frac{1}{2}}\int_{\mathbf{R}}|\phi(\eta)|\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta\leq Ct^{\frac{1}{2}}\left\|\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\phi\right\|_{\mathbf{L}^1(\mathbf{R})}.$$

In particular we find

$$\begin{aligned}
& \|\mathcal{Q}(1-\theta)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}+\|\mathcal{D}_{-1}\mathcal{Q}(-t)(1-\theta)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
\leq\ & Ct^{\frac{1}{2}}\left\|\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\phi\right\|_{\mathbf{L}^1(\mathbf{R}_-)}.
\end{aligned} \tag{2.11}$$

The next lemma says that the main term of the asymptotics of $\mathcal{Q}\phi$ in the positive half-line is $A^*(t)\phi$.

Lemma 2.3. *Let $\alpha>-2$, $\alpha_1>-\frac{3}{2}$, $\beta\leq\frac{3}{4}-\alpha$, $\beta_1<\frac{5}{2}-\alpha_1$. Then the estimate*

$$\begin{aligned}
& \|\mathcal{Q}\theta\phi-\theta A^*(t)\phi\|_{\mathbf{L}^\infty(\mathbf{R}_+)}+\|\mathcal{Q}(-t)\theta\phi\|_{\mathbf{L}^\infty(\mathbf{R}_-)} \\
\leq\ & C\max\left(t^{-\frac{\alpha_1}{3}},t^{-\frac{1}{2}}\log\langle t\rangle\right)\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
& +C\max\left(t^{-\frac{1+2\alpha}{6}},t^{-\frac{1}{4}}\right)\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}
\end{aligned}$$

is valid for all $t \geq 1$.

Proof. For the case of $\xi \geq 0$, we write

$$\begin{aligned}
&\mathcal{Q}\theta\phi - \theta A^{*}(t)\phi \\
= &\sqrt{\frac{t}{2\pi i}}\int_{0}^{\infty} e^{itS(\xi,\eta)}\left(\phi(\eta) - \phi(\xi)\right)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)d\eta \\
&+\sqrt{\frac{t}{2\pi i}}\int_{0}^{\frac{2}{3}\xi} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)d\eta \\
&+\sqrt{\frac{t}{2\pi i}}\int_{\frac{3}{2}\xi}^{\infty} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)d\eta \\
= &I_1 + I_2 + I_3.
\end{aligned}$$

In the first integral I_1 by using the identity (2.3) we get

$$\begin{aligned}
I_1 = &-\sqrt{\frac{t}{2\pi i}}\int_{\frac{\xi}{3}}^{3\xi} e^{itS(\xi,\eta)}\left(\phi(\eta) - \phi(\xi)\right)(\eta - \xi) \\
&\qquad\times\partial_{\eta}\left(H_2\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)\right)d\eta \\
&-\sqrt{\frac{t}{2\pi i}}\int_{\frac{\xi}{3}}^{3\xi} e^{itS(\xi,\eta)}(\eta-\xi)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)H_2\phi_{\eta}(\eta)\,d\eta.
\end{aligned}$$

Then using (2.4) and the estimates

$$\left|\phi(\eta) - \phi(\xi)\right| \leq C\xi^{\alpha}\langle\xi\rangle^{\beta}\left|\eta - \xi\right|^{\frac{1}{2}}\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_{\eta}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

and

$$\left|H_2\right| \leq C\left(1 + t\xi\langle\xi\rangle^{2}(\xi-\eta)^{2}\right)^{-1}$$

in the domain $\frac{\xi}{3} \leq \eta \leq 3\xi$, we find

$$\begin{aligned}
&\left|I_1\right| \leq Ct^{\frac{1}{2}}\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_{\eta}\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\xi^{\frac{1}{2}+\alpha}\langle\xi\rangle^{1+\beta} \\
&\times\left(\int_{\frac{\xi}{3}}^{3\xi}\frac{|\eta-\xi|^{\frac{1}{2}}\,d\eta}{1+t\xi\langle\xi\rangle^{2}(\xi-\eta)^{2}} + \left(\int_{\frac{\xi}{3}}^{3\xi}\frac{(\eta-\xi)^{2}\,d\eta}{\left(1+t\xi\langle\xi\rangle^{2}(\xi-\eta)^{2}\right)^{2}}\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Changing $\eta = \xi y$ we get

$$\begin{aligned}
\int_{\frac{\xi}{3}}^{3\xi}\frac{|\eta-\xi|^{\frac{1}{2}}\,d\eta}{1+t\xi\langle\xi\rangle^{2}(\xi-\eta)^{2}} &\leq C\xi^{\frac{3}{2}}\int_{\frac{1}{3}}^{3}\frac{|y-1|^{\frac{1}{2}}\,dy}{1+t\Lambda(\xi)(1-y)^{2}} \\
&\leq C\xi^{\frac{3}{2}}\langle t\Lambda(\xi)\rangle^{-\frac{3}{4}}
\end{aligned}$$

and

$$\int_{\frac{\xi}{3}}^{3\xi} \frac{(\eta-\xi)^2 \, d\eta}{\left(1+t\xi\langle\xi\rangle^2(\xi-\eta)^2\right)^2} \le C\xi^3 \int_{\frac{1}{3}}^{3} \frac{(y-1)^2 \, dy}{\left(1+t\Lambda(\xi)(1-y)^2\right)^2} \le C\xi^3 \langle t\Lambda(\xi)\rangle^{-\frac{3}{2}}.$$

Then using

$$t^{\frac{1}{2}}\xi^{2+\alpha}\langle\xi\rangle^{1+\beta}\langle t\Lambda(\xi)\rangle^{-\frac{3}{4}} \le Ct^{\frac{1}{2}}\xi^{2+\alpha}\left\langle t^{\frac{1}{3}}\xi\right\rangle^{-\frac{9}{4}} \le C\left(t^{-\frac{2\alpha+1}{6}}+t^{-\frac{1}{4}}\right)$$

for $0<\xi<1$, if $\alpha\ge -2$ and

$$t^{\frac{1}{2}}\xi^{2+\alpha}\langle\xi\rangle^{1+\beta}\langle t\Lambda(\xi)\rangle^{-\frac{3}{4}} \le Ct^{\frac{1}{2}}\xi^{3+\alpha+\beta}\left\langle t^{\frac{1}{5}}\xi\right\rangle^{-\frac{15}{4}} \le Ct^{-\frac{1}{4}}$$

for $\xi\ge 1$, if $\beta\le\frac{3}{4}-\alpha$, we obtain

$$\begin{aligned} |I_1| &\le C\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} t^{\frac{1}{2}}\xi^{2+\alpha}\langle\xi\rangle^{1+\beta}\langle t\Lambda(\xi)\rangle^{-\frac{3}{4}} \\ &\le C\max\left(t^{-\frac{2\alpha+1}{6}},t^{-\frac{1}{4}}\right)\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

In the second integral I_2 using the identity $e^{itS(\xi,\eta)}=H_5\partial_\eta\left(\eta e^{itS(\xi,\eta)}\right)$ with $H_5=(1+it\eta S_\eta(\xi,\eta))^{-1}$ we integrate by parts

$$I_2=-\sqrt{\frac{t}{2\pi i}}\int_0^{\frac{2}{3}\xi} e^{itS(\xi,\eta)}\phi(\eta)\,\eta\partial_\eta\left(H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\right)d\eta$$
$$-\sqrt{\frac{t}{2\pi i}}\int_0^{\frac{2}{3}\xi} e^{itS(\xi,\eta)}\eta H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi_\eta(\eta)\,d\eta.$$

Since $S_\eta(\xi,\eta)=-\Lambda''(\eta)(\xi-\eta)$ for $\eta>0$, we have the estimates

$$\left|\eta H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\right| \le C\eta^{\frac{3}{2}}\langle\eta\rangle\left(1+t(\xi-\eta)\eta^2\langle\eta\rangle^2\right)^{-1}, \tag{2.12}$$

and

$$\left|\eta\partial_\eta\left(H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\right)\right| \le C\eta^{\frac{1}{2}}\langle\eta\rangle\left(1+t(\xi-\eta)\eta^2\langle\eta\rangle^2\right)^{-1}. \tag{2.13}$$

Hence in the domain $0<\eta<\frac{2}{3}\xi$ we get

$$\begin{aligned} |I_2| &\le Ct^{\frac{1}{2}}\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}\int_0^{\frac{2}{3}\xi}\frac{\eta^{\frac{1}{2}+\alpha_1}\langle\eta\rangle^{1+\beta_1}d\eta}{1+t\xi\eta^2\langle\eta\rangle^2} \\ &\quad +Ct^{\frac{1}{2}}\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\left(\int_0^{\frac{2}{3}\xi}\frac{\eta^{3+2\alpha}\langle\eta\rangle^{2+2\beta}d\eta}{\left(1+t\xi\eta^2\langle\eta\rangle^2\right)^2}\right)^{\frac{1}{2}}. \end{aligned}$$

We get

$$\int_0^{\frac{2}{3}\xi} \frac{\eta^{\frac{1}{2}+\alpha_1} \langle\eta\rangle^{1+\beta_1} d\eta}{1+t\xi\eta^2\langle\eta\rangle^2} \le C\int_0^{\infty} \frac{\eta^{\frac{1}{2}+\alpha_1} \langle\eta\rangle^{1+\beta_1} d\eta}{1+t\Lambda(\eta)}$$

$$\le C\int_0^1 \frac{\eta^{\frac{1}{2}+\alpha_1} d\eta}{1+t\eta^3} + Ct^{-1}\int_1^{\infty} \eta^{-\frac{7}{2}+\alpha+\beta} d\eta$$

$$\le Ct^{-\frac{1}{2}-\frac{\alpha_1}{3}} \int_0^{t^{\frac{1}{3}}} \frac{\eta^{\frac{1}{2}+\alpha_1} d\eta}{1+\eta^3} + Ct^{-1} \le C\left(t^{-\frac{1}{2}-\frac{\alpha_1}{3}} + t^{-1}\log\langle t\rangle\right),$$

if $\alpha_1 > -\frac{3}{2}$, $\alpha_1+\beta_1 < \frac{5}{2}$. Similarly

$$\int_0^{\frac{2}{3}\xi} \frac{\eta^{3+2\alpha} \langle\eta\rangle^{2+2\beta} d\eta}{\left(1+t\xi\eta^2\langle\eta\rangle^2\right)^2} \le C\int_0^{\infty} \frac{\eta^{3+2\alpha} \langle\eta\rangle^{2+2\beta} d\eta}{(1+t\Lambda(\eta))^2}$$

$$\le C\int_0^1 \frac{\eta^{3+2\alpha} d\eta}{1+t^2\eta^6} + Ct^{-2}\int_1^{\infty} \eta^{-5+2\alpha+2\beta} d\eta$$

$$\le Ct^{-\frac{4}{3}-\frac{2\alpha}{3}} \int_0^{t^{\frac{1}{3}}} \frac{\eta^{3+2\alpha} d\eta}{1+\eta^6} + Ct^{-2} \le C\left(t^{-\frac{4}{3}-\frac{2\alpha}{3}} + t^{-2}\log\langle t\rangle\right),$$

where $\alpha > -2$, $\alpha+\beta < 2$. Therefore

$$|I_2| \le C\max\left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}}\log\langle t\rangle\right) \left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)}$$
$$+C\max\left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}}\log\langle t\rangle\right) \left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

In the third integral I_3 using the identity $e^{itS(\xi,\eta)} = H_5\partial_\eta\left(\eta e^{itS(\xi,\eta)}\right)$ with $H_5 = (1+it\eta S_\eta(\xi,\eta))^{-1}$ we integrate by parts

$$I_3 = -\sqrt{\frac{t}{2\pi i}} \int_{\frac{3}{2}\xi}^{\infty} e^{itS(\xi,\eta)}\phi(\eta)\,\eta\partial_\eta\left(H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\right)d\eta$$
$$-\sqrt{\frac{t}{2\pi i}} \int_{\frac{3}{2}\xi}^{\infty} e^{itS(\xi,\eta)}\eta H_5\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(1-\chi\left(\xi\eta^{-1}\right)\right)\phi_\eta(\eta)\,d\eta.$$

Since $\frac{3}{2}\xi < \eta$ implies $\frac{1}{3}\eta < (\eta-\xi)$, then by (2.12) and (2.13) we get

$$|I_3| \le Ct^{\frac{1}{2}}\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \int_{\frac{3}{2}\xi}^{\infty} \frac{\eta^{\frac{1}{2}+\alpha_1}\langle\eta\rangle^{1+\beta_1} d\eta}{1+t\Lambda(\eta)}$$
$$+Ct^{\frac{1}{2}}\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \left(\int_{\frac{3}{2}\xi}^{\infty} \frac{\eta^{3+2\alpha}\langle\eta\rangle^{2+2\beta} d\eta}{(1+t\Lambda(\eta))^2}\right)^{\frac{1}{2}}.$$

In the same way as in the proof of the estimate for I_2 we obtain

$$\begin{aligned} |I_3| &\leq C\max\left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}}\log\langle t\rangle\right)\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &+C\max\left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}}\log\langle t\rangle\right)\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

Next consider $\mathcal{Q}(-t)\theta\phi$ for $\xi < 0$. Using the identity

$$e^{-itS(\xi,\eta)} = H_6\partial_\eta\left(\eta e^{-itS(\xi,\eta)}\right)$$

with $H_6 = (1 - it\eta S_\eta(\xi,\eta))^{-1}$, we integrate by parts

$$\mathcal{Q}(-t)\theta\phi = -\sqrt{\frac{it}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\eta)\,\eta\partial_\eta\left(H_6\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\right)d\eta$$
$$-\sqrt{\frac{it}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\eta H_6\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\phi_\eta(\eta)\,d\eta.$$

Since $S_\eta(\xi,\eta) = -\Lambda''(\eta)(\xi-\eta)$, $\xi < 0$, $\eta > 0$, then we have the estimates

$$\left|\eta H_6\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\right| \leq C\eta^{\frac{3}{2}}\langle\eta\rangle(1 + t\Lambda(\eta))^{-1},$$

and

$$\left|\eta\partial_\eta\left(H_6\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\right)\right| \leq C\eta^{\frac{1}{2}}\langle\eta\rangle(1 + t\Lambda(\eta))^{-1}.$$

Hence we get

$$\begin{aligned} |\mathcal{Q}(-t)\theta\phi| &\leq Ct^{\frac{1}{2}}\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R})}\int_0^\infty\frac{|\eta|^{\frac{1}{2}+\alpha_1}\langle\eta\rangle^{1+\beta_1}d\eta}{1+t\Lambda(\eta)} \\ &+Ct^{\frac{1}{2}}\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R})}\left(\int_0^\infty\frac{|\eta|^{3+2\alpha}\langle\eta\rangle^{2+2\beta}d\eta}{(1+t\Lambda(\eta))^2}\right)^{\frac{1}{2}}. \end{aligned}$$

Then in the same way as in the proof of the estimate for I_2 we obtain

$$\begin{aligned} |\mathcal{Q}(-t)\theta\phi| &\leq C\max\left(t^{-\frac{\alpha_1}{3}}, t^{-\frac{1}{2}}\log\langle t\rangle\right)\left\|\eta^{-\alpha_1}\langle\eta\rangle^{-\beta_1}\phi\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ &+C\max\left(t^{-\frac{1+2\alpha}{6}}, t^{-\frac{1}{2}}\log\langle t\rangle\right)\left\|\eta^{-\alpha}\langle\eta\rangle^{-\beta}\partial_\eta\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \end{aligned}$$

for all $\xi \leq 0$. Lemma 2.3 is proved. □

§3. Estimates in the $\mathbf{L}^2$ - norm

3.1. Estimate for derivatives of $\mathcal{Q}$

First we note that

$$\begin{aligned} \|\mathcal{Q}\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} &= Ct^{\frac{1}{2}} \left\| \int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)\xi} M\phi(\eta) \left|\Lambda''(\eta)\right|^{\frac{1}{2}} d\eta \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &= Ct^{\frac{1}{2}} \left\| \int_{\mathbf{R}} e^{-itx\xi} M\phi(\eta(x)) \left|\Lambda''(\eta(x))\right|^{-\frac{1}{2}} dx \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\le C \int_{\mathbf{R}} |\phi(\eta(x))|^2 \left|\Lambda''(\eta(x))\right|^{-1} dx = C\|\phi\|_{\mathbf{L}^2(\mathbf{R})}. \end{aligned} \tag{3.1}$$

Also we have the relation

$$i\xi\mathcal{Q}\phi = \mathcal{Q}\mathcal{A}\phi,$$

where

$$\mathcal{A} = \frac{\overline{M}}{t\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}\partial_\eta \frac{M}{\left|\Lambda''(\eta)\right|^{\frac{1}{2}}} = \mathcal{A}_0 + i\eta\theta(\eta)$$

with $\mathcal{A}_0 = \frac{1}{t|\Lambda''(\eta)|^{\frac{1}{2}}}\partial_\eta \frac{1}{|\Lambda''(\eta)|^{\frac{1}{2}}}$. In the next lemma we obtain the estimates through the operator $\mathcal{A}_0$.

Lemma 3.1. *The estimates*

$$\left\|\langle\xi\rangle^{-4} t\partial_t\mathcal{Q}(t)\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\sum_{l=0}^{4}\left\|t\mathcal{A}_0\eta^l\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + Ct\left\|\Lambda'\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)}.$$

and

$$\begin{aligned} \left\|\langle\xi\rangle^{-4} t\partial_t\mathcal{Q}(t)\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} &\le \left\|t\mathcal{A}_0^2\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \|\eta t\mathcal{A}_0\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &\quad + \sum_{l=1}^{4}\left\|t\mathcal{A}_0\eta^l\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + Ct\left\|\Lambda'\phi\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \end{aligned}$$

are true.

Proof. By a direct calculation we find

$$\begin{aligned} &t\partial_t\mathcal{Q}\phi = (it\Lambda(\xi)\mathcal{Q}\phi - it\mathcal{Q}\Lambda\theta\phi) \\ &+ \left(it\mathcal{Q}\eta\Lambda'\theta\phi - it\xi\mathcal{Q}\Lambda'\theta\phi\right) + it\xi\mathcal{Q}(1-\theta)\Lambda'\phi. \end{aligned}$$

Applying the commutator

$$i\xi\mathcal{Q}\phi - \mathcal{Q}i\eta\theta\phi = \mathcal{Q}(\mathcal{A} - i\eta\theta)\phi = \mathcal{Q}\mathcal{A}_0\phi$$

we obtain

$$it\mathcal{Q}\eta\Lambda'\theta\phi - it\xi\mathcal{Q}\Lambda'\theta\phi = -\mathcal{Q}t\mathcal{A}_0\Lambda'\theta\phi.$$

By the relation $\mathcal{A} = \mathcal{A}_0 + i\eta\theta\left(\eta\right)$ we get

$$\left(i\xi\right)^n \mathcal{Q}\phi - \mathcal{Q}\left(i\eta\right)^n \theta\phi = \sum_{l=0}^{n-1} \left(i\xi\right)^l \mathcal{Q}\mathcal{A}_0 \left(i\eta\right)^{n-1-l} \theta\phi.$$

Therefore

$$it\Lambda\left(\xi\right)\mathcal{Q}\phi - it\mathcal{Q}\Lambda\theta\phi = it\frac{a}{3}\sum_{l=0}^{2}\left(i\xi\right)^l \mathcal{Q}\mathcal{A}_0\left(i\eta\right)^{2-l}\theta\phi$$

$$+it\frac{b}{5}\sum_{l=0}^{4}\left(i\xi\right)^l \mathcal{Q}\mathcal{A}_0\left(i\eta\right)^{4-l}\theta\phi.$$

Then by inequality (3.1) we obtain the first estimate of the lemma. The second estimate is proved in the same way. Lemma 3.1 is proved. □

3.2. Auxiliary estimates in $\mathbf{L}^2$

In the next two lemmas we estimate some integrals to prove the estimate of the derivative of $\mathcal{V}\phi$. First, we consider the integral

$$\mathcal{I}_1\phi = \left|\Lambda''\left(\eta\right)\right|^{\frac{1}{2}} t^{\frac{1}{2}} \int_0^{\infty} e^{-itS(\xi,\eta)}\phi\left(\xi\right)\psi\left(\xi,\eta\right)d\xi.$$

Lemma 3.2. *Let $\beta > \alpha > 0$, $\delta = 0$ or $\beta > \alpha = 0$, $\delta > 0$. Suppose that*

$$\left|\left|\eta\right|^{-\alpha}\langle\eta\rangle^{\beta}\langle\xi\rangle^{\delta}\left(\eta\partial_\eta\right)^k\psi\left(\xi,\eta\right)\right| \le C$$

for all $k = 0, 1, 2$, $\eta \in \mathbf{R}$, $\xi > 0$. Then the estimate

$$\left\|\mathcal{I}_1\phi\right\|_{\mathbf{L}^2(\mathbf{R})} \le C\left\|\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\begin{cases} 1 \text{ for } \alpha > 0 \\ \log\langle t\rangle \text{ for } \alpha = 0\end{cases}$$

is true for all $t \ge 1$.

Proof. We have

$$
\begin{aligned}
&\|\mathcal{I}_1\phi\|^2_{\mathbf{L}^2(\mathbf{R})} = Ct\int_{\mathbf{R}} d\eta\left|\Lambda''(\eta)\right|\int_0^\infty e^{itS(\xi,\eta)}\overline{\phi(\xi)\psi(\xi,\eta)}d\xi \\
&\times\int_0^\infty e^{-itS(\zeta,\eta)}\phi(\zeta)\psi(\zeta,\eta)\,d\zeta \\
= \; & Ct\int_0^\infty e^{it\Lambda(\xi)}\overline{\phi(\xi)}d\xi\int_0^\infty e^{-it\Lambda(\zeta)}\phi(\zeta)\,d\zeta \\
&\times\int_{\mathbf{R}} d\eta\left|\Lambda''(\eta)\right|\overline{\psi(\xi,\eta)}\psi(\zeta,\eta)\,e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)} \\
= \; & C\int_0^\infty d\xi e^{it\Lambda(\xi)}\overline{\phi(\xi)}\int_0^\infty d\zeta e^{-it\Lambda(\zeta)}\phi(\zeta)\,K(t,\xi,\zeta),
\end{aligned}
$$

where

$$K(t,\xi,\zeta) = t\int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)}\left|\Lambda''(\eta)\right|\overline{\psi(\xi,\eta)}\psi(\zeta,\eta)\,d\eta.$$

We integrate two times by parts via the identity

$$e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)} = H_6\partial_\eta\left(\eta e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)}\right)$$

with $H_6 = \left(1 - it\eta\left|\Lambda''(\eta)\right|(\xi-\zeta)\right)^{-1}$ to get

$$K(t,\xi,\zeta) = t\int_{\mathbf{R}} e^{-it\frac{\eta}{|\eta|}\Lambda'(\eta)(\xi-\zeta)}\eta\partial_\eta\left(H_6\eta\partial_\eta\left(H_6\left|\Lambda''(\eta)\right|\overline{\psi(\xi,\eta)}\psi(\zeta,\eta)\right)\right)d\eta.$$

Then using the estimate

$$\left|\eta\partial_\eta\left(H_6\eta\partial_\eta\left(H_6\left|\Lambda''(\eta)\right|\overline{\psi(\xi,\eta)}\psi(\zeta,\eta)\right)\right)\right| \le \frac{C\left|\eta\right|^{1+2\alpha}\langle\eta\rangle^{2-2\beta}\langle\xi\rangle^{-\delta}\langle\zeta\rangle^{-\delta}}{\left(1+t\left|\Lambda'(\eta)\right|\left|\xi-\zeta\right|\right)^2},$$

we get

$$
\begin{aligned}
&\left|K(t,\xi,\zeta)\right| \le Ct\int_{\mathbf{R}}\left|\eta\partial_\eta\left(H_6\eta\partial_\eta\left(H_6\left|\Lambda''(\eta)\right|\overline{\psi(\xi,\eta)}\psi(\zeta,\eta)\right)\right)\right|d\eta \\
\le \; & Ct\langle\xi-\zeta\rangle^{-\delta}\left(\int_0^1\frac{\eta^{1+2\alpha}d\eta}{\left(1+t\eta^2\left|\xi-\zeta\right|\right)^2} + \int_1^\infty\frac{\eta^{3+2\alpha-2\beta}d\eta}{\left(1+t\eta^4\left|\xi-\zeta\right|\right)^2}\right) \\
\le \; & Ct\langle\xi-\zeta\rangle^{-\delta}\left|(\xi-\zeta)t\right|^{-1-\alpha}\int_0^{(t|\xi-\zeta|)^{\frac12}}\frac{y^{1+2\alpha}dy}{\left(1+y^2\right)^2} \\
&\qquad + Ct\left|(\xi-\zeta)t\right|^{\frac{\beta-\alpha}{2}-1}\int_{(t|\xi-\zeta|)^{\frac14}}^\infty\frac{y^{3+2\alpha-2\beta}dy}{\left(1+y^4\right)^2} \\
\le \; & Ct\langle\xi-\zeta\rangle^{-\delta}\langle(\xi-\zeta)t\rangle^{-1-\alpha} \\
&\qquad + Ct\left(\left|(\xi-\zeta)t\right|^{\frac{\beta-\alpha}{2}-1}+1\right)\langle(\xi-\zeta)t\rangle^{-\frac{\beta-\alpha}{2}-1}
\end{aligned}
$$

if $\beta > \alpha \geq 0$. Then by the Young inequality for convolutions we obtain

$$\begin{aligned}
&\|\mathcal{I}_1\phi\|^2_{\mathbf{L}^2(\mathbf{R})} \leq C\,\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty |K(t,\xi,\zeta)|\,|\phi(\zeta)|\,d\zeta \right\|_{\mathbf{L}^2} \\
\leq\ & C\,\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty t\,\langle \xi-\zeta\rangle^{-\delta} \langle (\xi-\zeta)\,t\rangle^{-1-\alpha} |\phi(\zeta)|\,d\zeta \right\|_{\mathbf{L}^2} \\
+\ & C\,\|\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \left\| \int_0^\infty t^{\frac{\beta-\alpha}{2}} |\xi-\zeta|^{\frac{\beta-\alpha}{2}-1} \left(|(\xi-\zeta)\,t|^{\frac{\beta-\alpha}{2}-1} + 1 \right) |\phi(\zeta)|\,d\zeta \right\|_{\mathbf{L}^2} \\
\leq\ & C\,\|\phi\|^2_{\mathbf{L}^2(\mathbf{R}_+)} \left(\left\| t\,\langle \xi t\rangle^{-1-\alpha} \langle \xi\rangle^{-\delta} \right\|_{\mathbf{L}^1} + \left\| t \left(|\xi t|^{\frac{\beta-\alpha}{2}-1} + 1 \right) \langle \xi t\rangle^{-\frac{\beta-\alpha}{2}-1} \right\|_{\mathbf{L}^1} \right) \\
\leq\ & C\,\|\phi\|^2_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 \text{ for } \alpha > 0 \\ \log\langle t\rangle \text{ for } \alpha = 0 \end{cases},
\end{aligned}$$

since in the case of $\alpha > 0$, $\delta = 0$ we have

$$\left\| t\,\langle \xi t\rangle^{-1-\alpha} \langle \xi\rangle^{-\delta} \right\|_{\mathbf{L}^1} \leq C \int_0^t \frac{d\xi}{(1+\xi)^{1+\alpha}} + C \int_1^\infty \xi^{-1-\alpha} d\xi \leq C,$$

and in the case of $\alpha = 0$, $\delta > 0$ we find

$$\left\| t\,\langle \xi t\rangle^{-1-\alpha} \langle \xi\rangle^{-\delta} \right\|_{\mathbf{L}^1} \leq C \int_0^t \frac{d\xi}{1+\xi} + C \int_1^\infty \xi^{-1-\delta} d\xi \leq C \log\langle t\rangle\,.$$

Lemma 3.2 is proved. □

In the next lemma we prove another estimate of the integral

$$\mathcal{I}_2\phi = t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \phi(\xi)\,\psi(\xi,\eta)\,d\xi$$

in $\mathbf{L}^2$.

Lemma 3.3. *Let $k = 0, 1, 2$, $k - 1 \leq \alpha < k + 1 - \beta$. Suppose that*

$$|\psi(\xi,\eta)| + |(\xi - \eta\theta(\eta))\,\partial_\xi \psi(\xi,\eta)| \leq |\xi + |\eta||^{-k} |\eta|^\alpha \langle \eta\rangle^\beta$$

for $\xi > 0$, $\eta \in \mathbf{R}$. Suppose that $\phi(0) = 0$. Then the estimate

$$\|\mathcal{I}_2\phi\|_{\mathbf{L}^2(\mathbf{R})} \leq C \max\left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right) \|\langle \xi\rangle\,\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

is true for all $t \geq 1$.

Proof. Consider $\eta > 0$. Using identity (2.1) we integrate by parts

$$\begin{aligned}
\mathcal{I}_2\phi &= -t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \phi(\xi)\,(\xi-\eta)\,\partial_\xi\left(H_1\psi(\xi,\eta)\right) d\xi \\
&\quad -t^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)}\,(\xi-\eta)\,H_1\psi(\xi,\eta)\,\phi_\xi(\xi)\,d\xi.
\end{aligned}$$

By the estimates $|H_1| \le C\left(1+t(\xi-\eta)^2(\eta+\xi)\left(1+\eta^2+\xi^2\right)\right)^{-1}$, and

$$|(\xi-\eta)\partial_\xi(H_1\psi(\xi,\eta))| \le \frac{C|\xi+|\eta||^{-k}|\eta|^\alpha\langle\eta\rangle^\beta}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}$$

we obtain

$$|\mathcal{I}_2\phi| \le Ct^{\frac12}\int_0^\infty \frac{|\phi(\xi)-\phi(0)||\xi+|\eta||^{-k}|\eta|^\alpha\langle\eta\rangle^\beta d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}$$
$$+Ct^{\frac12}\int_0^\infty \frac{|\xi-\eta||\xi+|\eta||^{-k}|\eta|^\alpha\langle\eta\rangle^\beta|\partial_\xi\phi|d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}.$$

Since $|\phi(\xi)-\phi(0)| \le C\xi^{\frac12}\langle\xi\rangle^{-\frac12}\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}$ we get

$$|\mathcal{I}_2\phi| \le Ct^{\frac12}\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\int_0^\infty \frac{|\xi+|\eta||^{-k}\xi^{\frac12}\langle\xi\rangle^{-\frac12}|\eta|^\alpha\langle\eta\rangle^\beta d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}$$
$$(3.2)\qquad +Ct^{\frac12}\int_0^\infty \frac{|\xi-\eta||\xi+|\eta||^{-k}|\eta|^\alpha\langle\eta\rangle^\beta|\partial_\xi\phi|d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}.$$

We estimate the first summand in (3.2) as

$$\int_0^\infty \frac{|\xi+|\eta||^{-k}\xi^{\frac12}\langle\xi\rangle^{-\frac12}|\eta|^\alpha\langle\eta\rangle^\beta d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}$$
$$(3.3)\qquad \le C\int_0^{2\eta}\frac{|\eta|^{\alpha-k+\frac12}\langle\eta\rangle^\beta d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2} + C|\eta|^\alpha\langle\eta\rangle^\beta\int_\eta^\infty \frac{\xi^{\frac12-k}\langle\xi\rangle^{-\frac12}d\xi}{1+t\Lambda(\xi)}.$$

Then we find

$$\int_0^{2\eta}\frac{|\eta|^{\alpha-k+\frac12}\langle\eta\rangle^\beta d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2} \le C\int_0^2 \frac{|\eta|^{\alpha-k+\frac32}\langle\eta\rangle^\beta dy}{1+t\Lambda(\eta)(y-1)^2}$$
$$\le C|\eta|^{\alpha-k+\frac32}\langle\eta\rangle^\beta\langle t\Lambda(\eta)\rangle^{-\frac12}$$

and

$$t^{\frac12}\left\||\eta|^{\alpha-k+\frac32}\langle\eta\rangle^\beta\langle t\Lambda(\eta)\rangle^{-\frac12}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le Ct^{\frac12}\left(\int_0^1\left\langle t^{\frac13}\eta\right\rangle^{-3}\eta^{2\alpha-2k+3}d\eta\right)^{\frac12}$$
$$(3.4)\quad +C\left(\int_1^\infty \eta^{2\alpha+2\beta-2k-2}d\eta\right)^{\frac12} \le C\max\left(1,t^{-\frac16-\frac{\alpha-k}{3}}\right)$$

if $\alpha > k-2$, $\alpha+\beta-k<1$. Also the second term of the right-hand side of (3.3) is estimated as

$$|\eta|^\alpha\langle\eta\rangle^\beta\int_\eta^\infty \frac{\xi^{\frac12-k}\langle\xi\rangle^{-\frac12}d\xi}{1+t\Lambda(\xi)} \le Ct^{-1}\eta^{\alpha+\beta}\int_\eta^\infty \xi^{-k-5}d\xi \le Ct^{-1}\langle\eta\rangle^{\alpha+\beta-k-4}$$

for $\eta > 1$, $t \geq 1$, and

$$|\eta|^{\alpha} \langle \eta \rangle^{\beta} \int_{\eta}^{\infty} \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \leq C \int_{\eta}^{1} \frac{\xi^{\alpha-k+\frac{1}{2}} d\xi}{1 + t\xi^{3}} + Ct^{-1} \int_{1}^{\infty} \xi^{-k-5} d\xi$$

$$\leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \int_{\eta t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha-k+\frac{1}{2}} d\xi}{1+\xi^{3}} + Ct^{-1}$$

$$\leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \max\left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{\alpha-k-\frac{3}{2}}\right) + Ct^{-1} \log t$$

for $0 < \eta \leq 1$. Hence

$$\left\| |\eta|^{\alpha} \langle \eta \rangle^{\beta} \int_{\eta}^{\infty} \frac{\xi^{\frac{1}{2}-k} \langle \xi \rangle^{-\frac{1}{2}} d\xi}{1 + t\Lambda(\xi)} \right\|_{\mathbf{L}^{2}(\mathbf{R}_{+})}$$

$$\leq Ct^{-\frac{1}{2}-\frac{\alpha-k}{3}} \left(\int_{0}^{1} \max\left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{2\alpha-2k-3}\right) d\eta \right)^{\frac{1}{2}} + Ct^{-1} \log t$$

$$+Ct^{-1} \left(\int_{1}^{\infty} \langle \eta \rangle^{2\alpha+2\beta-2k-8} d\eta \right)^{\frac{1}{2}} \leq C \max\left(t^{-1} \log t, t^{-\frac{2}{3}-\frac{\alpha-k}{3}}\right) \tag{3.5}$$

for $\alpha + \beta - k < \frac{7}{2}$. Thus by (3.3)-(3.5)

$$t^{\frac{1}{2}} \left\| \int_{0}^{\infty} \frac{||\xi + |\eta||^{-k} \xi^{\frac{1}{2}} \langle \xi \rangle^{-\frac{1}{2}} |\eta|^{\alpha} \langle \eta \rangle^{\beta} d\xi}{1 + t(\xi - \eta)^{2} (\eta + \xi)(1 + \eta^{2} + \xi^{2})} \right\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \leq C \max\left(1, t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right) \tag{3.6}$$

if $\alpha > k - 2$, $\alpha + \beta - k < 1$.

Consider the second summand in (3.2) for $0 < \eta < 1$

$$\int_{0}^{\infty} \frac{|\xi - \eta| \, ||\xi + |\eta||^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta} |\partial_{\xi} \phi| d\xi}{1 + t(\xi - \eta)^{2} (\eta + \xi)(1 + \eta^{2} + \xi^{2})}$$

$$\leq C \int_{0}^{2\eta} \frac{|\xi - \eta|^{\alpha+1-k} |\partial_{\xi} \phi| d\xi}{1 + t|\eta - \xi|^{3}}$$

$$+ C \|\langle \xi \rangle \phi_{\xi}\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \left(\int_{2\eta}^{\infty} \frac{\xi^{2\alpha+2-2k} d\xi}{(1 + t\Lambda(\xi))^{2}} \right)^{\frac{1}{2}}. \tag{3.7}$$

if $\alpha + 1 - k \geq 0$. By the Young inequality

$$\left\| \int_{0}^{2\eta} \frac{|\xi - \eta|^{\alpha+1-k} |\partial_{\xi} \phi| d\xi}{1 + t|\eta - \xi|^{3}} \right\|_{\mathbf{L}^{2}}$$

$$\leq C \|\phi_{\xi}\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \left\| |\eta|^{\alpha+1-k} \left\langle t^{\frac{1}{3}} \eta \right\rangle^{-3} \right\|_{\mathbf{L}^{1}(\mathbf{R})}$$

$$\leq C \max\left(t^{-1} \log t, t^{-\frac{\alpha-k+2}{3}}\right) \|\phi_{\xi}\|_{\mathbf{L}^{2}(\mathbf{R}_{+})}. \tag{3.8}$$

Let us consider the second term of the right hand side of (3.7). We have

$$\begin{aligned}\int_{2\eta}^{\infty}\frac{\xi^{2\alpha+2-2k}d\xi}{(1+t\Lambda(\xi))^2} &\le \int_{2\eta}^{2}\frac{\xi^{2\alpha+2-2k}d\xi}{(1+t\xi^3)^2}+Ct^{-2}\int_2^{\infty}\xi^{2\alpha-2k-8}d\xi\\ &\le Ct^{-\frac{2\alpha-2k+3}{3}}\int_{2\eta t^{\frac13}}^{2t^{\frac13}}\frac{\xi^{2\alpha+2-2k}d\xi}{(1+\xi^3)^2}+Ct^{-2}\\ &\le Ct^{-1-\frac{2\alpha-2k}{3}}\max\left(1,\left\langle t^{\frac13}\eta\right\rangle^{2\alpha-2k-3}\right)+Ct^{-2}\end{aligned}$$

if $\alpha-k<\frac72$ and then

$$\begin{aligned}&\left\|\left(\int_{2\eta}^{\infty}\frac{\xi^{2\alpha+2-2k}d\xi}{(1+t\Lambda(\xi))^2}\right)^{\frac12}\right\|_{\mathbf{L}^2(0,1)}\\ &\le Ct^{-\frac12-\frac{\alpha-k}{3}}\left(\int_0^1\max\left(1,\left\langle t^{\frac13}\eta\right\rangle^{2\alpha-2k-3}\right)d\eta\right)^{\frac12}+Ct^{-1}\\ &\le C\max\left(t^{-\frac12},t^{-\frac23-\frac{\alpha-k}{3}}\right). \qquad (3.9)\end{aligned}$$

By (3.7)-(3.9)

$$\begin{aligned}&t^{\frac12}\left\|\int_0^{\infty}\frac{|\xi-\eta|\,|\xi+|\eta||^{-k}|\eta|^{\alpha}\langle\eta\rangle^{\beta}|\partial_\xi\phi|\,d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}\right\|_{\mathbf{L}^2(0,1)}\\ &\le C\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\max\left(1,t^{-\frac16-\frac{\alpha-k}{3}}\right). \qquad (3.10)\end{aligned}$$

For $\eta>1$ we have

$$\begin{aligned}&\int_0^{\infty}\frac{|\xi-\eta|\,|\xi+|\eta||^{-k}|\eta|^{\alpha}\langle\eta\rangle^{\beta}|\partial_\xi\phi|\,d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}\\ &\le C\eta^{\alpha+\beta-k}\int_0^{2\eta}\frac{|\xi-\eta|\,|\partial_\xi\phi|\,d\xi}{1+t\eta^3(\xi-\eta)^2}+Ct^{-1}\eta^{\alpha+\beta}\int_\eta^{\infty}\xi^{-k-4}|\partial_\xi\phi|\,d\xi\\ &\le C\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\eta^{\alpha+\beta-k}\left(\int_0^{2\eta}\frac{(\xi-\eta)^2d\xi}{\left(1+t\eta^3(\xi-\eta)^2\right)^2}\right)^{\frac12}\\ &\quad+Ct^{-1}\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\eta^{\alpha+\beta}\left(\int_\eta^{\infty}\xi^{-2k-8}d\xi\right)^{\frac12}\\ &\le C\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}t^{-\frac34}\eta^{\alpha+\beta-k-\frac94},\end{aligned}$$

and then

$$\left\| \int_0^\infty \frac{|\xi-\eta|\,|\xi+|\eta||^{-k}\,|\eta|^\alpha \langle\eta\rangle^\beta\,|\partial_\xi\phi|\,d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \right\|_{\mathbf{L}^2(1,\infty)}$$

$$\leq Ct^{-\frac{3}{4}}\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\left\|\eta^{\alpha+\beta-k-\frac{9}{4}}\right\|_{\mathbf{L}^2(1,\infty)} \leq Ct^{-\frac{3}{4}}\|\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)} \tag{3.11}$$

if $\alpha+\beta-k<\frac{7}{4}$. Therefore by (3.10) and (3.11)

$$t^{\frac{1}{2}}\left\| \int_0^\infty \frac{|\xi-\eta|\,|\xi+|\eta||^{-k}\,|\eta|^\alpha \langle\eta\rangle^\beta\,|\partial_\xi\phi|\,d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

$$\leq C\max\left(1,t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right)\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}. \tag{3.12}$$

By (3.6) and (3.12)

$$\|\mathcal{I}_2\phi\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\max\left(1,t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right)\|\langle\xi\rangle\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

To estimate $\mathcal{I}_2\phi$ for $\eta<0$, we integrate by parts via the identity (2.10)

$$\mathcal{I}_2\phi = -t^{\frac{1}{2}}\int_0^\infty e^{-itS(\xi,\eta)}\left(\phi(\xi)-\phi(0)\right)\xi\partial_\xi\left(H_4\psi(\xi,\eta)\right)d\xi$$
$$-t^{\frac{1}{2}}\int_0^\infty e^{-itS(\xi,\eta)}\xi H_4\psi(\xi,\eta)\,\phi_\xi(\xi)\,d\xi.$$

Using the estimates

$$|H_4| \leq C\left(1+t\xi\left(\xi^2\langle\xi\rangle^2+\eta^2\langle\eta\rangle^2\right)\right)^{-1}, |\psi(\xi,\eta)| \leq C\,|\xi+|\eta||^{-k}\,|\eta|^\alpha\langle\eta\rangle^\beta$$

and

$$|\xi\partial_\xi\left(H_4\psi(\xi,\eta)\right)| \leq C\,|\xi+|\eta||^{-k}\,|\eta|^\alpha\langle\eta\rangle^\beta\left(1+t\xi\left(\xi^2\langle\xi\rangle^2+\eta^2\langle\eta\rangle^2\right)\right)^{-1},$$

we obtain

$$|\mathcal{I}_2\phi| \le Ct^{\frac{1}{2}} \|\langle\xi\rangle \phi_\xi\|_{\mathbf{L}^2} \int_0^\infty \frac{\xi^{\frac{1}{2}} \langle\xi\rangle^{-\frac{1}{2}} |\xi+|\eta||^{-k} |\eta|^\alpha \langle\eta\rangle^\beta d\xi}{1+t\xi\left(\xi^2\langle\xi\rangle^2+\eta^2\langle\eta\rangle^2\right)}$$

$$+Ct^{\frac{1}{2}} \int_0^\infty \frac{|\xi+|\eta||^{-k} |\eta|^\alpha \langle\eta\rangle^\beta \phi_\xi(\xi)\xi d\xi}{1+t\xi\left(\xi^2\langle\xi\rangle^2+\eta^2\langle\eta\rangle^2\right)}$$

$$\le Ct^{\frac{1}{2}} \|\langle\xi\rangle \phi_\xi\|_{\mathbf{L}^2} \left(\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k}\langle\eta\rangle^\beta d\xi}{1+t\xi\eta^2\langle\eta\rangle^2} + \int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k}\langle\xi\rangle^{\beta-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)}\right)$$

$$+Ct^{\frac{1}{2}} \|\langle\xi\rangle \phi_\xi\|_{\mathbf{L}^2}$$

$$\times\left(\left(\int_0^{|\eta|} \frac{|\eta|^{2\alpha-2k}\langle\eta\rangle^{2\beta}\xi^2 d\xi}{\left(1+t\eta^2\langle\eta\rangle^2\xi\right)^2}\right)^{\frac{1}{2}} + \left(\int_{|\eta|}^\infty \frac{\xi^{2\alpha-2k+2}\langle\xi\rangle^{2\beta} d\xi}{(1+t\Lambda(\xi))^2}\right)^{\frac{1}{2}}\right). \tag{3.13}$$

For $|\eta|<1$

$$\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k}\langle\eta\rangle^\beta d\xi}{1+t\xi\eta^2\langle\eta\rangle^2} \le C\int_0^1 \frac{|\eta|^{\alpha+\frac{3}{2}-k}\langle\eta\rangle^\beta dy}{1+t\Lambda(\eta)y}$$

$$\le C|\eta|^{\alpha+\frac{3}{2}-k}\int_0^1 \frac{dy}{1+t|\eta|^3 y} \le C|\eta|^{\alpha+\frac{3}{2}-k}\left\langle t|\eta|^3\right\rangle^{-\frac{3}{4}}$$

and for $|\eta|\ge 1$

$$\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k}\langle\eta\rangle^\beta d\xi}{1+t\xi\eta^2\langle\eta\rangle^2} \le C|\eta|^{\alpha+\beta+\frac{3}{2}-k}\int_0^1 \frac{dy}{1+t|\eta|^5 y} \le Ct^{-\frac{3}{4}}|\eta|^{\alpha+\beta-k-\frac{9}{4}}.$$

Therefore

$$\left\|\int_0^{|\eta|} \frac{|\eta|^{\alpha+\frac{1}{2}-k}\langle\eta\rangle^\beta d\xi}{1+t\xi\eta^2\langle\eta\rangle^2}\right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\left(\int_0^1 |\eta|^{2\alpha+3-2k}\left\langle t^{\frac{1}{3}}|\eta|\right\rangle^{-\frac{9}{2}} d|\eta|\right)^{\frac{1}{2}}$$

$$+Ct^{-\frac{3}{4}}\left(\int_1^\infty |\eta|^{2\alpha+2\beta-2k-\frac{9}{2}} d|\eta|\right)^{\frac{1}{2}} \le C\max\left(t^{-\frac{3}{4}}\log t, t^{-\frac{\alpha-k+2}{3}}\right) \tag{3.14}$$

if $\alpha+\beta-k<\frac{5}{2}$. In the same manner

$$\begin{aligned}
&\left\| \left(\int_0^{|\eta|} \frac{|\eta|^{2\alpha-2k} \langle\eta\rangle^{2\beta} \xi^2 d\xi}{\left(1+t\eta^2 \langle\eta\rangle^2 \xi\right)^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
\le\ & \left\| \left(\int_0^{1} \frac{|\eta|^{2\alpha-2k+3} \langle\eta\rangle^{2\beta} y^2 dy}{\left(1+t\Lambda(\eta) y\right)^2} \right)^{\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\
\le\ & C\left(\int_0^1 |\eta|^{2\alpha-2k+3} \left\langle t|\eta|^3 \right\rangle^{-2} d|\eta| \right)^{\frac{1}{2}} \\
& + Ct^{-1}\left(\int_1^\infty \eta^{2\alpha+2\beta-2k-7} d|\eta| \right)^{\frac{1}{2}} \\
\le\ & C \max\left(t^{-1}\log t, t^{-\frac{\alpha-k+2}{3}} \right). \qquad (3.15)
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle\xi\rangle^{\beta-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)} \\
\le\ & Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \int_{|\eta|t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha+\frac{1}{2}-k} d\xi}{1+\xi^3} + Ct^{-1}\int_1^\infty \xi^{\alpha+\beta-k-5} d\xi \\
\le\ & Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \min\left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{\alpha-k-\frac{3}{2}}\right) + Ct^{-1}
\end{aligned}$$

for $|\eta|<1$ if $\alpha-k>-\frac{3}{2}$, $\alpha+\beta-k<4$. And

$$\int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle\xi\rangle^{\beta-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)} \le Ct^{-1}\int_{|\eta|}^\infty \xi^{\alpha+\beta-k-5} d\xi \le Ct^{-1}|\eta|^{\alpha+\beta-k-4}$$

for $|\eta|\ge 1$. Therefore

$$\begin{aligned}
&\left\| \int_{|\eta|}^\infty \frac{\xi^{\alpha+\frac{1}{2}-k} \langle\xi\rangle^{\beta-\frac{1}{2}} d\xi}{1+t\Lambda(\xi)} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\le\ & Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}} \left(\int_0^1 \min\left(1, \left\langle t^{\frac{1}{3}}\eta \right\rangle^{2\alpha-2k-3}\right) d\eta \right)^{\frac{1}{2}} \\
& + Ct^{-1}\left(\int_0^1 d\eta \right)^{\frac{1}{2}} + Ct^{-1}\left(\int_1^\infty |\eta|^{2\alpha+2\beta-2k-8} d\eta \right)^{\frac{1}{2}} \\
\le\ & C\min\left(t^{-1}, t^{-\frac{\alpha-k+2}{3}} \right) \qquad (3.16)
\end{aligned}$$

if $\alpha+\beta-k<\frac{7}{2}$. Similarly,

$$\begin{aligned}
&\left\|\left(\int_{|\eta|}^{\infty}\frac{\xi^{2\alpha-2k+2}\langle\xi\rangle^{2\beta}\,d\xi}{(1+t\Lambda(\xi))^{2}}\right)^{\frac{1}{2}}\right\|_{\mathbf{L}^2(\mathbf{R}_-)}\\
\leq\;& Ct^{-\frac{\alpha-k}{3}-\frac{1}{2}}\left(\int_0^1\min\left(1,\left\langle t^{\frac{1}{3}}\eta\right\rangle^{2\alpha-2k-3}\right)d\eta\right)^{\frac{1}{2}}\\
&+Ct^{-1}\left(\int_1^{\infty}d\eta\right)^{\frac{1}{2}}+Ct^{-1}\left(\int_1^{\infty}\eta^{2\alpha+2\beta-2k-9}d\eta\right)^{\frac{1}{2}}\\
\leq\;& C\min\left(t^{-1},t^{-\frac{\alpha-k+2}{3}}\right).
\end{aligned}\tag{3.17}$$

Thus we get by (3.13)-(3.17)

$$\|\mathcal{I}_2\phi\|_{\mathbf{L}^2(\mathbf{R}_-)}\leq C\min\left(1,t^{-\frac{1}{6}-\frac{\alpha-k}{3}}\right)\|\langle\xi\rangle\,\phi_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Lemma 3.3 is proved. □

Next we estimate

$$\mathcal{I}_3=t^{\frac{1}{2}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi(\xi,\eta)\,d\xi$$

in $\mathbf{L}^\infty$.

Lemma 3.4. *Let $k=0,1,2,\ k-1\leq\alpha<k+4,\ \alpha+\beta<1+k$. Suppose that*

$$|\psi(\xi,\eta)|+|(\xi-\eta\theta(\eta))\,\partial_\xi\psi(\xi,\eta)|\leq|\xi+|\eta||^{-k}|\eta|^{\alpha}\langle\eta\rangle^{\beta}$$

for $\xi>0$, $\eta\in\mathbf{R}$. Then the estimate

$$\|\mathcal{I}_3\|_{\mathbf{L}^2(\mathbf{R})}\leq C\max\left(1,t^{-\frac{\alpha-k}{3}}\right)$$

is true for all $t\geq1$.

Proof. Consider $\eta>0$. Using identity (2.1) we integrate by parts

$$\begin{aligned}
\mathcal{I}_3\;=\;&-t^{\frac{1}{2}}\int_0^{\infty}e^{-itS(\xi,\eta)}(\xi-\eta)\,\partial_\xi(H_1\psi(\xi,\eta))\,d\xi\\
&+O\left(t^{\frac{1}{2}}\eta\psi(0,\eta)\langle t\Lambda(\eta)\rangle^{-1}\right).
\end{aligned}$$

Using the estimate

$$|(\xi-\eta)\,\partial_\xi(H_1\psi(\xi,\eta))|\leq\frac{C|\xi+|\eta||^{-k}|\eta|^{\alpha}\langle\eta\rangle^{\beta}}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}$$

we obtain

$$
\begin{aligned}
&|\mathcal{I}_3| \leq Ct^{\frac{1}{2}} |\eta|^{\alpha+1-k} \langle\eta\rangle^{\beta} \langle t\Lambda(\eta)\rangle^{-1} \\
&+Ct^{\frac{1}{2}} \int_0^{\infty} \frac{|\xi+|\eta||^{-k} |\eta|^{\alpha} \langle\eta\rangle^{\beta} d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)}.
\end{aligned}
\tag{3.18}
$$

We estimate the first summand as

$$
\begin{aligned}
& \left\| t^{\frac{1}{2}} |\eta|^{\alpha+1-k} \langle\eta\rangle^{\beta} \langle t\Lambda(\eta)\rangle^{-1} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq\ & Ct^{\frac{1}{2}} \left\| \eta^{\alpha+1-k} \left\langle t^{\frac{1}{3}}\eta \right\rangle^{-3} \right\|_{\mathbf{L}^2(0,1)} \\
& +Ct^{-\frac{1}{2}} \left\| \eta^{\alpha+\beta-k-4} \right\|_{\mathbf{L}^2(1,\infty)} \leq C\max\left(t^{-\frac{\alpha-k}{3}}, t^{-\frac{1}{2}}\right)
\end{aligned}
\tag{3.19}
$$

if $\alpha \geq k-1$, $\alpha+\beta-k < \frac{7}{2}$. For the second summand we get

$$
\begin{aligned}
& \int_0^{\infty} \frac{|\xi+|\eta||^{-k} |\eta|^{\alpha} \langle\eta\rangle^{\beta} d\xi}{1+t(\xi-\eta)^2(\eta+\xi)(1+\eta^2+\xi^2)} \\
\leq\ & C\int_0^{2\eta} \frac{|\eta|^{\alpha-k} \langle\eta\rangle^{\beta} d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2} + C|\eta|^{-\frac{1}{3}} \langle\eta\rangle^{\beta} \int_{\eta}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\Lambda(\xi)}.
\end{aligned}
$$

Then we find

$$
\begin{aligned}
& \int_0^{2\eta} \frac{|\eta|^{\alpha-k} \langle\eta\rangle^{\beta} d\xi}{1+t\eta\langle\eta\rangle^2(\xi-\eta)^2} \leq C\int_0^{2} \frac{\eta^{\alpha+1-k} \langle\eta\rangle^{\beta} dy}{1+t\Lambda(\eta)(y-1)^2} \\
\leq\ & C\eta^{\alpha+1-k} \langle\eta\rangle^{\beta} \langle t\Lambda(\eta)\rangle^{-\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\| \eta^{\alpha+1-k} \langle\eta\rangle^{\beta} \langle t\Lambda(\eta)\rangle^{-\frac{1}{2}} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\left\| \eta^{\alpha+1-k} \left\langle t^{\frac{1}{3}}\eta \right\rangle^{-\frac{3}{2}} \right\|_{\mathbf{L}^2(0,1)} \\
& +Ct^{-\frac{1}{2}} \left\| \eta^{\alpha+\beta+1-k-\frac{5}{2}} \right\|_{\mathbf{L}^2(1,\infty)} \leq C\max\left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}}, t^{-\frac{1}{2}}\right)
\end{aligned}
$$

for $\alpha \geq k-1$, $\alpha+\beta-k < 1$. Also the second term is estimated as

$$
|\eta|^{-\frac{1}{3}} \langle\eta\rangle^{\beta} \int_{\eta}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\Lambda(\xi)} \leq Ct^{-1}\eta^{\beta} \int_{\eta}^{\infty} \xi^{\alpha-k-5} d\xi \leq Ct^{-1}\eta^{\alpha+\beta-k-4}
$$

for $\eta \geq 1$, $t \geq 1$, $\alpha - k < 4$ and

$$\begin{aligned} & |\eta|^{-\frac{1}{3}} \langle \eta \rangle^{\beta} \int_{\eta}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\Lambda(\xi)} \\ \leq & \; C |\eta|^{-\frac{1}{3}} \int_{\eta}^{1} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t\xi^{3}} + Ct^{-1} |\eta|^{-\frac{1}{3}} \int_{1}^{\infty} \xi^{\alpha-k-5+\frac{1}{3}} d\xi \\ \leq & \; Ct^{-\frac{\alpha}{3}-\frac{1}{9}} |\eta|^{-\frac{1}{3}} \int_{\eta t^{\frac{1}{3}}}^{t^{\frac{1}{3}}} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+\xi^{3}} + Ct^{-1} |\eta|^{-\frac{1}{3}} \\ \leq & \; Ct^{-\frac{\alpha+1-k}{3}-\frac{1}{9}} |\eta|^{-\frac{1}{3}} \min\left(1, \left\langle t^{\frac{1}{3}} \eta \right\rangle^{\alpha-k-2+\frac{1}{3}}\right) + C |\eta|^{-\frac{1}{3}} t^{-1} \log t \end{aligned}$$

for $0 < \eta \leq 1$, $k - 1 \leq \alpha < k + 4$. Hence for the second summand

$$\begin{aligned} & t^{\frac{1}{2}} \left\| \int_{0}^{\infty} \frac{|\eta|^{\alpha} \left|\xi + |\eta|\right|^{-1} \langle \eta \rangle^{\beta} \langle \xi \rangle^{\delta} d\xi}{1 + t(\xi-\eta)^{2}(\eta+\xi)(1+\eta^{2}+\xi^{2})} \right\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \\ \leq & \; C \max\left(1, t^{-\frac{\alpha-k}{3}}\right). \end{aligned} \tag{3.20}$$

We apply (3.19) and (3.20) to (3.18) to get

$$\|\mathcal{I}_{3}\|_{\mathbf{L}^{2}(\mathbf{R}_{+})} \leq C \max\left(1, t^{-\frac{\alpha-k}{3}}\right)$$

for all $t \geq 1$, if $k - 1 \leq \alpha < k + 4$, $\alpha + \beta - k < 1$.

To estimate $\mathcal{I}_{3}$ for $\eta < 0$, we integrate by parts via the identity (2.10)

$$\mathcal{I}_{3} = -t^{\frac{1}{2}} \int_{0}^{\infty} e^{-itS(\xi,\eta)} \xi \partial_{\xi} \left(H_{4} \psi(\xi,\eta)\right) d\xi.$$

Using the estimate

$$\left|\xi \partial_{\xi} \left(H_{4} \psi(\xi,\eta)\right)\right| \leq \frac{C \left|\xi + |\eta|\right|^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta}}{1 + t\xi \left(\xi^{2} \langle \xi \rangle^{2} + \eta^{2} \langle \eta \rangle^{2}\right)},$$

we obtain

$$\begin{aligned} |\mathcal{I}_{3}| \leq & \; Ct^{\frac{1}{2}} \int_{0}^{\infty} \frac{\left|\xi + |\eta|\right|^{-k} |\eta|^{\alpha} \langle \eta \rangle^{\beta} d\xi}{1 + t\xi \left(\xi^{2} \langle \xi \rangle^{2} + \eta^{2} \langle \eta \rangle^{2}\right)} \\ \leq & \; Ct^{\frac{1}{2}} \int_{0}^{|\eta|} \frac{|\eta|^{\alpha-k} \langle \eta \rangle^{\beta} d\xi}{1 + t\xi\eta^{2} \langle \eta \rangle^{2}} + Ct^{\frac{1}{2}} |\eta|^{-\frac{1}{3}} \int_{|\eta|}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} \langle \xi \rangle^{\beta} d\xi}{1 + t\Lambda(\xi)}. \end{aligned} \tag{3.21}$$

We have

$$\int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle\eta\rangle^{\beta} d\xi}{1+t\xi\eta^2 \langle\eta\rangle^2} \le C|\eta|^{\alpha+1-k} \int_0^1 \frac{dy}{1+t|\eta|^3 y}$$

$$\le \quad Ct^{-\frac{\alpha+1-k}{3}} \min\left(1, \left\langle t^{\frac{1}{3}}\eta\right\rangle^{-2}\right) + Ct^{-1}\log t$$

for $|\eta| < 1$ if $\alpha \ge k-1$, and

$$\int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle\eta\rangle^{\beta} d\xi}{1+t\xi\eta^2 \langle\eta\rangle^2} \quad \le \quad C|\eta|^{\alpha+1-k+\beta} \int_0^1 \frac{dy}{1+t|\eta|^5 y} \quad \le \quad Ct^{-1}\log t$$

for $|\eta| \ge 1$ if $\alpha+\beta-k < 4$. Therefore

$$\left\| \int_0^{|\eta|} \frac{|\eta|^{\alpha-k} \langle\eta\rangle^{\beta} d\xi}{1+t\xi\eta^2 \langle\eta\rangle^2} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}} + t^{-1}\log t\right). \tag{3.22}$$

Next we find

$$|\eta|^{-\frac{1}{3}} \int_{|\eta|}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} \langle\xi\rangle^{\beta} d\xi}{1+t\Lambda(\xi)} \le Ct^{-1}|\eta|^{-\frac{1}{3}} \int_{|\eta|}^{\infty} \xi^{\alpha+\beta-k-5+\frac{1}{3}} d\eta$$

$$\le \quad Ct^{-1}|\eta|^{\alpha+\beta-k-4}$$

for $|\eta| \ge 1$ if $\alpha+\beta-k < 4$ and

$$|\eta|^{-\frac{1}{3}} \int_{|\eta|}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} \langle\xi\rangle^{\beta} d\xi}{1+t\Lambda(\xi)}$$

$$\le \quad C|\eta|^{-\frac{1}{3}} \int_{|\eta|}^{1} \frac{\xi^{\alpha-k+\frac{1}{3}} d\xi}{1+t|\xi|^3} + C|\eta|^{-\frac{1}{3}} \int_1^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} \langle\xi\rangle^{\beta} d\xi}{1+t\Lambda(\xi)}$$

$$\le \quad C|\eta|^{-\frac{1}{3}} \left(t^{-\frac{\alpha+1-k}{3}+\frac{1}{9}} \min\left(1, \left\langle t^{\frac{1}{3}}\eta\right\rangle^{\alpha-k-2+\frac{1}{3}}\right) + t^{-1}\log t\right)$$

for $|\eta| < 1$. Therefore

$$\left\| |\eta|^{-\frac{1}{3}} \int_{|\eta|}^{\infty} \frac{\xi^{\alpha-k+\frac{1}{3}} \langle\xi\rangle^{\beta} d\xi}{1+t\Lambda(\xi)} \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\left(t^{-\frac{\alpha+1-k}{3}-\frac{1}{6}} + t^{-1}\log t\right). \tag{3.23}$$

By (3.21)-(3.23)

$$\|\mathcal{I}_3\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C\max\left(1, t^{-\frac{\alpha-k}{3}}\right)$$

if $\alpha \ge k-1$, $\alpha+\beta-k < 3$. Lemma 3.4 is proved. □

3.3. Estimate for derivative of $\mathcal{V}$

In the next lemma we estimate the derivative $\partial_\eta \mathcal{V}$ through the operator $\mathcal{A}_0$.

Lemma 3.5. *Let $\alpha+\beta+j<3$, and $\alpha\geq 0$ if $j=1,2,3$, $\alpha\geq 1$ if $j=0$. Then the estimate*

$$\begin{aligned} &\left\| |\eta|^{\alpha} \langle \eta \rangle^{\beta} t\mathcal{A}_0 \mathcal{V}\xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \\ \leq\ & C \left\| \langle \xi \rangle \phi_\xi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 \text{ for } \alpha>0, j=1,2,3 \text{ or } \alpha>1, j=0 \\ \log\langle t \rangle \text{ for } \alpha=0, j=1,2,3 \text{ or } \alpha=1, j=0 \end{cases} \\ & + C\max\left(1, t^{\frac{1}{2}-\frac{\alpha+j}{3}}\right) |\phi(0)| \end{aligned}$$

is true for all $t\geq 1$.

When $j=2,3$, $\left\| \langle \xi \rangle \phi_\xi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$ can be replaced by $\left\| \xi \phi_\xi \right\|_{\mathbf{L}^2(\mathbf{R}_+)}$.

Proof. We have

$$t\mathcal{A}_0 \mathcal{V}\xi^j \phi = \left|\Lambda''(\eta)\right|^{-\frac{1}{2}} \partial_\eta \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \xi^j \phi(\xi)\, d\xi.$$

By the identity

$$\begin{aligned} \partial_\eta e^{-itS(\xi,\eta)} &= \frac{S_\eta(\xi,\eta)}{S_\xi(\xi,\eta)} \partial_\xi e^{-itS(\xi,\eta)} \\ &= -\frac{\left|\Lambda''(\eta)\right| (\xi-\eta\theta(\eta))}{\Lambda'(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)} \partial_\xi e^{-itS(\xi,\eta)} \end{aligned}$$

we integrate by parts

$$\begin{aligned} & |\eta|^{\alpha} \langle \eta \rangle^{\beta} t\mathcal{A}_0 \mathcal{V}\xi^j \phi \\ =\ & |\eta|^{\alpha} \langle \eta \rangle^{\beta} \left|\Lambda''(\eta)\right|^{\frac{1}{2}} \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \partial_\xi \left(\frac{\xi-\eta\theta(\eta)}{\Lambda'(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)} \xi^j \phi(\xi) \right) d\xi \\ =\ & \sqrt{\frac{it}{2\pi}} \left|\Lambda''(\eta)\right|^{\frac{1}{2}} \int_0^\infty e^{-itS(\xi,\eta)} \psi_1(\xi,\eta) \phi_\xi(\xi)\, d\xi \\ & + \sqrt{\frac{it}{2\pi}} \int_0^\infty e^{-itS(\xi,\eta)} \psi_2(\xi,\eta) \phi(\xi)\, d\xi, \end{aligned}$$

where

$$\begin{aligned} \psi_1(\xi,\eta) &= |\eta|^{\alpha} \langle \eta \rangle^{\beta} \frac{(\xi-\eta\theta(\eta))\xi^j}{\Lambda'(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)} \\ &= \begin{cases} \frac{|\eta|^{\alpha}\langle\eta\rangle^{\beta}\xi^j}{(\eta+\xi)(a+b\eta^2+b\xi^2)} \text{ for } \eta>0 \\ \frac{|\eta|^{\alpha}\langle\eta\rangle^{\beta}\xi^{j+1}}{\Lambda'(\xi)+\Lambda'(\eta)} \text{ for } \eta<0 \end{cases} \end{aligned}$$

and

$$\begin{aligned}\psi_2(\xi,\eta) &= |\eta|^{\alpha}\langle\eta\rangle^{\beta}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\partial_\xi\frac{(\xi-\eta\theta(\eta))\,\xi^j}{\Lambda'(\xi)-\frac{\eta}{|\eta|}\Lambda'(\eta)}\\ &= |\eta|^{\alpha}\langle\eta\rangle^{\beta}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\begin{cases}\partial_\xi\frac{\xi^j}{(\eta+\xi)(a+b\eta^2+b\xi^2)} \text{ for } \eta>0\\ \partial_\xi\frac{\xi^{j+1}}{\Lambda'(\xi)+\Lambda'(\eta)} \text{ for } \eta<0\end{cases}.\end{aligned}$$

By a direct calculation

$$\begin{aligned}\left|(\eta\partial_\eta)^k\psi_1(\xi,\eta)\right| &\le C\frac{|\eta|^{\alpha}\langle\eta\rangle^{\beta}\xi^j}{(|\eta|+\xi)\left(\langle\xi\rangle^2+\langle\eta\rangle^2\right)}\\ &\le C\begin{cases}|\eta|^{\alpha}\langle\eta\rangle^{\beta+j-3},\ j=1,2,3\\ |\eta|^{\alpha-1}\langle\eta\rangle^{\beta+\delta-2}\langle\xi\rangle^{-\delta},\ j=0\end{cases}\end{aligned}$$

for all $k=0,1,2$, $\eta\in\mathbf{R}$, $\xi>0$. Then we apply Lemma 3.2 with β replaced by $-\beta-j+3$ for $j=1,2,3$, and with α replaced by $\alpha-1$, and β replaced by $-\beta-\delta+2$ for $j=0$, and $\delta>0$ small, to find

$$\begin{aligned}&\left\|\sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^\infty e^{-itS(\xi,\eta)}\psi_1(\xi,\eta)\langle\xi\rangle\phi_\xi(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\\ \le\ & C\left\|\langle\xi\rangle\phi_\xi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\begin{cases}1 \text{ for } \alpha>0, j=1,2,3 \text{ or } \alpha>1, j=0\\ \log\langle t\rangle \text{ for } \alpha=0, j=1,2,3 \text{ or } \alpha=1, j=0\end{cases}\end{aligned}$$

if $-\beta-j+3>\alpha\ge0$ for $j=1,2,3$, and $-\beta+2>\alpha-1\ge0$ for $j=0$. Also we have

$$\begin{aligned}&|\psi_2(\xi,\eta)|+|(\xi-\eta\theta(\eta))\,\partial_\xi\psi_2(\xi,\eta)|\\ &\le C|\eta|^{\alpha+\frac{1}{2}}|\xi|^j|\xi+|\eta||^{-2}\langle\eta\rangle^{\beta+1}(\langle\xi\rangle+\langle\eta\rangle)^{-2}.\end{aligned}$$

Then we apply Lemma 3.3 with α,β replaced by $\alpha+\frac{1}{2},\beta-1$, respectively, and $k=2-j$, $j=0,1,2$ and $k=0$, $j=3$, then

$$\left\|\sqrt{\frac{it}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\psi_2(\xi,\eta)(\phi(\xi)-\phi(0))\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\le C\left\|\langle\xi\rangle\phi_\xi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

In the same manner by Lemma 3.4 we get

$$\left\|\sqrt{\frac{it}{2\pi}}\phi(0)\int_0^\infty e^{-itS(\xi,\eta)}\psi_2(\xi,\eta)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\le C\max\left(1,t^{\frac{1}{2}-\frac{\alpha+j}{3}}\right)|\phi(0)|.$$

Lemma 3.5 is proved. □

In the next lemma we estimate the derivative $\partial_t \mathcal{V}\phi$.

Lemma 3.6. *Let $\alpha > 0$, $\alpha + \beta < -1$. Then the estimate*

$$\left\| |\eta|^{\alpha} \langle \eta \rangle^{\beta} t \partial_t \mathcal{V}\phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \left\| \langle \xi \rangle \phi_{\xi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C \left| \phi(0) \right|$$

is true for all $t \geq 1$.

Proof. We have

$$\partial_a \mathcal{V}\phi = |\eta| \left| \Lambda''(\eta) \right|^{-1} \mathcal{V}\phi$$
$$- it \sqrt{\frac{it \left| \Lambda''(\eta) \right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \partial_a S(\xi,\eta) \phi(\xi) \, d\xi$$

and

$$\partial_b \mathcal{V}\phi = 2 |\eta|^3 \left| \Lambda''(\eta) \right|^{-1} \mathcal{V}\phi$$
$$- it \sqrt{\frac{it \left| \Lambda''(\eta) \right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \partial_b S(\xi,\eta) \phi(\xi) \, d\xi.$$

Using the identity

$$e^{-itS(\xi,\eta)} = -\frac{1}{itS_{\xi}(\xi,\eta)} \partial_{\xi} e^{-itS(\xi,\eta)},$$

we integrate by parts

$$\partial_a \mathcal{V}\phi = |\eta| \left| \Lambda''(\eta) \right|^{-\frac{1}{2}} \mathcal{V}\phi$$
$$- \sqrt{\frac{it \left| \Lambda''(\eta) \right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \frac{\partial_a S(\xi,\eta)}{S_{\xi}(\xi,\eta)} \phi_{\xi}(\xi) \, d\xi$$
$$- \sqrt{\frac{it \left| \Lambda''(\eta) \right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \left(\partial_{\xi} \frac{\partial_a S(\xi,\eta)}{S_{\xi}(\xi,\eta)} \right) \phi(\xi) \, d\xi.$$

Thus we get

$$|\eta|^{\alpha} \langle \eta \rangle^{\beta} \partial_a \mathcal{V}\phi = \sqrt{\frac{it}{2\pi}} \left| \Lambda''(\eta) \right|^{\frac{1}{2}} \int_0^{\infty} e^{-itS(\xi,\eta)} \psi_1(\xi,\eta) \langle \xi \rangle \phi_{\xi}(\xi) \, d\xi$$
$$+ \sqrt{\frac{it}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \psi_2(\xi,\eta) \phi(\xi) \, d\xi,$$

where

$$\begin{aligned} \psi_1(\xi,\eta) &= -|\eta|^{\alpha} \langle \eta \rangle^{\beta} \langle \xi \rangle^{-1} \frac{\partial_a S(\xi,\eta)}{S_{\xi}(\xi,\eta)} \\ &= -\frac{1}{3} |\eta|^{\alpha} \langle \eta \rangle^{\beta} \langle \xi \rangle^{-1} \begin{cases} \frac{(2\eta+\xi)(\xi-\eta)}{(\eta+\xi)(a+b\eta^2+b\xi^2)} & \text{for } \eta > 0 \\ \frac{\xi^2 (\xi^2+3\eta^2)}{\Lambda'(\xi)+\Lambda'(\eta)} & \text{for } \eta < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned}\psi_2(\xi,\eta) &= |\eta|^{\alpha+1}\langle\eta\rangle^{\beta} - |\eta|^{\alpha}\langle\eta\rangle^{\beta}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(\partial_\xi \frac{\partial_a S(\xi,\eta)}{S_\xi(\xi,\eta)}\right)\\ &= |\eta|^{\alpha+1}\langle\eta\rangle^{\beta} - \frac{1}{3}|\eta|^{\alpha}\langle\eta\rangle^{\beta}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\begin{cases}\partial_\xi \frac{(2\eta+\xi)(\xi-\eta)}{(\eta+\xi)(a+b\eta^2+b\xi^2)} \text{ for } \eta>0\\ \partial_\xi \frac{\xi\left(\xi^2+3\eta^2\right)}{\Lambda'(\xi)+\Lambda'(\eta)} \text{ for } \eta<0\end{cases}.\end{aligned}$$

We have

$$\left|(\eta\partial_\eta)^k\psi_1(\xi,\eta)\right| \le C|\eta|^{\alpha}\langle\eta\rangle^{\beta}$$

for all $k=0,1,2$, $\eta\in\mathbf{R}$, $\xi>0$. Then applying Lemma 3.2 we get for $\alpha+\beta<0$, $\alpha>0$

$$\begin{aligned}&\left\|\sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^\infty e^{-itS(\xi,\eta)}\psi_1(\xi,\eta)\langle\xi\rangle\phi_\xi(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\\ \le\ & C\left\|\langle\xi\rangle\phi_\xi\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\end{aligned}$$

Also we have

$$|\psi_2(\xi,\eta)| + |(\xi-\eta\theta(\eta))\partial_\xi\psi_2(\xi,\eta)| \le |\eta|^{\alpha+\frac{1}{2}}\langle\eta\rangle^{\beta-1}$$

for $\xi>0$, $\eta\in\mathbf{R}$. We apply Lemma 3.3 and Lemma 3.4 with α and β replaced by $\alpha+\frac{1}{2}$ and $1-\beta$, respectively, to obtain

$$\begin{aligned}&\left\|\sqrt{\frac{it}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\psi_2(\xi,\eta)\phi(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\\ \le\ & C\left\|\langle\xi\rangle\phi_\xi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C|\phi(0)|.\end{aligned}$$

Similarly, using the identity

$$e^{-itS(\xi,\eta)} = -\frac{1}{itS_\xi(\xi,\eta)}\partial_\xi e^{-itS(\xi,\eta)},$$

we integrate by parts

$$\begin{aligned}&\partial_b\mathcal{V}\phi = 2|\eta|^3\left|\Lambda''(\eta)\right|^{-1}\mathcal{V}\phi\\ &-\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\frac{\partial_b S(\xi,\eta)}{S_\xi(\xi,\eta)}\phi_\xi(\xi)\,d\xi\\ &-\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\left(\partial_\xi\frac{\partial_b S(\xi,\eta)}{S_\xi(\xi,\eta)}\right)\phi(\xi)\,d\xi.\end{aligned}$$

Thus we get

$$|\eta|^{\alpha}\langle\eta\rangle^{\beta}\partial_b\mathcal{V}\phi=\sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_3(\xi,\eta)\langle\xi\rangle\phi_{\xi}(\xi)\,d\xi$$
$$+\sqrt{\frac{it}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_4(\xi,\eta)\langle\xi\rangle\phi(\xi)\,d\xi,$$

where

$$\psi_3(\xi,\eta)=-|\eta|^{\alpha}\langle\eta\rangle^{\beta}\langle\xi\rangle^{-1}\frac{\partial_bS(\xi,\eta)}{S_{\xi}(\xi,\eta)}$$
$$=-\frac{1}{5}|\eta|^{\alpha}\langle\eta\rangle^{\beta}\langle\xi\rangle^{-1}\begin{cases}\frac{\left(4\eta^3+\xi^3+2\eta\xi^2+3\eta^2\xi\right)(\xi-\eta)}{(\eta+\xi)(a+b\eta^2+b\xi^2)}\text{ for }\eta>0\\ \frac{\xi\left(5\eta^4+\xi^4\right)}{\Lambda'(\xi)+\Lambda'(\eta)}\text{ for }\eta<0\end{cases}$$

and

$$\psi_4(\xi,\eta)=2|\eta|^{\alpha+3}\langle\eta\rangle^{\beta}\xi\langle\xi\rangle^{-1}-|\eta|^{\alpha}\langle\eta\rangle^{\beta}\langle\xi\rangle^{-1}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\left(\partial_{\xi}\frac{\partial_bS(\xi,\eta)}{S_{\xi}(\xi,\eta)}\xi\right)$$
$$=2|\eta|^{\alpha+3}\langle\eta\rangle^{\beta}\xi\langle\xi\rangle^{-1}$$
$$-\frac{1}{5}|\eta|^{\alpha}\langle\eta\rangle^{\beta}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\langle\xi\rangle^{-1}\begin{cases}\partial_{\xi}\frac{\left(4\eta^3+\xi^3+2\eta\xi^2+3\eta^2\xi\right)(\xi-\eta)\xi}{(\eta+\xi)(a+b\eta^2+b\xi^2)}\text{ for }\eta>0\\ \partial_{\xi}\frac{\xi^2\left(5\eta^4+\xi^4\right)}{\Lambda'(\xi)+\Lambda'(\eta)}\text{ for }\eta<0\end{cases}.$$

We have

$$\left|(\eta\partial_{\eta})^k\psi_3(\xi,\eta)\right|\le C|\eta|^{\alpha}\langle\eta\rangle^{\beta+1}$$

for all $k=0,1,2$, $\eta\in\mathbf{R}$, $\xi>0$. Then applying Lemma 3.2 we get for $\beta<-\alpha-1$, $\alpha>0$

$$\left\|\sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_3(\xi,\eta)\langle\xi\rangle\phi_{\xi}(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\le C\left\|\langle\xi\rangle\phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.$$

Also we have

$$|\psi_4(\xi,\eta)|+|(\xi-\eta\theta(\eta))\partial_{\xi}\psi_4(\xi,\eta)|\le|\eta|^{\alpha+\frac{1}{2}}\langle\eta\rangle^{\beta+2}$$

for $\xi>0$, $\eta\in\mathbf{R}$, $\alpha+\beta<-1$. We apply Lemma 3.3 and Lemma 3.4 to get

$$\left\|\sqrt{\frac{it}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_4(\xi,\eta)\phi(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}$$
$$\le C\left\|\langle\xi\rangle\phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}+C|\phi(0)|.$$

Using the above estimates and the following relation

$$t\partial_t \mathcal{V}\phi = a\partial_a \mathcal{V}\phi + b\partial_b \mathcal{V}\phi - |\eta| \left|\Lambda''(\eta)\right|^{-1} \left(1+2\eta^2\right) \mathcal{V}\phi$$

we find

$$\left\| |\eta|^{\alpha} \langle\eta\rangle^{\beta} t\partial_t \mathcal{V}\phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C \left\|\langle\xi\rangle \phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left|\phi(0)\right|.$$

Lemma 3.6 is proved. □

In the next lemma we estimate the commutator $[\eta, \mathcal{V}]\, \xi^j \phi$, for $j = 2, 3$.

Lemma 3.7. *Let* $\alpha \in [0,1)$, $\beta > \alpha$, $j = 2, 3$. *Then the estimate*

$$\left\| |\eta|^{\alpha} \langle\eta\rangle^{-\beta} [\eta, \mathcal{V}]\, \xi^j \phi \right\|_{\mathbf{L}^2(\mathbf{R})} \leq C t^{\frac{1}{6}-\frac{\alpha}{3}} \left\|\langle\xi\rangle \phi\right\|_{\mathbf{L}^{\infty}(\mathbf{R}_+)}$$
$$+ C \left\|\langle\xi\rangle \phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \begin{cases} 1 \text{ for } \alpha > 0 \\ \log\langle t\rangle \text{ for } \alpha = 0 \end{cases}$$

is true for all $t \geq 1$.

Proof. We have

$$[\eta, \mathcal{V}]\, \xi^j \phi = \sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} (\eta - \xi)\, \xi^j \phi(\xi)\, d\xi.$$

Using the identity

$$e^{-itS(\xi,\eta)} = -\frac{1}{itS_{\xi}(\xi,\eta)} \partial_{\xi} e^{-itS(\xi,\eta)},$$

we integrate by parts

$$[t\eta, \mathcal{V}]\, \xi^j \phi = t\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} (\eta - \xi)\, \xi^j(\xi)\, d\xi$$
$$= -\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \frac{(\eta-\xi)\,\xi^j}{iS_{\xi}(\xi,\eta)} \phi_{\xi}(\xi)\, d\xi$$
$$-\sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \left(\partial_{\xi} \frac{(\eta-\xi)\,\xi^j}{iS_{\xi}(\xi,\eta)}\right) \phi(\xi)\, d\xi.$$

Thus we get

$$|\eta|^{\alpha} \langle\eta\rangle^{-\beta} [\eta, \mathcal{V}]\, \xi^j \phi = \sqrt{\frac{it}{2\pi}} \left|\Lambda''(\eta)\right|^{\frac{1}{2}} \int_0^{\infty} e^{-itS(\xi,\eta)} \psi_5(\xi,\eta) \langle\xi\rangle \phi_{\xi}(\xi)\, d\xi$$
$$+\sqrt{\frac{it}{2\pi}} \int_0^{\infty} e^{-itS(\xi,\eta)} \psi_6(\xi,\eta)\, \phi(\xi)\, d\xi,$$

where

$$\begin{aligned}\psi_5(\xi,\eta) &= -|\eta|^{\alpha}\langle\eta\rangle^{-\beta}\langle\xi\rangle^{-1}\frac{(\eta-\xi)\xi^j}{iS_{\xi}(\xi,\eta)}\\ &= -|\eta|^{\alpha}\langle\eta\rangle^{-\beta}\langle\xi\rangle^{-1}\begin{cases}\frac{-\xi^j}{(\eta+\xi)(a+b\eta^2+b\xi^2)} \text{ for } \eta>0\\ \frac{\xi^{j+1}}{\Lambda'(\xi)+\Lambda'(\eta)} \text{ for } \eta<0\end{cases}\end{aligned}$$

and

$$\begin{aligned}\psi_6(\xi,\eta) &= -|\eta|^{\alpha}\langle\eta\rangle^{-\beta}\left|\Lambda''(\eta)\right|^{\frac12}\left(\partial_{\xi}\frac{(\eta-\xi)\xi^j}{iS_{\xi}(\xi,\eta)}\right)\\ &= -|\eta|^{\alpha}\langle\eta\rangle^{-\beta}\left|\Lambda''(\eta)\right|^{\frac12}\begin{cases}\partial_{\xi}\frac{-\xi^j}{(\eta+\xi)(a+b\eta^2+b\xi^2)} \text{ for } \eta>0\\ \partial_{\xi}\frac{\xi^{j+1}}{\Lambda'(\xi)+\Lambda'(\eta)} \text{ for } \eta<0\end{cases}.\end{aligned}$$

We have

$$\left|(\eta\partial_{\eta})^k\psi_5(\xi,\eta)\right| \le C|\eta|^{\alpha}\langle\eta\rangle^{-\beta}\langle\xi\rangle^{-1}$$

for all $k=0,1,2$, $\eta\in\mathbf{R}$, $\xi>0$. Then applying Lemma 3.2 we get for $\beta>\alpha\ge 0$

$$\begin{aligned}&\left\|\sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac12}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_5(\xi,\eta)\langle\xi\rangle\phi_{\xi}(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\\ \le\;& C\left\|\langle\xi\rangle\phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\begin{cases}1 \text{ for } \alpha>0\\ \log\langle t\rangle \text{ for } \alpha=0\end{cases}.\end{aligned}$$

Also we have

$$|\psi_6(\xi,\eta)| + |(\xi-\eta\theta(\eta))\partial_{\xi}\psi_6(\xi,\eta)| \le |\eta|^{\alpha+\frac12}\langle\eta\rangle^{-\beta+\frac12}\langle\xi\rangle$$

for $\xi>0$, $\eta\in\mathbf{R}$. We apply Lemma 3.3 with $\alpha\in[0,1)$, $\alpha-\beta<0$

$$\begin{aligned}&\left\|\sqrt{\frac{it}{2\pi}}\int_0^{\infty}e^{-itS(\xi,\eta)}\psi_6(\xi,\eta)\phi(\xi)\,d\xi\right\|_{\mathbf{L}^2(\mathbf{R})}\\ \le\;& Ct^{\frac16-\frac{\alpha}{3}}\|\phi\|_{\mathbf{L}^{\infty}(\mathbf{R}_+)} + C\left\|\langle\xi\rangle\phi_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.\end{aligned}$$

Lemma 3.7 is proved. □

§4. Estimate for the nonlinearity

In the next lemma we study the large time behavior of the nonlinearity $\mathcal{F}\mathcal{U}(-t)(u_x)^3$. Note that

$$\|\widehat{\varphi}\|_{\mathbf{Z}} = \left\|\langle\xi\rangle^2\widehat{\varphi}\right\|_{\mathbf{L}^{\infty}(\mathbf{R}_+)} + t^{-\gamma}\left\|\langle\xi\rangle\widehat{\varphi}_{\xi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$

is bounded in time for small $\gamma>0$.

Lemma 4.1. *Assume that* $\|\widehat{\varphi}\|_{\mathbf{Z}} \leq \varepsilon$. *Then the asymptotics*

$$\mathcal{FU}(-t)\,\partial_x^j (u_x)^3 = -i^{\frac{3}{2}} t^{-1} e^{it\Omega} \mathcal{D}_3 \left|\Lambda''(\xi)\right|^{-1} (i\xi)^{j+3} \widehat{\varphi}^3(\xi)$$
$$+3it^{-1}\xi^3 \left|\Lambda''(\xi)\right|^{-1} |\widehat{\varphi}(\xi)|^2 (i\xi)^j \widehat{\varphi}(\xi) + O\left(\varepsilon^3 t^{-\frac{9}{8}}\right)$$

is true for all $t \geq 1$, $\xi \geq 0$, $j = 0, 1, 2$, *where* $\widehat{\varphi}(t) = \mathcal{FU}(-t)\,u(t)$.

Proof. We have $\partial_x (u_x)^3 = 3(u_x)^2 u_{xx}$ and

$$\partial_x^2 (u_x)^3 = 3(u_x)^2 u_{xxx} + 6u_x (u_{xx})^2.$$

So we need to consider the term $\mathcal{FU}(-t)\left(\partial_x^j u\right)^2 \partial_x^k u$ with $j = 1, 2$, $k = 1, 2, 3$ for $3 \leq 2j + k \leq 5$. In view of (1.10) we find for the new dependent variable $\widehat{\varphi} = \mathcal{FU}(-t)\,u(t)$

$$\begin{aligned}
&\mathcal{FU}(-t)\left(\partial_x^j u\right)^2 \partial_x^k u \\
= \;& -i^{\frac{3}{2}} t^{-1} e^{it\Omega} \mathcal{D}_3 \mathcal{Q}(3t) \frac{1}{|\Lambda''|} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\
& + t^{-1}\mathcal{Q}(t) \frac{1}{|\Lambda''|} \left(2\left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)\left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)\left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right. \\
& \qquad \left. + \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right)\right) \\
& + 3i^{\frac{3}{2}} t^{-1} \mathcal{D}_{-1} \mathcal{Q}(-t) \frac{1}{|\Lambda''|} \left(\left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right. \\
& \left. + 2\left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)\left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)\left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right)\right) \\
& - i^{\frac{1}{2}} t^{-1} e^{it\Omega} \mathcal{D}_{-3} \mathcal{Q}(-3t) \frac{1}{|\Lambda''|} \left(\overline{\mathcal{V}(i\xi)^j \widehat{\varphi}}\right)^2 \left(\overline{\mathcal{V}(i\xi)^k \widehat{\varphi}}\right),
\end{aligned}$$

where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$.

By Lemma 2.3 with $\alpha_1 = \frac{1}{2}$, $\alpha = -\frac{1}{8}$, $\beta_1 < 2$, $\beta = \frac{7}{8}$ we have

$$\begin{aligned}
&\mathcal{Q}(3t) \frac{\theta}{|\Lambda''|} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\
= \;& \theta A^*(3t) \left|\Lambda''\right|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \\
& + Ct^{-\frac{1}{6}} \left\|\eta^{-\frac{1}{2}} \langle\eta\rangle^{-\beta_1} \left|\Lambda''\right|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
& + Ct^{-\frac{1}{8}} \left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \partial_\eta \left|\Lambda''\right|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

By Lemma 2.2 with $\alpha = -\frac{1}{2}$, $j = 1, 2, 3$

$$\left|\mathcal{V}(i\xi)^j \widehat{\varphi}\right| \leq C|\eta|^j |\widehat{\varphi}| + C|\eta|^{\frac{1}{2}} \langle\eta\rangle^{j-\frac{9}{4}} t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}. \tag{4.1}$$

Hence for $3 \leq 2j + k \leq 5$

$$\begin{aligned} &\left\|\eta^{-\frac{1}{2}} \langle\eta\rangle^{-\beta_1} \left|\Lambda''\right|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ \leq\ & C\left(\left\|\langle\xi\rangle^2 \widehat{\varphi}\right\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} + t^{-\frac{1}{2}} \|\widehat{\varphi}\|^3_{\mathbf{Z}_1}\right) \leq C\varepsilon^3. \end{aligned}$$

Using the relation

$$\partial_\eta \left|\Lambda''\right|^{-1} = \left|\Lambda''\right| t\mathcal{A}_0 \left|\Lambda''\right|^{-1} + \frac{\Lambda'''}{2\left|\Lambda''\right|^2}$$

and the Leibnitz rule

$$\begin{aligned} \mathcal{A}_0\left(\left|\Lambda''\right|^{-1} \phi_1\phi_2\phi_3\right) &= \left|\Lambda''\right|^{-1} \phi_2\phi_3\mathcal{A}_0\phi_1 \\ &\quad + \left|\Lambda''\right|^{-1} \phi_1\phi_3\mathcal{A}_0\phi_2 + \left|\Lambda''\right|^{-1} \phi_1\phi_2\mathcal{A}_0\phi_3 \end{aligned}$$

we get

$$\begin{aligned} &\left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \partial_\eta \left|\Lambda''\right|^{-1} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C\left\|\eta^{\frac{1}{8}-2} \langle\eta\rangle^{-\frac{7}{8}-2} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C\left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(t\mathcal{A}_0\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C\left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \left(t\mathcal{A}_0\mathcal{V}(i\xi)^j \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)}. \end{aligned}$$

By (4.1) we have

$$\begin{aligned} &\left\|\eta^{\frac{1}{8}-2} \langle\eta\rangle^{-\frac{7}{8}-2} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C\left(\left\|\langle\xi\rangle^2 \widehat{\varphi}\right\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} + t^{-\frac{1}{2}} \|\widehat{\varphi}\|^3_{\mathbf{Z}_1}\right) \leq C\varepsilon^3 \end{aligned}$$

and

$$\begin{aligned} &\left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right)^2 \left(t\mathcal{A}_0\mathcal{V}(i\xi)^k \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + C\left\|\eta^{\frac{1}{8}} \langle\eta\rangle^{-\frac{7}{8}} \left(\mathcal{V}(i\xi)^j \widehat{\varphi}\right) \left(\mathcal{V}(i\xi)^k \widehat{\varphi}\right) \left(t\mathcal{A}_0\mathcal{V}(i\xi)^j \widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C\|\widehat{\varphi}\|^2_{\mathbf{Z}}\left(\left\|\eta^{\frac{1}{2}} t\mathcal{A}_0\mathcal{V}(i\xi)^j \widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\|\eta^{\frac{1}{2}} \langle\eta\rangle^{-1} t\mathcal{A}_0\mathcal{V}(i\xi)^3 \widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\right). \end{aligned}$$

By Lemma 3.5 with $\alpha = \frac{1}{2}$

$$\begin{aligned} & \left\| \eta^{\frac{1}{2}} t \mathcal{A}_0 \mathcal{V} (i\xi)^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\| \eta^{\frac{1}{2}} \langle \eta \rangle^{-1} t \mathcal{A}_0 \mathcal{V} (i\xi)^3 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq \ & C \left(\| \langle \xi \rangle \phi_\xi \|_{\mathbf{L}^2(\mathbf{R}_+)} + |\phi(0)| \right) \end{aligned}$$

which implies

$$\begin{aligned} & \left\| \eta^{\frac{1}{8}} \langle \eta \rangle^{-\frac{7}{8}} \partial_\eta \left| \Lambda'' \right|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq \ & C\varepsilon^3 + \|\widehat{\varphi}\|_{\mathbf{Z}}^2 \left(\| \langle \xi \rangle \phi_\xi \|_{\mathbf{L}^2(\mathbf{R}_+)} + |\phi(0)| \right) \leq C\varepsilon^3 t^\gamma . \end{aligned}$$

By (2.11)

$$\begin{aligned} & \left\| \mathcal{Q}(3t)(1-\theta) \left| \Lambda'' \right|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ \leq \ & C t^{\frac{1}{2}} \left\| \left| \Lambda'' \right|^{-\frac{1}{2}} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^1(\mathbf{R}_-)} . \end{aligned}$$

Applying estimate of Lemma 2.2 for $\eta < 0$ with $\alpha = 0$, we have

$$\left| \mathcal{V} \xi^j \widehat{\varphi} \right| \leq C \max \left(t^{-\frac{1}{2}} \log \langle t \rangle , t^{-\frac{j}{3}} \right) \langle \eta \rangle^{j-\frac{5}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1} ,$$

which implies

$$\begin{aligned} & \left\| \mathcal{Q}(3t)(1-\theta) \left| \Lambda'' \right|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \\ \leq \ & C t^{\frac{1}{2}} \left\| \left| \Lambda'' \right|^{-\frac{1}{2}} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \right\|_{\mathbf{L}^1(\mathbf{R}_-)} \\ \leq \ & C t^{-\frac{1}{2}} \|\widehat{\varphi}\|_{\mathbf{Z}_1}^3 \left\| |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-1} \right\|_{\mathbf{L}^1(\mathbf{R}_-)} \leq C t^{-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}}^3 \leq C t^{-\frac{1}{6}} \varepsilon^3 . \end{aligned}$$

Thus we find

$$\begin{aligned} & \mathcal{Q}(3t) \frac{1}{|\Lambda''|} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) \\ = \ & \theta A^* (3t) \left| \Lambda'' \right|^{-1} \left(\mathcal{V} (i\xi)^j \widehat{\varphi} \right)^2 \left(\mathcal{V} (i\xi)^k \widehat{\varphi} \right) + O \left(t^{-\frac{1}{8}} \varepsilon^3 \right) . \end{aligned}$$

Next by Lemma 2.1 $A(t,\xi) = 1 + O\left(\left\langle t\xi^3 \right\rangle^{-1} \right)$ and $A^*(t,\xi) = 1 + O\left(\left\langle t\xi^3 \right\rangle^{-1} \right)$ for $t^{\frac{1}{3}} \xi \to \infty$. And by Lemma 2.2 with $\alpha = 0$

$$\mathcal{V} (i\xi)^j \phi = A(t,\xi) (i\xi)^j \phi(\xi) + O\left(t^{-\frac{1}{4}} \langle \xi \rangle^{j-\frac{7}{4}} \|\phi\|_{\mathbf{Z}_1} \right) .$$

Hence

$$\begin{aligned}&\mathcal{Q}(3t)\frac{1}{|\Lambda''|}\left(\mathcal{V}(i\xi)^j\widehat{\varphi}\right)^2\left(\mathcal{V}(i\xi)^k\widehat{\varphi}\right)\\=&\left|\Lambda''(\xi)\right|^{-1}(i\xi)^{2j+k}\widehat{\varphi}^3(\xi)+O\left(t^{-\frac{1}{8}}\varepsilon^3\right).\end{aligned}$$

In the same manner we find for the second summand

$$\begin{aligned}&\mathcal{Q}(t)\frac{1}{|\Lambda''|}\left(2\left(\overline{\mathcal{V}(i\xi)^j\widehat{\varphi}}\right)\left(\mathcal{V}(i\xi)^j\widehat{\varphi}\right)\left(\mathcal{V}(i\xi)^k\widehat{\varphi}\right)+\left(\mathcal{V}(i\xi)^j\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}(i\xi)^k\widehat{\varphi}}\right)\right)\\=&\frac{i^k\xi^{2j+k}}{|\Lambda''(\xi)|}\left(2+(-1)^{j-k}\right)|\widehat{\varphi}(\xi)|^2\widehat{\varphi}(\xi)+O\left(t^{-\frac{1}{8}}\varepsilon^3\right).\end{aligned}$$

In the case of the nonlinearity $(u_x)^3$ we have $j=k=1$, for the nonlinearity $\partial_x(u_x)^3=3(u_x)^2u_{xx}$ we have $j=1$, $k=2$, finally, in the case of the nonlinearity $\partial_x^2(u_x)^3=3(u_x)^2u_{xxx}+6u_x(u_{xx})^2$ we have $j=1$, $k=3$ or $j=2$, $k=1$.

Next applying the second estimate of Lemma 2.3 as above we get for the third and fourth terms

$$\begin{aligned}&\mathcal{D}_{-1}\mathcal{Q}(-t)\frac{1}{|\Lambda''|}\left(\left(\overline{\mathcal{V}(i\xi)^j\widehat{\varphi}}\right)^2\left(\mathcal{V}(i\xi)^k\widehat{\varphi}\right)\right.\\&\qquad\left.+2\left(\mathcal{V}(i\xi)^j\widehat{\varphi}\right)\left(\overline{\mathcal{V}(i\xi)^j\widehat{\varphi}}\right)\left(\overline{\mathcal{V}(i\xi)^k\widehat{\varphi}}\right)\right)=O\left(t^{-\frac{1}{8}}\varepsilon^3\right)\end{aligned}$$

and

$$\mathcal{D}_{-3}\mathcal{Q}(-3t)\frac{1}{|\Lambda''|}\left(\overline{\mathcal{V}(i\xi)^j\widehat{\varphi}}\right)^2\left(\overline{\mathcal{V}(i\xi)^k\widehat{\varphi}}\right)=O\left(t^{-\frac{1}{8}}\varepsilon^3\right).$$

Lemma 4.1 is proved. □

§5. A priori estimates

Local existence and uniqueness of solutions to the Cauchy problem (1.6) was shown in paper [19]. However we do not have any local result involving the operator $\mathcal{J}=\mathcal{U}(t)x\mathcal{U}(-t)$.

Theorem 5.1. *Assume that the initial data $u_0\in\mathbf{H}^3\cap\mathbf{H}^{1,1}$ are real-valued with a sufficiently small norm $\|u_0\|_{\mathbf{H}^3\cap\mathbf{H}^{1,1}}<\infty$. Then there exists a time T such that the Cauchy problem (1.6) has a unique local solution $\mathcal{U}(-t)u\in\mathbf{C}\left([0,T];\mathbf{H}^3\cap\mathbf{H}^{1,1}\right)$.*

Proof. Let us consider the linearized equation of (1.6) such that

$$\partial_t u-\frac{a}{3}\partial_x^3u+\frac{b}{5}\partial_x^5u=(v_x)^3, \tag{5.1}$$

$$\partial_t u_x - \frac{a}{3}\partial_x^3 u_x + \frac{b}{5}\partial_x^5 u_x = 3\left(v_x\right)^2 \partial_x u_x, \tag{5.2}$$

where $v \in \mathbf{C}\left([0,T];\mathbf{H}^3 \cap \mathbf{H}^{1,1}\right)$ and

$$\sup_{t\in[0,T]} \|v(t)\|_{\mathbf{H}^3\cap\mathbf{H}^{1,1}} \le 2\rho, \|u_0\|_{\mathbf{H}^3\cap\mathbf{H}^{1,1}} \le \rho.$$

By applying the usual energy method to (5.1) and (5.2), we have

$$\begin{aligned}\|u(t)\|_{\mathbf{H}^3} &\le \|u_0\|_{\mathbf{H}^3} + C\int_0^t \|v(s)\|_{\mathbf{H}^1}^3 ds + C\int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|u(s)\|_{\mathbf{H}^3} ds \\ &\le \|u_0\|_{\mathbf{H}^3} + \rho^3 CT + \rho^2 CT \sup_{t\in[0,T]} \|u(t)\|_{\mathbf{H}^3}.\end{aligned}$$

Therefore we have $\mathbf{H}^3$ solution. Note that $\|xu(t)\|_{\mathbf{H}^1} \le C\left\|\langle i\partial_x\rangle^{-1} xu(t)\right\|_{\mathbf{L}^2} + C\|xu_x(t)\|_{\mathbf{L}^2} + C\|u(t)\|_{\mathbf{L}^2}$. Multiplying both sides of (5.1) by $\langle i\partial_x\rangle^{-1} x$, we get

$$\begin{aligned}&\partial_t \langle i\partial_x\rangle^{-1} xu - \frac{a}{3}\partial_x^3 \langle i\partial_x\rangle^{-1} xu + \frac{b}{5}\partial_x^5 \langle i\partial_x\rangle^{-1} xu \\ = \ & -a\langle i\partial_x\rangle^{-1}\partial_x^2 u + b\langle i\partial_x\rangle^{-1}\partial_x^4 u + \langle i\partial_x\rangle^{-1}\left(\left(v_x\right)^2 xv_x\right)\end{aligned}$$

from which with the energy method it follows that

$$\begin{aligned}&\left\|\langle i\partial_x\rangle^{-1} xu(t)\right\|_{\mathbf{L}^2} \\ \le\ & \left\|\langle i\partial_x\rangle^{-1} xu_0\right\|_{\mathbf{L}^2} + C\int_0^t \left(\|u(s)\|_{\mathbf{H}^3} + \|v(s)\|_{\mathbf{H}^1}^2 \|xv_x(s)\|_{\mathbf{L}^2}\right) ds \\ \le\ & \left\|\langle i\partial_x\rangle^{-1} xu_0\right\|_{\mathbf{L}^2} + CT\sup_{t\in[0,T]}\|u(t)\|_{\mathbf{H}^3} + \rho^3 CT.\end{aligned}$$

In order to avoid the derivative loss we consider the identity $\mathcal{P} = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I}$, where the modified dilation operator $\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a$, and $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$. Similarly to (5.2) we consider

$$\partial_t \mathcal{P}u - \frac{a}{3}\partial_x^3 \mathcal{P}u + \frac{b}{5}\partial_x^5 \mathcal{P}u = 3\left(v_x\right)^2 \partial_x \mathcal{P}u \tag{5.3}$$

and

$$\partial_t \mathcal{I}u - \frac{a}{3}\partial_x^3 \mathcal{I}u + \frac{b}{5}\partial_x^5 \mathcal{I}u = 3\left(v_x\right)^2 \partial_x \mathcal{I}u + t\partial_x^3\left(v_x\right)^2 \partial_x u - t\left(v_x\right)^2 \partial_x^4 u. \tag{5.4}$$

By applying the usual energy method to (5.3) and (5.4) we have

$$\begin{aligned}\|\mathcal{P}u(t)\|_{\mathbf{L}^2} &\le \|x\partial_x u_0\|_{\mathbf{L}^2} + C\int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|\mathcal{P}u(s)\|_{\mathbf{L}^2} ds \\ &\le \|u_0\|_{\mathbf{H}^{1,1}} + \rho^2 CT \sup_{t\in[0,T]} \|\mathcal{P}u(t)\|_{\mathbf{L}^2}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{I}u(t)\|_{\mathbf{L}^2} &\le C\int_0^t \|v(s)\|_{\mathbf{H}^3}^2 \|\mathcal{I}u(t)\|_{\mathbf{L}^2}\,ds \\ &\le \rho^2 CT \sup_{t\in[0,T]} \|\mathcal{I}u(t)\|_{\mathbf{L}^2} + \rho^2 CT \sup_{t\in[0,T]} \|u(t)\|_{\mathbf{H}^3}.\end{aligned}$$

Hence by equation (5.1) we find $\|\mathcal{J}\partial_x u(t)\|_{\mathbf{L}^2} \le 5\|\mathcal{P}u(t)\|_{\mathbf{L}^2} + 2a\|\mathcal{I}u(t)\|_{\mathbf{L}^2} + 5t\|\mathcal{L}u(t)\|_{\mathbf{L}^2} \le \|u_0\|_{\mathbf{H}^{1,1}} + \rho^3 CT$. Therefore we have $\mathbf{H}^{1,1}$ solution. □

We can take $T > 1$ if the data are small in $\mathbf{H}^3 \cap \mathbf{H}^{1,1}$ and we may assume that

$$\sup_{t\in[0,1]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}^{0,2}_\infty} + \|\mathcal{J}u(t)\|_{\mathbf{H}^1}\right) \le \varepsilon. \tag{5.5}$$

To get the desired results, we prove the a priori estimates of solutions uniformly in time. Define the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]} \left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}^{0,2}_\infty} + t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1}\right).$$

Lemma 5.2. *Assume that*

$$\sup_{t\in[1,T]} \|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}^{0,2}_\infty} = \sup_{t\in[1,T]} \left\|\langle\xi\rangle^2 \widehat{\varphi}\right\|_{\mathbf{L}^\infty} \le \varepsilon$$

holds. Then there exists an ε such that the estimate

$$\sup_{t\in[1,T]} t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1} < 100\varepsilon$$

is true for all $T > 1$.

Proof. Arguing by the contradiction, we can assume that there exists a time $T > 0$ such that $\sup_{t\in[1,T]} t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1} = 100\varepsilon$. We have the identity

$$\|\mathcal{J}u(t)\|_{\mathbf{L}^2} + \|\partial_x\mathcal{J}u(t)\|_{\mathbf{L}^2} = \|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}.$$

Since

$$\begin{aligned}\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}^2 &= \|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2(|\xi|<1)}^2 + \|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2(|\xi|\ge 1)}^2 \\ &\le C\left\|\langle\xi\rangle^{-4}\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2}^2 + \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}^2,\end{aligned}$$

we obtain

$$\|\mathcal{J}u(t)\|_{\mathbf{H}^1} = \|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \le C\left\|\langle\xi\rangle^{-4}\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} + 2\|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}. \tag{5.6}$$

To estimate the norm $\left\| \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} \right\|_{\mathbf{L}^2}$ we use equation (1.10). We have

$$\begin{aligned} & \partial_t \langle \xi \rangle^{-4} \partial_\xi \widehat{\varphi} = \langle \xi \rangle^{-4} \partial_\xi \mathcal{F} \mathcal{U} (-t) u_x^3 \\ = \;& (it)^{-1} \langle \xi \rangle^{-4} \partial_\xi \mathcal{Q} \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} (i\xi) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} (i\xi) \widehat{\varphi}} \right)^3 . \end{aligned}$$

Since $\partial_\xi S(\xi, \eta) = \Lambda'(\xi) - \frac{\eta}{|\eta|} \Lambda'(\eta) \xi$, $\Lambda'(\xi) = a\xi^2 + b\xi^4$, we find

$$\begin{aligned} \partial_\xi \mathcal{Q} \phi &= it \Lambda'(\xi) \mathcal{Q} \phi - it \mathcal{Q} \frac{\eta}{|\eta|} \Lambda'(\eta) \phi \\ &= ita \left(\xi^2 \mathcal{Q} - \mathcal{Q} \eta |\eta| \right) \phi + itb \left(\xi^4 \mathcal{Q} - \mathcal{Q} \eta^3 |\eta| \right) \phi. \end{aligned}$$

Since $i\xi \mathcal{Q} \phi = \mathcal{Q} \mathcal{A} \phi$, and $\mathcal{A} = \mathcal{A}_0 + i\eta\theta(\eta)$, we get $[i\xi, \mathcal{Q}] = \mathcal{Q}(\mathcal{A} - i\eta) = \mathcal{Q}\mathcal{A}_0 - i\mathcal{Q}\eta(1-\theta)$. Hence we find

$$\begin{aligned} & \left(\xi^2 \mathcal{Q} - \mathcal{Q} \eta |\eta| \right) \phi = -\mathcal{Q} \left(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0 \right) \phi + \mathcal{Q} \left(\eta^2 \theta \phi - \eta |\eta| \right) \phi \\ & = -\mathcal{Q} \left(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0 \right) \phi + \mathcal{Q} \eta^2 (1-\theta) \phi \end{aligned}$$

and

$$\begin{aligned} & \left(\xi^4 \mathcal{Q} - \mathcal{Q} \eta^3 |\eta| \right) \phi = \xi^2 \left(\xi^2 \mathcal{Q} - \mathcal{Q} \eta |\eta| \right) \phi + \left(\xi^2 \mathcal{Q} - \mathcal{Q} \eta^2 \right) \eta |\eta| \phi \\ & = -\left(1 + \xi^2 \right) \mathcal{Q} \left(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0 \right) \phi + \xi^2 \mathcal{Q} \eta^2 (1-\theta) \phi + \mathcal{Q} (1-\theta) \eta^4 \phi. \end{aligned}$$

Hence we get

$$\begin{aligned} & \partial_\xi \mathcal{F} \mathcal{U} (-t) u_x^3 = \partial_\xi \mathcal{Q} \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1} u_x^3 \\ = \;& (it)^{-1} \partial_\xi \mathcal{Q} \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} (i\xi) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} (i\xi) \widehat{\varphi}} \right)^3 \\ = \;& -\left(a + b + b\xi^2 \right) \mathcal{Q} \left(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0 \right) \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} (i\xi) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} (i\xi) \widehat{\varphi}} \right)^3 + R_0 \end{aligned}$$

with

$$\begin{aligned} & R_0 = \left(a + b\xi^2 \right) \mathcal{Q} \eta^2 (1-\theta) \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} (i\xi) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} (i\xi) \widehat{\varphi}} \right)^3 \\ & + b \mathcal{Q} (1-\theta) \eta^4 \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} (i\xi) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} (i\xi) \widehat{\varphi}} \right)^3 . \end{aligned}$$

By inequality (3.1)

$$\begin{aligned} & \left\| \langle \xi \rangle^{-4} R_0 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C \left\| \eta^2 \langle \eta \rangle^2 \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} i\xi \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} i\xi \widehat{\varphi}} \right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \\ \le \;& C \left\| |\eta| \langle \eta \rangle^{-1} \left| \mathcal{V} \xi \widehat{\varphi} \right|^3 \right\|_{\mathbf{L}^2(\mathbf{R}_-)} \le C t^{-\frac{9}{8}} \left\| \widehat{\varphi} \right\|_{\mathbf{Z}_1}^3 \le C \varepsilon^3 t^{-\frac{9}{8}}, \end{aligned}$$

since by the second estimate of Lemma 2.2 with $j = 1$, $\alpha = \frac{1}{8}$ we have $|\mathcal{V}\xi\widehat{\varphi}| \le Ct^{-\frac{3}{8}} |\eta|^{-\frac{1}{8}} \langle\eta\rangle^{-1} \|\widehat{\varphi}\|_{\mathbf{Z}_1}$ for all $\eta < 0$. Note that

$$\begin{aligned}
&\mathcal{A} \left|\Lambda''\right|^{-1} \overline{M} \left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\
= \;& \frac{\overline{M}}{t\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}\partial_\eta \left(\frac{M}{\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{\frac{M}{\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\
= \;& 3\left|\Lambda''\right|^{-1} \overline{M} \left(M\psi + i\overline{M\psi}\right)^2 \left(M\psi_1 + i\overline{M\psi_1}\right),
\end{aligned}$$

where we denote $\psi = \mathcal{V}(i\xi)\widehat{\varphi}$, $\psi_1 = \mathcal{V}(i\xi)^2\widehat{\varphi}$

$$\begin{aligned}
&(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0)\left|\Lambda''\right|^{-1} \overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\
= \;& (\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0)\left|\Lambda''\right|^{-1} \overline{M}\left(M\psi + i\overline{M\psi}\right)^3 \\
= \;& 3\mathcal{A}_0\overline{M}\left|\Lambda''\right|^{-1}\left(M\psi + i\overline{M\psi}\right)^2\left(M\psi_1 + i\overline{M\psi_1}\right) \\
& + i\eta\theta\mathcal{A}_0\overline{M}\left|\Lambda''\right|^{-1}\left(M\psi + i\overline{M\psi}\right)^3
\end{aligned}$$

We have the commutator $[\mathcal{A}_0, M^n] = in\eta\theta M^n$, and the Leibnitz rule

$$\begin{aligned}
\mathcal{A}_0\left(\left|\Lambda''\right|^{-1}\phi_1\phi_2\phi_3\right) \;=\; & \left|\Lambda''\right|^{-1}\phi_2\phi_3\mathcal{A}_0\phi_1 \\
& + \left|\Lambda''\right|^{-1}\phi_1\phi_3\mathcal{A}_0\phi_2 + \left|\Lambda''\right|^{-1}\phi_1\phi_2\mathcal{A}_0\phi_3
\end{aligned}$$

then we get

$$\begin{aligned}
&(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0)\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\
= \;& 3\mathcal{A}_0\left(M^2\left|\Lambda''\right|^{-1}\psi^2\psi_1 + \left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi_1}\right)\right) \\
& - 3\mathcal{A}_0\left(\overline{M}^2\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_1 + 2\psi\overline{\psi\psi_1}\right) + i\overline{M}^4\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_1}\right) \\
& + i\eta\theta\mathcal{A}_0\left(M^2\left|\Lambda''\right|^{-1}\psi^3 + 3i\psi^2\overline{\psi} - 3\overline{M}^2\left|\Lambda''\right|^{-1}\psi\overline{\psi}^2 - i\overline{M}^4\left|\Lambda''\right|^{-1}\overline{\psi}^3\right) \\
= \;& 6i\eta\theta M^2\left|\Lambda''\right|^{-1}\psi^2\psi_1 - 2\eta^2\theta M^2\left|\Lambda''\right|^{-1}\psi^3 \\
& + 6i\eta\theta\overline{M}^2\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_1 + 2\psi\overline{\psi\psi_1}\right) - 6\eta^2\theta\overline{M}^2\left|\Lambda''\right|^{-1}\psi\overline{\psi}^2 \\
& - 12\eta\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_1} - 4i\eta^2\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{\psi}^3 + R_1,
\end{aligned}$$

where

$$\begin{aligned}
R_1 = \;& 3M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\psi_1 + 3\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_1 + i\psi^2\overline{\psi_1}\right) \\
& - 3\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_1 + 2\psi\overline{\psi\psi_1}\right) - 3i\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_1} \\
& + i\eta\theta M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^3 - 3\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\overline{\psi} \\
& - 3i\eta\theta\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi\overline{\psi}^2 + \eta\theta\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^3.
\end{aligned}$$

By inequality (3.1)

$$\begin{aligned}
&\left\| \langle \xi \rangle^{-4} \left(a + b\xi^2 \right) \mathcal{Q} R_1 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| R_1 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq\;& C \left\| |\eta|^{-1} \langle \eta \rangle^{-2} \left| \mathcal{V} \xi \widehat{\varphi} \right|^2 \mathcal{A}_0 \mathcal{V} \xi^2 \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ C \left\| |\eta|^{-1} \langle \eta \rangle^{-2} \left| \mathcal{V} \xi \widehat{\varphi} \right| \left| \mathcal{V} \xi^2 \widehat{\varphi} \right| \mathcal{A}_0 \mathcal{V} \xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ C \left\| \langle \eta \rangle^{-2} \left| \mathcal{V} \xi \widehat{\varphi} \right|^2 \mathcal{A}_0 \mathcal{V} \xi \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

By Lemma 2.2 with $\alpha = -\frac{1}{2}$, $j = 1, 2$ we have $\left| \mathcal{V} \left(i\xi \right)^j \widehat{\varphi} \right| \leq C \left| \eta \right|^j \left| \widehat{\varphi} \right| + C \left| \eta \right|^{\frac{1}{2}} \langle \eta \rangle^{j - \frac{9}{4}} t^{-\frac{1}{6}} \left\| \widehat{\varphi} \right\|_{\mathbf{Z}_1}$. Using Lemma 3.5 with $\alpha = 0$ and $\alpha = 1$ we obtain for $j = 1, 2$

$$\left\| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V} \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t^{\frac{1}{6}} \left\| \widehat{\varphi} \right\|_{\mathbf{Z}_1} \leq C \varepsilon t^{\frac{1}{6}}$$

and

$$\left\| |\eta| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V} \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \left\| \langle \xi \rangle \widehat{\varphi}_\xi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C \left| \widehat{\varphi}\left(0\right) \right| \leq C \varepsilon t^{\gamma}.$$

Therefore

$$\begin{aligned}
&\left\| \langle \xi \rangle^{-4} \left(a + b\xi^2 \right) \mathcal{Q} R_1 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C t^{-1} \left\| \widehat{\varphi} \right\|^2_{\mathbf{Z}_1} \left\| |\eta| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V} \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ C t^{-\frac{4}{3} + 2\gamma} \left\| \widehat{\varphi} \right\|^2_{\mathbf{Z}_1} \left\| \langle \eta \rangle^{-2} t \mathcal{A}_0 \mathcal{V} \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma - 1}.
\end{aligned}$$

Using the relation $\psi_1 = \mathcal{V} \left(i\xi \right)^2 \widehat{\varphi} = \mathcal{A} \mathcal{V} \left(i\xi \right) \widehat{\varphi} = i\eta\theta \mathcal{V} \left(i\xi \right) \widehat{\varphi} + \mathcal{A}_0 \mathcal{V} \left(i\xi \right) \widehat{\varphi} = i\eta\theta\psi + \mathcal{A}_0 \psi$ we get

$$\begin{aligned}
&\left(\mathcal{A}_0 \mathcal{A} + i\eta\theta \mathcal{A}_0 \right) \left| \Lambda'' \right|^{-1} \overline{M} \left(M \mathcal{V} \left(i\xi \right) \widehat{\varphi} + i \overline{M} \overline{\mathcal{V} \left(i\xi \right) \widehat{\varphi}} \right)^3 \\
=\;& -8\eta^2 \theta M^2 \left| \Lambda'' \right|^{-1} \psi^3 + 8 i \eta^2 \theta \overline{M}^4 \left| \Lambda'' \right|^{-1} \overline{\psi}^3 + R_2,
\end{aligned}$$

where

$$\begin{aligned}
&R_2 = 6 i\eta\theta M^2 \left| \Lambda'' \right|^{-1} \psi^2 \mathcal{A}_0 \psi \\
&+ 6 i \eta\theta \overline{M}^2 \left| \Lambda'' \right|^{-1} \left(\overline{\psi}^2 \left(\mathcal{A}_0 \psi \right) + 2 \psi \overline{\psi} \overline{\left(\mathcal{A}_0 \psi \right)} \right) \\
&+ 12 i\eta\theta \overline{M}^4 \left| \Lambda'' \right|^{-1} \overline{\psi}^2 \overline{\mathcal{A}_0 \psi} + R_1,
\end{aligned}$$

As above

$$\begin{aligned}
&\left\| \langle \xi \rangle^{-4} \left(a + b\xi^2 \right) \mathcal{Q} R_2 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq\;& C \left\| \widehat{\varphi} \right\|^2_{\mathbf{Z}_1} \left\| |\eta| \langle \eta \rangle^{-2} \mathcal{A}_0 \mathcal{V} \xi^j \widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C \varepsilon^3 t^{\gamma - 1}.
\end{aligned}$$

In the same manner we use $\psi^3 = -i\eta^3\theta\left(\mathcal{V}\widehat{\varphi}\right)^3 + R_3$, with

$$\begin{aligned} R_3 &= -\eta^2\theta\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right) + i\eta\theta\left(\mathcal{V}i\xi\widehat{\varphi}\right)\left(\mathcal{V}\widehat{\varphi}\right)\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right) \\ &\quad + \left(\mathcal{V}i\xi\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right), \end{aligned}$$

which can be estimates as

$$\begin{aligned} &\left\|\langle\xi\rangle^{-4}\left(a+b\xi^2\right)\mathcal{Q}\eta^2\theta M^2\left|\Lambda''\right|^{-1}R_3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\left\|\eta\langle\eta\rangle^{-2}R_3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & C\left\|\eta^3\langle\eta\rangle^{-2}\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & +C\left\|\eta^2\langle\eta\rangle^{-2}\left(\mathcal{V}i\xi\widehat{\varphi}\right)\left(\mathcal{V}\widehat{\varphi}\right)\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & +C\left\|\eta\langle\eta\rangle^{-2}\left(\mathcal{V}i\xi\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq\ & Ct^{-1}\left\|\widehat{\varphi}\right\|_{\mathbf{Z}}^2\left\|\eta^2 t\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}, \end{aligned}$$

since by Lemma 3.5 with $\alpha = 2$, $j = 0$ we have

$$\left\|\eta^2 t\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\left\|\langle\xi\rangle\widehat{\varphi}_\xi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} + C\left|\widehat{\varphi}(0)\right| \leq C\varepsilon t^\gamma.$$

Therefore

$$\begin{aligned} &\left(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\ =\ & 8i\eta^5\theta M^2\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3 - 8\eta^5\theta\overline{M}^4\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 \\ & -8\eta^2\theta M^2\left|\Lambda''\right|^{-1}R_3 + 8i\eta^2\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{R_3} + R_2. \end{aligned}$$

Next consider

$$\begin{aligned} &\mathcal{Q}\left(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\ =\ & 8i\mathcal{Q}M^2\eta^5\theta\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3 - 8\mathcal{Q}\overline{M}^4\eta^5\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + R_4, \end{aligned}$$

where the remainder

$$R_4 = -8\mathcal{Q}\eta^2\theta M^2\left|\Lambda''\right|^{-1}R_3 + 8i\mathcal{Q}\eta^2\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{R_3} + \mathcal{Q}R_2$$

has the estimate $\left\|\langle\xi\rangle^{-4}\left(a+b\xi^2\right)R_4\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}$. We use the identities $\mathcal{Q}(t)M^2\phi = i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\phi$ and $\mathcal{Q}(t)\overline{M}^4\phi = i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\phi$, where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$. Then we get

$$\begin{aligned} &\mathcal{Q}\left(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\ =\ & 8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\eta^5\theta\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3 \\ & -8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\eta^5\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + R_4. \end{aligned}$$

Define $\chi_1 \in \mathbf{C}^{\infty}(\mathbf{R})$ such that $\chi_1(x) = 1$ for ≤ 1 and $\chi_1(x) = 0$ for $x \geq 2$, $\chi_2(x) = 1 - \chi_1(x)$ and write

$$\begin{aligned} & \mathcal{Q}\left(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^3 \\ = \;& 8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_3\xi^3\mathcal{Q}(3t)\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\xi^3\mathcal{Q}(-3t)\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + R_5, \end{aligned}$$

where

$$\begin{aligned} & R_5 = 8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\chi_1(t^{\nu}\eta)\eta^5\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\chi_1(t^{\nu}\eta)\eta^5\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 \\ & +8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_3\left[\xi^3, \mathcal{Q}(3t)\right]\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\left[\xi^3, \mathcal{Q}(-3t)\right]\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + R_4. \end{aligned}$$

Note that

$$\begin{aligned} & \left\|\langle\xi\rangle^{-4}\left(a + b\xi^2\right)\mathcal{Q}(3t)\chi_1(t^{\nu}\eta)\eta^5\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \leq \;& C\varepsilon^3 t^{\frac{1}{2}}\int_0^{t^{-\nu}}\eta^{\frac{9}{2}}d\eta \leq C\varepsilon^3 t^{\frac{1}{2} - \frac{11}{2}\nu} \leq C\varepsilon^3 t^{\gamma - 1} \end{aligned}$$

if we choose $\nu = \frac{3}{11}$. Also we write

$$\begin{aligned} & \left[\xi^3, \mathcal{Q}(3t)\right]\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ = \;& -\xi\mathcal{Q}(3t)\mathcal{A}_0^2\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -\xi\mathcal{Q}(3t)\eta\theta\mathcal{A}_0\chi_2(t^{\nu}\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -i\xi\mathcal{Q}(3t)\mathcal{A}_0\chi_2(t^{\nu}\eta)\eta^3\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3 \\ & -i\mathcal{Q}(3t)\mathcal{A}_0\chi_2(t^{\nu}\eta)\eta^4\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3. \end{aligned} \tag{5.7}$$

Using the identity $\mathcal{A} = \mathcal{A}_0 + i\eta\theta$ and the commutator $[\mathcal{A}_0, \mu] = \frac{1}{t|\Lambda''(\eta)|}\partial_\eta\mu$

by a direct calculation we get

$$
\begin{aligned}
& \mathcal{A}_0^2 \chi_2 (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\
= \; & t^{2\nu-2} \chi_2'' (t^\nu \eta) \frac{\eta^2 \theta}{|\Lambda''|^3} (\mathcal{V}\widehat{\varphi})^3 + t^{\nu-2} \chi_2' (t^\nu \eta) \left(\frac{\eta^2 \theta}{|\Lambda''|} \right)' \left|\Lambda''\right|^{-2} (\mathcal{V}\widehat{\varphi})^3 \\
& + t^{\nu-2} \chi_2' (t^\nu \eta) \frac{2\eta\theta}{|\Lambda''|^3} (\mathcal{V}\widehat{\varphi})^3 + t^{-2} \chi_2 (t^\nu \eta) \left(\frac{2\eta\theta}{|\Lambda''|} \right)' \left|\Lambda''\right|^{-2} (\mathcal{V}\widehat{\varphi})^3 \\
& + 6 t^{\nu-1} \chi_2' (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& + 6 t^{-1} \chi_2 (t^\nu \eta) 2\eta\theta \left|\Lambda''\right|^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& + 3 \chi_2 (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi}) (\mathcal{V} i\xi\widehat{\varphi}) (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& - 6 i \chi_2 (t^\nu \eta) \eta^3 \theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V}\widehat{\varphi}) \\
& - 3 i t^{-1} \chi_2 (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-2} (\mathcal{V}\widehat{\varphi})^3 - 3 \chi_2 (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0 \mathcal{V} i\xi\widehat{\varphi}) .
\end{aligned}
$$

Then we estimate

$$
\begin{aligned}
& \left\| \langle\xi\rangle^{-4} \left(a + b\xi^2\right) \xi \mathcal{Q}(3t) \mathcal{A}_0^2 \chi_2 (t^\nu \eta) \eta^2 \theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\leq \; & C t^{\nu-2} \left(t^\nu \left\| \chi_2'' (t^\nu \eta) \frac{\eta^2 \theta}{|\Lambda''|^3} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + \left\| \chi_2' (t^\nu \eta) \left(\frac{\eta^2 \theta}{|\Lambda''|} \right)' \frac{1}{|\Lambda''|^2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right) \\
& \times \|\mathcal{V}\widehat{\varphi}\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
+ \; & C t^{\nu-2} \left(\left\| \chi_2' (t^\nu \eta) \frac{\eta\theta}{|\Lambda''|^3} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} + t^{-2} \left\| \chi_2 (t^\nu \eta) \left(\frac{2\eta\theta}{|\Lambda''|} \right)' \frac{1}{|\Lambda''|^2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \right) \\
& \times \|\mathcal{V}\widehat{\varphi}\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
+ \; & C t^{\nu-1} \left\| \chi_2' (t^\nu \eta) \eta^{\frac{1}{2}} \theta \left|\Lambda''\right|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \; & C t^{-1} \left\| \chi_2 (t^\nu \eta) \eta^{-\frac{1}{2}} \theta \left|\Lambda''\right|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \; & C \left\| \chi_2 (t^\nu \eta) \eta\theta \left|\Lambda''\right|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{-\frac{1}{2}} \mathcal{V}\xi\widehat{\varphi} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \; & C \left\| \eta\theta \left|\Lambda''\right|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^2 \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
+ \; & C t^{-1} \left\| \eta^2 \theta \left|\Lambda''\right|^{-2} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
+ \; & C \left\| \eta\theta \left|\Lambda''\right|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \|\mathcal{V}\widehat{\varphi}\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta \mathcal{A}_0 \mathcal{V}\xi\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \leq C\varepsilon^3 t^{\gamma-1}.
\end{aligned}
$$

In the same manner we estimate

$$\begin{aligned}
&\left\| \langle\xi\rangle^{-4} \left(a+b\xi^2\right) \xi \mathcal{Q}\left(3t\right) \eta\theta \mathcal{A}_0 \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ \left\| \langle\xi\rangle^{-4} \left(a+b\xi^2\right) i\xi \mathcal{Q}\left(3t\right) \mathcal{A}_0 \chi_2\left(t^\nu \eta\right) \eta^3 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ \left\| \langle\xi\rangle^{-4} \left(a+b\xi^2\right) i \mathcal{Q}\left(3t\right) \mathcal{A}_0 \chi_2\left(t^\nu \eta\right) \eta^4 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\le\ & Ct^{\nu-1} \left\| \chi_2'\left(t^\nu \eta\right) \eta^3 \theta \left|\Lambda''\right|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\|\mathcal{V}\widehat{\varphi}\right\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
&+ Ct^{-1} \left\| \chi_2\left(t^\nu \eta\right) \eta^2 \left|\Lambda''\right|^{-2} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\|\mathcal{V}\widehat{\varphi}\right\|^3_{\mathbf{L}^\infty(\mathbf{R}_+)} \\
&+ C \left\| \chi_2\left(t^\nu \eta\right) \eta^{\frac{3}{2}} \theta \left|\Lambda''\right|^{-1} \right\|_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\|\mathcal{V}\widehat{\varphi}\right\|^2_{\mathbf{L}^\infty(\mathbf{R}_+)} \left\| \eta^{\frac{3}{2}} \mathcal{A}_0 \mathcal{V}\widehat{\varphi} \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
\le\ & C\varepsilon^3 t^{\gamma-1}.
\end{aligned}$$

Thus we get $\left\| \langle\xi\rangle^{-4} \left(a+b\xi^2\right) R_5 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\varepsilon^3 t^{\gamma-1}$. Therefore we obtain the equation $\partial_t \langle\xi\rangle^{-4} \partial_\xi \widehat{\varphi} = \xi^3 e^{it\Omega}\Phi + O\left(\varepsilon^3 t^{\gamma-1}\right)$, where

$$\begin{aligned}
\Phi \ =\ & \langle\xi\rangle^{-4} \left(a+b\xi^2\right) 8 i^{\frac{3}{2}} 3^{-3} \mathcal{D}_3 \mathcal{Q}\left(3t\right) \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \\
&+ \langle\xi\rangle^{-4} \left(a+b\xi^2\right) 8 i^{\frac{1}{2}} 3^{-3} \mathcal{D}_{-3} \mathcal{Q}\left(-3t\right) \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3.
\end{aligned}$$

In the first summand of the right-hand side of the above equation we represent

$$\xi^3 e^{it\Omega}\Phi = \partial_t \left(\frac{\xi^3}{i\Omega} e^{it\Omega}\Phi \right) - \frac{\xi^3}{i\Omega} e^{it\Omega} \partial_t \Phi.$$

Hence

$$\partial_t \left(\langle\xi\rangle^{-4} \partial_\xi \widehat{\varphi} - \frac{\xi^3}{i\Omega} e^{it\Omega}\Phi \right) = -\frac{\xi^3}{i\Omega} e^{it\Omega} \partial_t \Phi + O\left(\varepsilon^3 t^{\gamma-1}\right).$$

By Lemma 2.2

$$\left\| \frac{\xi^3}{i\Omega} e^{it\Omega}\Phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C \left\| \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\varepsilon^3. \tag{5.8}$$

Also we find

$$\begin{aligned}
&\left\| \langle\xi\rangle^{-2} \partial_t \Phi \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C \left\| \langle\xi\rangle^{-4} \left(\partial_t \mathcal{Q}\left(3t\right)\right) \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ Ct^{\nu-1} \left\| \chi_2'\left(t^\nu \eta\right) \eta^3 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\
&+ C \left\| \chi_2\left(t^\nu \eta\right) \eta^2 \theta \left|\Lambda''\right|^{-1} \left(\mathcal{V}\widehat{\varphi}\right)^2 \left(\partial_t \mathcal{V}\widehat{\varphi}\right) \right\|_{\mathbf{L}^2(\mathbf{R}_+)}.
\end{aligned}$$

By the second estimate of Lemma 3.1, as above in (5.7), we have

$$
\begin{aligned}
&\left\|\langle\xi\rangle^{-4}\partial_t\mathcal{Q}(3t)\phi\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\left\|\mathcal{A}_0^2\chi_2(t^\nu\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
&+\left\|\eta\theta\mathcal{A}_0\chi_2(t^\nu\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
&+\sum_{l=1}^{4}\left\|\mathcal{A}_0\eta^l\theta\chi_2(t^\nu\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\varepsilon^3 t^{\gamma-1}.
\end{aligned}
$$

Then by Lemma 3.6 and Lemma 2.2 we find

$$
\begin{aligned}
&\left\|\chi_2(t^\nu\eta)\eta^2\theta\left|\Lambda''\right|^{-1}(\mathcal{V}\widehat{\varphi})^2(\partial_t\mathcal{V}\widehat{\varphi})\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le\ & C\left\|\langle\eta\rangle\mathcal{V}\widehat{\varphi}\right\|^2_{\mathbf{L}^\infty(\mathbf{R})}\left\|\eta\langle\eta\rangle^{-3}\partial_t\mathcal{V}\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R})} \le C\varepsilon^3 t^{\gamma-1}.
\end{aligned}
$$

Hence we get $\partial_t\left(\langle\xi\rangle^{-4}\partial_\xi\widehat{\varphi}-\frac{\xi^3}{i\Omega}e^{it\Omega}\Phi\right)=O\left(\varepsilon^3 t^{\gamma-1}\right)$. Integrating in time we find

$$
\left\|\langle\xi\rangle^{-4}\partial_\xi\widehat{\varphi}\right\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^3 t^\gamma. \tag{5.9}
$$

We next consider the estimate of $\|\partial_x\mathcal{J}u(t)\|_{\mathbf{L}^2}=\|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}$. In order to avoid the derivative loss we consider the modified dilation operator $\mathcal{P}=t\partial_t+\frac{1}{5}x\partial_x-\frac{2}{5}a\partial_a$. Then $\mathcal{P}=t\mathcal{L}+\frac{1}{5}\mathcal{J}\partial_x+\frac{2a}{5}\mathcal{I}$, where $\mathcal{I}=-\partial_a+\frac{1}{3}t\partial_x^3$. Note that the commutators $[\mathcal{P},\mathcal{L}]=-\mathcal{L}$ and $[\mathcal{P},\partial_x]=-\frac{1}{5}\partial_x$ are true. Also we have $[\mathcal{I},\mathcal{L}]=0$. Note that

$$
\begin{aligned}
\mathcal{I}u &= \mathcal{I}\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi}=-\mathcal{F}^{-1}\left(\partial_a+\frac{1}{3}it\xi^3\right)E\widehat{\varphi}\\
&= -\mathcal{F}^{-1}E\partial_a\widehat{\varphi}=\mathcal{F}^{-1}E\widetilde{\mathcal{I}}\widehat{\varphi}=\mathcal{U}(t)\mathcal{F}^{-1}\widetilde{\mathcal{I}}\widehat{\varphi},
\end{aligned}
$$

where $\widetilde{\mathcal{I}}=-\partial_a$. Hence $\|\mathcal{I}u\|_{\mathbf{L}^2}=\left\|\widetilde{\mathcal{I}}\widehat{\varphi}\right\|_{\mathbf{L}^2}$ .To estimate $\|\mathcal{P}u\|_{\mathbf{L}^2}$ we apply $\mathcal{P}$ to equation (1.6) to obtain

$$
\mathcal{L}\mathcal{P}u=(\mathcal{P}+1)\mathcal{L}u=(\mathcal{P}+1)u_x^3=3u_x^2\mathcal{P}\partial_x u+u_x^3.
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\|\mathcal{P}u\|^2_{\mathbf{L}^2} &= 3\left(\mathcal{P}u,u_x^2\mathcal{P}\partial_x u\right)+\left(\mathcal{P}u,u_x^3\right)\\
&\le C\|u_x u_{xx}\|_{\mathbf{L}^\infty}\|\mathcal{P}u\|^2_{\mathbf{L}^2}+\|\mathcal{P}u\|_{\mathbf{L}^2}\|u_x\|^3_{\mathbf{L}^6}.
\end{aligned}
$$

We have

$$
\begin{aligned}
|u_x|+|u_{xx}| &= \left|2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}i\xi\widehat{\varphi}\right|+\left|2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}(i\xi)^2\widehat{\varphi}\right|\\
&\le Ct^{-\frac{1}{2}}|\eta|^{-\frac{1}{2}}\langle\eta\rangle^{-1}\left(\left|\mathcal{V}i\xi\widehat{\varphi}\right|+\left|\mathcal{V}(i\xi)^2\widehat{\varphi}\right|\right).
\end{aligned}
$$

Therefore by the second estimate of Lemma 2.2

$$|u_x| + |u_{xx}| \le Ct^{-\frac{1}{2}-\frac{1}{6}} \|\widehat{\varphi}\|_{\mathbf{Z}_1} \le Ct^{-\frac{1}{2}-\frac{1}{6}} \left(\varepsilon + \|\langle\xi\rangle\, \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\right)$$

for $x < 0$ and by

$$\left|\mathcal{V}\xi^j \widehat{\varphi}\right| \le C\,|\eta|\,\left|\eta^{j-1}\widehat{\varphi}\right| + C\,|\eta|^{\frac{1}{2}}\,\langle\eta\rangle^{-\frac{1}{4}}\,t^{-\frac{1}{6}}\,\|\widehat{\varphi}\|_{\mathbf{Z}_1}$$

which follows from the first estimate of Lemma 2.2

$$|u_x| + |u_{xx}| \le C\varepsilon t^{-\frac{1}{2}} + Ct^{-\frac{1}{2}-\frac{1}{6}} \left(\varepsilon + \|\langle\xi\rangle\, \widehat{\varphi}_\xi\|_{\mathbf{L}^2(\mathbf{R}_+)}\right)$$

for $x \ge 0$. We can use the estimates $\|u_x u_{xx}\|_{\mathbf{L}^\infty} \le Ct^{-1}\left(\varepsilon^2 + t^{-\frac{1}{3}}\,\|\langle\xi\rangle\,\widehat{\varphi}_\xi\|^2_{\mathbf{L}^2}\right)$ and

$$\|u_x\|^3_{\mathbf{L}^6} \le \|u_x\|^2_{\mathbf{L}^\infty}\,\|u_x\|_{\mathbf{L}^2} \le Ct^{-1}\left(\varepsilon^2 + t^{-\frac{1}{3}}\,\|\langle\xi\rangle\,\widehat{\varphi}_\xi\|^2_{\mathbf{L}^2}\right)\|u_x\|_{\mathbf{L}^2}\,.$$

Then we have

$$\begin{aligned} \frac{d}{dt}\|\mathcal{P}u\|_{\mathbf{L}^2} &\le Ct^{-1}\left(\varepsilon^2 + t^{-\frac{1}{3}}\,\|\langle\xi\rangle\,\widehat{\varphi}_\xi\|^2_{\mathbf{L}^2}\right)\|\mathcal{P}u\|_{\mathbf{L}^2} \\ &\quad + Ct^{-1}\left(\varepsilon^2 + t^{-\frac{1}{3}}\,\|\langle\xi\rangle\,\widehat{\varphi}_\xi\|^2_{\mathbf{L}^2}\right)\|u_x\|_{\mathbf{L}^2} \end{aligned} \tag{5.10}$$

from which it follows that

$$\|\mathcal{P}u\|_{\mathbf{L}^2} \le \varepsilon + C\varepsilon^3 t^\gamma. \tag{5.11}$$

We have

$$\begin{aligned} \|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} &= \|\partial_x\mathcal{J}u\|_{\mathbf{L}^2} \le 5\,\|\mathcal{P}u\|_{\mathbf{L}^2} + t\,\|\mathcal{L}u\|_{\mathbf{L}^2} + \frac{2a}{5}\,\|\mathcal{I}u\|_{\mathbf{L}^2} \\ &\le 5\,\|\mathcal{P}u\|_{\mathbf{L}^2} + Ct\,\|u_x\|^3_{\mathbf{L}^6} + C\,\|\mathcal{I}u\|_{\mathbf{L}^2} \\ &\le 5\varepsilon + C\varepsilon^3 t^\gamma + C\,\|\mathcal{I}u\|_{\mathbf{L}^2}\,. \end{aligned} \tag{5.12}$$

To estimate the operator $\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3$ we use the commutator $[\mathcal{I},\mathcal{L}] = 0$, then we get

$$\begin{aligned} \mathcal{L}\mathcal{I}u &= \mathcal{I}\mathcal{L}u = \mathcal{I}u_x^3 = -\partial_a u_x^3 + \frac{1}{3}t\partial_x^3 u_x^3 \\ &= -3u_x^2\partial_a u_x + t\partial_x^2\left(u_x^2 u_{xx}\right) \\ &= 3u_x^2\mathcal{I}\partial_x u + t\left(\partial_x^2\left(u_x^2 u_{xx}\right) - u_x^2\partial_x^4 u\right). \end{aligned}$$

Consider the estimate of the term $N = t\left(\partial_x^2\left(u_x^2 u_{xx}\right) - u_x^2\partial_x^4 u\right)$. Using the factorization formula as above we find with

$$
\begin{aligned}
& \mathcal{FU}(-t)\, N = -t\xi^2 \mathcal{FU}(-t)\left(u_x^2 u_{xx}\right) - t\mathcal{FU}(-t)\left(u_x^2\partial_x^4 u\right) \\
= \;& -\xi^2 \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u_x^2 u_{xx}\right) - \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\left(u_x^2\partial_x^4 u\right) \\
= \;& -(it)^{-1}\xi^2\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
& \times\left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) \\
& -(it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
& \times\left(M\mathcal{V}(i\xi)^4\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)^4\widehat{\varphi}}\right) \\
= \;& -(it)^{-1}\left[\xi^2,\mathcal{Q}\right]\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
\times\;& \left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) + R_6 \\
= \;& -\mathcal{Q}\left(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi + i\overline{M\psi}\right)^2\left(M\psi_1 + i\overline{M\psi_1}\right) + R_6,
\end{aligned}
$$

where $\psi = \mathcal{V}(i\xi)\widehat{\varphi}$, $\psi_1 = \mathcal{V}(i\xi)^2\widehat{\varphi}$, $\psi_2 = \mathcal{V}(i\xi)^3\widehat{\varphi}$

$$
\begin{aligned}
R_6 \;=\;& -(it)^{-1}\mathcal{Q}\eta^2\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
& \times\left(M\mathcal{V}(i\xi)^2\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)^2\widehat{\varphi}}\right) \\
& -(it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
& \times\left(M\mathcal{V}(i\xi)^4\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)^4\widehat{\varphi}}\right) \\
=\;& -(it)^{-1}\mathcal{Q}\left|\Lambda''\right|^{-1}\overline{M}\left(M\mathcal{V}(i\xi)\widehat{\varphi} + i\overline{M}\overline{\mathcal{V}(i\xi)\widehat{\varphi}}\right)^2 \\
\times\;& \left(M\left[\eta^2,\mathcal{V}\right](i\xi)^2\widehat{\varphi} + i\overline{M}\overline{\left[\eta^2,\mathcal{V}\right](i\xi)^2\widehat{\varphi}}\right).
\end{aligned}
$$

We have by (3.1), (4.1) and Lemma 3.7

$$
\begin{aligned}
\left\|R_6\right\|_{\mathbf{L}^2} \;\le\;& Ct^{-1}\left\|\left|\Lambda''\right|^{-1}\left|\mathcal{V}\xi\widehat{\varphi}\right|^2\left(\left[\eta^2,\mathcal{V}\right]\xi^2\widehat{\varphi}\right)\right\|_{\mathbf{L}^2(\mathbf{R})} \\
\le\;& C\varepsilon^2 t^{-1}\left\|\left|\eta\right|\langle\eta\rangle^{-4}\left[\eta,\mathcal{V}\right]\xi^2\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R})} \\
& + C\varepsilon^2 t^{-\frac{4}{3}}\left\|\langle\eta\rangle^{-3}\left[\eta,\mathcal{V}\right]\xi^3\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R})} \le C\varepsilon^3 t^{\gamma-1}.
\end{aligned}
$$

As above we write

$$\begin{aligned} & (\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right) \\ = \;& 2\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)\left(M\psi_1+i\overline{M\psi_1}\right)^2 \\ & +\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_2+i\overline{M\psi_2}\right) \\ & +i\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right). \end{aligned}$$

Then by the commutator $[\mathcal{A}_0, M^n] = int\eta\theta M^n$, and the Leibnitz rule

$$\begin{aligned} & \mathcal{A}_0\left(\left|\Lambda''\right|^{-1}\phi_1\phi_2\phi_3\right) \\ = \;& \left|\Lambda''\right|^{-1}\phi_2\phi_3\mathcal{A}_0\phi_1+\left|\Lambda''\right|^{-1}\phi_1\phi_3\mathcal{A}_0\phi_2+\left|\Lambda''\right|^{-1}\phi_1\phi_2\mathcal{A}_0\phi_3 \end{aligned}$$

we get

$$\begin{aligned} & (\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right) \\ = \;& 2\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)\left(M\psi_1+i\overline{M\psi_1}\right)^2 \\ & +\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_2+i\overline{M\psi_2}\right) \\ & +i\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right), \end{aligned}$$

$$\begin{aligned} & 2\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)\left(M\psi_1+i\overline{M\psi_1}\right)^2 \\ = \;& 4it\eta\theta\left|\Lambda''\right|^{-1}M^2\psi\psi_1^2+4it\eta\theta\left|\Lambda''\right|^{-1}\overline{M}^2\left(\psi\overline{\psi_1}^2+2\overline{\psi}\psi_1\overline{\psi_1}\right) \\ & -4t\eta\theta\left|\Lambda''\right|^{-1}\overline{M}^4\overline{\psi\psi_1}^2 \\ & +2M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi\psi_1^2+\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(i\overline{\psi}\psi_1^2+2i\psi\psi_1\overline{\psi_1}\right) \\ & -2\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\psi\overline{\psi_1}^2+2\overline{\psi}\psi_1\overline{\psi_1}\right)-i\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi\psi_1}^2, \end{aligned}$$

$$\begin{aligned} & \mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_2+i\overline{M\psi_2}\right) \\ = \;& 2it\eta\theta\left|\Lambda''\right|^{-1}M^2\psi^2\psi_2+2it\eta\theta\left|\Lambda''\right|^{-1}\overline{M}^2\left(\overline{\psi}^2\psi_2+2\psi\overline{\psi\psi_2}\right) \\ & -4t\eta\theta\left|\Lambda''\right|^{-1}\overline{M}^4\overline{\psi}^2\overline{\psi_2} \\ & +M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\psi_2+\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_2+i\psi^2\overline{\psi_2}\right) \\ & -\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_2+2\psi\overline{\psi\psi_2}\right)-i\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_2}, \end{aligned}$$

$$\begin{aligned}
& i\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right)\\
= & -2t\eta^2\theta\left|\Lambda''\right|^{-1}M^2\psi^2\psi_1-2t\eta^2\theta\left|\Lambda''\right|^{-1}\overline{M}^2\left(\overline{\psi}^2\psi_1+2\psi\overline{\psi\psi_1}\right)\\
& -4it\eta^2\theta\left|\Lambda''\right|^{-1}\overline{M}^4\overline{\psi}^2\overline{\psi_1}\\
& +i\eta\theta M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\psi_1+i\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_1+i\psi^2\overline{\psi_1}\right)\\
& -i\eta\theta\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_1+2\psi\overline{\psi\psi_1}\right)+\eta\theta\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_1}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \left(\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right)\\
= & \; 2it\eta\theta M^2\left|\Lambda''\right|^{-1}\left(2\psi\psi_1^2+\psi^2\psi_2+i\eta\psi^2\psi_1\right)+2it\eta\theta\overline{M}^2\\
& \times\left|\Lambda''\right|^{-1}\left(2\psi\overline{\psi_1}^2+4\overline{\psi}\psi_1\overline{\psi_1}+\overline{\psi}^2\psi_2+2\psi\overline{\psi\psi_2}+i\eta\overline{\psi}^2\psi_1+2i\eta\psi\overline{\psi\psi_1}\right)\\
& -4t\eta\theta\overline{M}^4\left|\Lambda''\right|^{-1}\left(\overline{\psi\psi_1}^2+\overline{\psi}^2\overline{\psi_2}+i\eta\overline{\psi}^2\overline{\psi_1}\right)+R_7,
\end{aligned}$$

where

$$\begin{aligned}
& R_7=2M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi\psi_1^2+\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(i\overline{\psi}\psi_1^2+2i\psi\psi_1\overline{\psi_1}\right)\\
& -2\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\psi\overline{\psi_1}^2+2\overline{\psi}\psi_1\overline{\psi_1}\right)-i\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi\psi_1}^2\\
& +M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\psi_2+\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_2+i\psi^2\overline{\psi_2}\right)\\
& -\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_2+2\psi\overline{\psi\psi_2}\right)-i\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_2}\\
& +i\eta\theta M^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\psi^2\psi_1+i\eta\theta\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(2i\psi\overline{\psi}\psi_1+i\psi^2\overline{\psi_1}\right)\\
& -i\eta\theta\overline{M}^2\mathcal{A}_0\left|\Lambda''\right|^{-1}\left(\overline{\psi}^2\psi_1+2\psi\overline{\psi\psi_1}\right)+\eta\theta\overline{M}^4\mathcal{A}_0\left|\Lambda''\right|^{-1}\overline{\psi}^2\overline{\psi_1}.
\end{aligned}$$

As above

$$\begin{aligned}
& \left\|R_7\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\le C\left\||\eta|^{-1}\langle\eta\rangle^{-2}\left|\mathcal{V}\xi\langle\xi\rangle\widehat{\varphi}\right|^2\mathcal{A}_0\mathcal{V}\xi^2\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
& +C\left\|\langle\eta\rangle^{-2}\left|\mathcal{V}\xi\langle\xi\rangle\widehat{\varphi}\right|^2\mathcal{A}_0\mathcal{V}\xi\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le & \; Ct^{-1}\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_1}^2\left\||\eta|\langle\eta\rangle^{-2}t\mathcal{A}_0\mathcal{V}\xi^2\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
& +Ct^{-\frac{4}{3}+2\gamma}\left\|\widehat{\varphi}\right\|_{\mathbf{Z}_1}^2\left\|\langle\eta\rangle^{-2}t\mathcal{A}_0\mathcal{V}\xi^2\langle\xi\rangle\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\\
\le & \; C\varepsilon^3t^{\gamma-1}.
\end{aligned}$$

Using the relations $\psi_1=i\eta\theta\psi+\mathcal{A}_0\psi$, $\psi_1^2=-\eta^2\theta\psi^2+i\eta\theta\psi\mathcal{A}_0\psi+\psi_1\mathcal{A}_0\psi$ and

$$\psi_2=\mathcal{V}\left(i\xi\right)^3\widehat{\varphi}=-\eta^2\theta\psi+i\eta\theta\mathcal{A}_0\psi+\mathcal{A}_0\psi_1,$$

we get

$$\begin{aligned} & \left(\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right) \\ = \quad & -8i\eta^3\theta M^2\left|\Lambda''\right|^{-1}\psi^3+4\eta^3\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{\psi}^3+R_8, \end{aligned}$$

where

$$\begin{aligned} R_8 &= 2i\eta\theta M^2\left|\Lambda''\right|^{-1}\left(2\psi\left(i\eta\theta\psi\mathcal{A}_0\psi+\psi_1\mathcal{A}_0\psi\right)\right. \\ &\qquad \left.+\psi^2\left(i\eta\theta\mathcal{A}_0\psi+\mathcal{A}_0\psi_1\right)+i\eta\psi^2\left(\mathcal{A}_0\psi\right)\right) \\ &+2i\eta\theta\overline{M}^2\left|\Lambda''\right|^{-1}\begin{pmatrix} 2\psi\left(-i\eta\theta\overline{\psi\mathcal{A}_0\psi}+\overline{\psi_1\mathcal{A}_0\psi}\right)+4\overline{\psi}\left(i\eta\theta\psi\overline{\mathcal{A}_0\psi}+\overline{\psi_1}\mathcal{A}_0\psi\right) \\ +\overline{\psi}^2\left(i\eta\theta\mathcal{A}_0\psi+\mathcal{A}_0\psi_1\right)+2\psi\overline{\psi}\left(-i\eta\theta\overline{\mathcal{A}_0\psi}+\overline{\mathcal{A}_0\psi_1}\right) \\ +i\eta\overline{\psi}^2\mathcal{A}_0\psi+2i\eta\psi\overline{\psi}\left(\overline{\mathcal{A}_0\psi}\right) \end{pmatrix} \\ &-4\eta\theta\overline{M}^4\left|\Lambda''\right|^{-1}\left(\overline{\psi}\left(-i\eta\theta\overline{\psi\mathcal{A}_0\psi}+\overline{\psi_1\mathcal{A}_0\psi}\right)\right. \\ &+\left.\overline{\psi}^2\left(-i\eta\theta\overline{\mathcal{A}_0\psi}+\overline{\mathcal{A}_0\psi_1}\right)+i\eta\overline{\psi}^2\left(\overline{\mathcal{A}_0\psi}\right)\right) \\ &+R_7. \end{aligned}$$

We have

$$\begin{aligned} \|R_8\|_{\mathbf{L}^2(\mathbf{R}_+)} &\leq C\left\|\eta\left\langle\eta\right\rangle^{-2}\left|\mathcal{V}\xi\widehat{\varphi}\right|^2\mathcal{A}_0\mathcal{V}\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\left\|\left\langle\eta\right\rangle^{-2}\left|\mathcal{V}\xi\widehat{\varphi}\right|\left|\mathcal{V}\xi^2\widehat{\varphi}\right|\mathcal{A}_0\mathcal{V}\xi\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ &+C\left\|\left\langle\eta\right\rangle^{-2}\left|\mathcal{V}\xi\widehat{\varphi}\right|^2\mathcal{A}_0\mathcal{V}\xi^2\widehat{\varphi}\right\|_{\mathbf{L}^2(\mathbf{R}_+)}\leq C\varepsilon^3t^{\gamma-1}. \end{aligned}$$

Then applying $\psi^3=-i\eta^3\theta\left(\mathcal{V}\widehat{\varphi}\right)^3+R_3$, with

$$R_3=-\eta^2\theta\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)+i\eta\theta\left(\mathcal{V}i\xi\widehat{\varphi}\right)\left(\mathcal{V}\widehat{\varphi}\right)\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)+\left(\mathcal{V}i\xi\widehat{\varphi}\right)^2\left(\mathcal{A}_0\mathcal{V}\widehat{\varphi}\right)$$

we obtain

$$\begin{aligned} & \left(\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right) \\ = \quad & -8\eta^6\theta M^2\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3+8i\eta^6\theta\overline{M}^4\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3+R_9, \end{aligned}$$

with $R_9=-8i\eta^3\theta M^2\left|\Lambda''\right|^{-1}R_3+4\eta^3\theta\overline{M}^4\left|\Lambda''\right|^{-1}\overline{R_3}+R_8$. Then as above

$$\begin{aligned} & \mathcal{Q}\left(\mathcal{A}_0\mathcal{A}+i\eta\theta\mathcal{A}_0\right)\left|\Lambda''\right|^{-1}\overline{M}\left(M\psi+i\overline{M\psi}\right)^2\left(M\psi_1+i\overline{M\psi_1}\right) \\ = \quad & -8\mathcal{Q}M^2\eta^6\theta\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3+8i\mathcal{Q}\overline{M}^4\eta^6\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3+\mathcal{Q}R_9 \\ = \quad & -8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_3\mathcal{Q}\left(3t\right)\eta^6\theta\left|\Lambda''\right|^{-1}\left(\mathcal{V}\widehat{\varphi}\right)^3 \\ & +8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}\left(-3t\right)\eta^6\theta\left|\Lambda''\right|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3+\mathcal{Q}R_9, \end{aligned}$$

where $\Omega = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right)$ and by Lemma 3.5

$$\begin{aligned} & \|\mathcal{Q}R_9\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C \left\| \eta^2 \langle\eta\rangle^{-2} R_3 \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ \le\ & C \left\| \eta^4 \langle\eta\rangle^{-2} (\mathcal{V}\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + \left\| \eta^3 \langle\eta\rangle^{-2} (\mathcal{V}\widehat{\varphi}) (\mathcal{V}\xi\widehat{\varphi}) (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \\ & + \left\| \eta^2 \langle\eta\rangle^{-2} (\mathcal{V}\xi\widehat{\varphi})^2 (\mathcal{A}_0\mathcal{V}\widehat{\varphi}) \right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\varepsilon^3 t^{\gamma-1}. \end{aligned}$$

Since $\frac{\xi^2}{1+\xi^2} + \frac{1}{1+\xi^2} = 1$ we write

$$\begin{aligned} & \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) \left|\Lambda''\right|^{-1} \overline{M} \left(M\psi + i\overline{M\psi}\right)^2 \left(M\psi_1 + i\overline{M\psi_1}\right) \\ =\ & -8t\mathcal{Q}M^2\eta^6\theta \left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 + 8it\mathcal{Q}\overline{M}^4\eta^6\theta\left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + \mathcal{Q}R_9 \\ =\ & -\frac{8i^{\frac{1}{2}}t\xi^2}{1+\xi^2} e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\eta^6\theta\left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & + \frac{8i^{\frac{3}{2}}t\xi^2}{1+\xi^2} e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\eta^6\theta\left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 \\ & - \frac{8i^{\frac{1}{2}}t}{1+\xi^2} e^{it\Omega}\mathcal{D}_3\mathcal{Q}(3t)\eta^6\theta\left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & + \frac{8i^{\frac{3}{2}}t}{1+\xi^2} e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}(-3t)\eta^6\theta\left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 + \mathcal{Q}R_9. \end{aligned}$$

Next we use $\mathcal{Q}(i\eta)^n\theta = (i\xi)^n\mathcal{Q} - \sum_{l=0}^{n-1}(i\xi)^{n-1-l}\mathcal{Q}\mathcal{A}_0(i\eta)^l\theta$, then

$$\begin{aligned} & \mathcal{Q}(\mathcal{A}_0\mathcal{A} + i\eta\theta\mathcal{A}_0) \left|\Lambda''\right|^{-1} \overline{M} \left(M\psi + i\overline{M\psi}\right)^2 \left(M\psi_1 + i\overline{M\psi_1}\right) \\ =\ & \xi^3 e^{it\Omega}\Phi_1 + R_{10}, \end{aligned}$$

where

$$\begin{aligned} \Phi_1 =\ & \frac{8}{1+\xi^2}\left(\frac{i}{3}\right)^3 i^{\frac{3}{2}}\mathcal{D}_3\mathcal{Q}(3t)\eta^5\theta\left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & + \frac{8}{1+\xi^2} i^{\frac{1}{2}}\left(\frac{i}{3}\right)^3 \mathcal{D}_{-3}\mathcal{Q}(-3t)\eta^5\theta\left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 \\ & + \frac{8}{1+\xi^2}\left(\frac{i}{3}\right)^3 i^{\frac{3}{2}}\mathcal{D}_3\mathcal{Q}(3t)\eta^3\theta\left|\Lambda''\right|^{-1} (\mathcal{V}\widehat{\varphi})^3 \\ & + \frac{8}{1+\xi^2} i^{\frac{1}{2}}\left(\frac{i}{3}\right)^3 \mathcal{D}_{-3}\mathcal{Q}(-3t)\eta^3\theta\left|\Lambda''\right|^{-1} \left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3 \end{aligned}$$

and

$$R_{10} = -\frac{8}{1+\xi^2}\sum_{l=0}^{2}(i\xi)^{2-l}\,8i^{\frac{3}{2}}e^{it\Omega}\mathcal{D}_3\mathcal{Q}\,(3t)\,\mathcal{A}_0\,(i\eta)^l\,\eta^3\theta\,|\Lambda''|^{-1}\,(\mathcal{V}\widehat{\varphi})^3$$
$$-\ \tfrac{8}{1+\xi^2}\sum_{l=0}^{2}(i\xi)^{2-l}\,8i^{\frac{1}{2}}e^{it\Omega}\mathcal{D}_{-3}\mathcal{Q}\,(-3t)\,\mathcal{A}_0\,(i\eta)^l\,\eta^3\theta\,|\Lambda''|^{-1}\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3+\mathcal{Q}R_9.$$

We have

$$\|R_{10}\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\left\|\eta^2\,(\mathcal{V}\widehat{\varphi})^2\,(\mathcal{A}_0\mathcal{V}\widehat{\varphi})\right\|_{\mathbf{L}^2(\mathbf{R}_+)}$$
$$+Ct^{-1}\left\|\eta\,\langle\eta\rangle^{-2}\,(\mathcal{V}\widehat{\varphi})^3\right\|_{\mathbf{L}^2(\mathbf{R}_+)} \le C\varepsilon^3t^{\gamma-1}.$$

In the same way as above we represent

$$\xi^3e^{it\Omega}\Phi_1=\partial_t\left(e^{it\Omega}\frac{\xi^3}{i\Omega}\Phi_1\right)-e^{it\Omega}\frac{\xi^3}{i\Omega}\partial_t\Phi_1.$$

And using Lemma 3.1 and Lemma 3.6 we find

$$N=\mathcal{U}\,(t)\,\mathcal{F}^{-1}\partial_t\left(\frac{\xi^3}{i\Omega}e^{it\Omega}\Phi_1\right)+O\left(\varepsilon^3t^{\gamma-1}\right)=\mathcal{L}\Psi+O\left(\varepsilon^3t^{\gamma-1}\right),$$

where we denote $\Psi=\mathcal{U}\,(t)\,\mathcal{F}^{-1}\left(\frac{\xi^3}{i\Omega}e^{it\Omega}\Phi_1\right)$. Thus we get

$$\begin{aligned}\mathcal{L}\mathcal{I}u &= 3u_x^2\mathcal{I}\partial_xu+\mathcal{U}\,(t)\,\mathcal{F}^{-1}\left(t\xi^3e^{it\Omega}\Phi\right)+\mathcal{U}\,(t)\,\mathcal{F}^{-1}R_{10}\\ &= 3u_x^2\mathcal{I}\partial_xu+\mathcal{U}\,(t)\,\mathcal{F}^{-1}\partial_t\left(e^{it\Omega}\frac{\xi^3}{i\Omega}\Phi_1\right)\\ &\quad -\mathcal{U}\,(t)\,\mathcal{F}^{-1}e^{it\Omega}\frac{\xi^3}{i\Omega}\partial_t\Phi_1+\mathcal{U}\,(t)\,\mathcal{F}^{-1}R_{10}\\ &= 3u_x^2\mathcal{I}\partial_xu+O\left(\varepsilon^3t^{-1}\right)-\mathcal{L}\Psi.\end{aligned}$$

Hence

$$\mathcal{L}\left(\mathcal{I}u+\Psi\right)=3u_x^2\partial_x\left(\mathcal{I}u+\Psi\right)+O\left(\varepsilon^3t^{-1}\right)-3u_x^2\partial_x\Psi.$$

We have $\|\Psi\|_{\mathbf{L}^2}+\|\partial_x\Psi\|_{\mathbf{L}^2}\le C\varepsilon^3t^{\gamma-1}$, then we get $\|\mathcal{I}u\|_{\mathbf{L}^2}\le C\varepsilon^3t^{\gamma-1}$. Therefore by (5.12)

$$\|\xi\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}\le 5\varepsilon+C\varepsilon^3t^\gamma. \tag{5.13}$$

We apply (5.9) and (5.13) to (5.6) to get $t^{-\gamma}\,\|\mathcal{J}u\,(t)\|_{\mathbf{H}^1}\le 2\left(6\varepsilon+C\varepsilon^3\right)\le 20\varepsilon$. This is the desired contradiction. Lemma 5.2 is proved. □

Finally we estimate the norm $\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{H}^{0,2}_\infty}$.

Lemma 5.3. *Assume that* $\sup_{t\in[1,T]} t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1}\le\varepsilon$ *holds for small* $\gamma>0$. *Then there exists an* ε *such that the estimate*

$$\sup_{t\in[1,T]}\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{H}^{0,2}_\infty}=\sup_{t\in[1,T]}\left\|\langle\xi\rangle^2\,\widehat{\varphi}\right\|_{\mathbf{L}^\infty}<100\varepsilon$$

is true for all $T>1$.

Proof. Assume that there exists a time $T>0$ such that $\sup_{t\in[1,T]}\left\|\langle\xi\rangle^2\,\widehat{\varphi}\right\|_{\mathbf{L}^\infty}$ $=100\varepsilon$. By equation (1.10) for $\widehat{\varphi}=\mathcal{F}\mathcal{U}(-t)\,u(t)$, using Lemma 4.1 we get

$$\begin{aligned}&\partial_t\,(i\xi)^j\,\widehat{\varphi}(t,\xi)=\mathcal{F}\mathcal{U}(-t)\,\partial_x^j\,(u_x)^3\\ =\;&-i^{\frac{3}{2}}t^{-1}e^{it\Omega}\mathcal{D}_3\left|\Lambda''(\xi)\right|^{-1}(i\xi)^{j+3}\,\widehat{\varphi}^3(t,\xi)\\ &+3it^{-1}\xi^3\left|\Lambda''(\xi)\right|^{-1}|\widehat{\varphi}(t,\xi)|^2\,(i\xi)^j\,\widehat{\varphi}(t,\xi)+O\left(\varepsilon^3t^{-\frac{9}{8}}\right).\end{aligned}$$

Choosing $\Psi_1(t,\xi)=\exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3\int_1^t|\widehat{\varphi}(t,\xi)|^2\frac{d\tau}{\tau}\right)$, we get

$$\begin{aligned}&\partial_t\left((i\xi)^j\,\widehat{\varphi}(t,\xi)\,\Psi_1(t,\xi)\right)\\ =\;&-t^{-1}e^{it\Omega}\,(i\xi)^j\,\mathcal{D}_3\left|\Lambda''(\xi)\right|^{-1}\xi^3\widehat{\varphi}^3(t,\xi)\,\Psi_1(t,\xi)+O\left(\varepsilon^3t^{-\frac{7}{6}+3\gamma}\right).\end{aligned}$$

Integrating in time, we obtain

$$\begin{aligned}&\left|(i\xi)^j\,\widehat{\varphi}(t,\xi)\,\Psi_1(t,\xi)\right|\\ \le\;&\left|(i\xi)^j\,\widehat{\varphi}(1,\xi)\right|+C\left|\int_1^t e^{i\tau\Omega}\,(i\xi)^j\,\mathcal{D}_3\left|\Lambda''(\xi)\right|^{-1}\xi^3\widehat{\varphi}^3(\tau,\xi)\,\Psi_1(\tau,\xi)\,\frac{d\tau}{\tau}\right|\\ +\;&O\left(\varepsilon^3\right).\end{aligned}$$

Integrating by parts we get $\left|(i\xi)^j\,\widehat{\varphi}(t,\xi)\right|\le\varepsilon+O\left(\varepsilon^3\right)<2\varepsilon$ for $\xi>0$. Since the solution u is real, we have $\overline{\widehat{\varphi}(t,\xi)}=\widehat{\varphi}(t,-\xi)$. Therefore $\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{H}^{0,2}_\infty}<10\varepsilon$. This is the desired contradiction. Lemma 5.3 is proved. □

§6. Proof of Theorem 1.1

By Lemma 5.2 and Lemma 5.3, we see that a priori estimate

$$\sup_{t\in[1,T]}t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1}+\sup_{t\in[1,T]}\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{H}^{0,2}_\infty}\le C\varepsilon \tag{6.1}$$

is true for all $T>0$. We also get $\sup_{t\in[1,T]} t^{-\gamma}\|u(t)\|_{\mathbf{H}^3}\le C\varepsilon$ by the energy method and (6.1). Therefore the global existence of solutions of the Cauchy problem (1.6) follows by a standard continuation argument by the local existence Theorem 5.1. Thus we have global in time of solutions to the Cauchy problem (1.6). Time decay of solutions in $\mathbf{L}^\infty$ follows from Lemma 2.2.

Now we turn to the proof of the asymptotic formula (1.7) for the solutions u of the Cauchy problem (1.6). We need to compute the asymptotics of the function $\widehat{\varphi}(t,\xi)$. As in the proof of Lemma 5.2 we get

$$\begin{aligned}&\partial_t(\widehat{\varphi}(t,\xi)\Psi(t,\xi))\\ =\ &-t^{-1}e^{it\Omega}\mathcal{D}_3\left|\Lambda''(\xi)\right|^{-1}\xi^3\widehat{\varphi}^3(t,\xi)\Psi(t,\xi)+O\left(\varepsilon^3 t^{-\frac{7}{6}+\gamma}\right).\end{aligned}$$

Integrating by parts implies for $y(t,\xi)=\widehat{\varphi}(t,\xi)\Psi(t,\xi)$

$$\begin{aligned}&y(t,\xi)-y(s,\xi)\\ =\ &\int_s^t e^{i\tau\Omega}\mathcal{D}_3\left|\Lambda''(\xi)\right|^{-1}\xi^3\widehat{\varphi}^3(\tau,\xi)\Psi(\tau,\xi)\frac{d\tau}{\tau}+O\left(\varepsilon^3\int_s^t\tau^{-\frac{7}{6}+\gamma}d\tau\right).\end{aligned}$$

Hence we obtain

$$\|y(t)-y(s)\|_{\mathbf{L}^\infty}\le C\varepsilon^3\int_s^t\tau^{-\frac{7}{6}+\gamma}d\tau\le C\varepsilon^3 s^{-\frac{1}{6}+\gamma}$$

for any $t>s>0$. Therefore there exists a unique final state $y_+\in\mathbf{L}^\infty$ such that

$$\|y(t)-y_+\|_{\mathbf{L}^\infty}\le C\varepsilon^3 t^{-\frac{1}{6}+\gamma} \tag{6.2}$$

for all $t>0$. We write

$$\int_1^t|\widehat{\varphi}(\tau,\xi)|^2\frac{d\tau}{\tau}=\int_1^t|y(\tau,\xi)|^2\frac{d\tau}{\tau}=|y_+|^2\log t+\Phi_2(t). \tag{6.3}$$

We study the asymptotics in time of the remainder term $\Phi_2(t)$. We have

$$\Phi_2(t)-\Phi_2(s)=\int_s^t\left(|y(\tau)|^2-|y(t)|^2\right)\frac{d\tau}{\tau}+\left(|y(t)|^2-|y_+|^2\right)\log\frac{t}{s}.$$

By (6.2) we obtain $\|\Phi_2(t)-\Phi_2(s)\|_{\mathbf{L}^\infty}\le C\varepsilon^3 s^{-\delta}$ for any $t>s>0$. Hence there exists a unique real-valued function $\Phi_+\in\mathbf{L}^\infty$ such that

$$\|\Phi_2(t)-\Phi_+\|_{\mathbf{L}^\infty}\le C\varepsilon^3 t^{-\delta} \tag{6.4}$$

for all $t>0$. Representation (6.3) and estimate (6.4) yield

$$\begin{aligned}&\left\|\Psi(t,\xi)-\exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3|y_+|^2\log t+3i\left|\Lambda''(\xi)\right|^{-1}\xi^3\Phi_+\right)\right\|_{\mathbf{L}^\infty}\\ \le\ &Ct^{-\frac{1}{6}+\gamma}\end{aligned}$$

for all $t > 0$. Thus we get the large time asymptotics

$$\|\widehat{\varphi}(t,\xi) - y_+ \Psi(t,\xi)\|_{\mathbf{L}^\infty} = \|y(t) - y_+\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{6}+\gamma}$$

and

$$\begin{aligned}
&\left\| y_+ \Psi(t,\xi) - y_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |y_+|^2 \log t + 3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 \Phi_+\right)\right\|_{\mathbf{L}^\infty} \\
\le\ & Ct^{-\frac{1}{6}+\gamma}.
\end{aligned}$$

Therefore we obtain the estimate

$$\left\|\widehat{\varphi}(t,\xi) - W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{6}+\gamma}$$

with $W_+ = y_+ e^{3i|\Lambda''(\xi)|^{-1}\xi^3\Phi_+}$. Similarly, we get

$$\left\|\xi\widehat{\varphi}(t,\xi) - \xi W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{6}+\gamma}.$$

Using the factorization of $\mathcal{U}(t)$ we have

$$\begin{aligned}
&\left\|\partial_x u(t) - 2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}i\xi\left(W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right)\right\|_{\mathbf{L}^\infty} \\
\le\ & C\left\|\mathcal{D}_t\mathcal{B}M\left(i\xi\widehat{\varphi}(t) - i\xi W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right)\right\|_{\mathbf{L}^\infty} \\
\le\ & Ct^{-\frac{1}{2}-\frac{1}{6}+\gamma}.
\end{aligned}$$

By Lemma 2.2 we have

$$\begin{aligned}
&\left|2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}i\xi\left(W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right)\right| \\
\le\ & Ct^{-\frac{1}{2}}|\eta|^{-\frac{1}{2}}\langle\eta\rangle^{-1}\left|\mathcal{V}i\xi\left(W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right)\right| \\
\le\ & Ct^{-\frac{1}{2}-\frac{1}{6}}\left(\|W_+\|_{\mathbf{L}^\infty} + \left\|\xi\partial_\xi W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right\|_{\mathbf{L}^2}\right) \\
\le\ & C\varepsilon t^{-\frac{1}{2}-\frac{1}{6}+\gamma}
\end{aligned}$$

for $x < 0$ and

$$\begin{aligned}
&\Big|2\mathrm{Re}\mathcal{D}_t\mathcal{B}M\mathcal{V}i\xi\left(W_+ \exp\left(3i\left|\Lambda''(\xi)\right|^{-1}\xi^3 |W_+|^2 \log t\right)\right) \\
&-2\mathrm{Re}\mathcal{D}_t\mathcal{B}Mi\eta\left(W_+ \exp\left(3i\left|\Lambda''(\eta)\right|^{-1}\eta^3 |W_+|^2 \log t\right)\right)\Big| \\
\le\ & C\varepsilon t^{-\frac{1}{2}-\frac{1}{6}+\gamma}
\end{aligned}$$

for $x \ge 0$. This completes the proof of asymptotics (1.7). Theorem 1.1 is proved.

§7. Appendix

We collect definitions of operators used in this paper.

$$\mathcal{L} = \partial_t - \frac{a}{3}\partial_x^3 + \frac{b}{5}\partial_x^5,$$

$$\left(\mathcal{D}_t\phi\right)(x) = (it)^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right), \mathcal{D}_t^{-1}\phi = (it)^{\frac{1}{2}}\phi(xt),$$

$$\mathcal{U}(t) = \mathcal{F}^{-1}E\mathcal{F},\ E = E(t,\xi) = e^{-it\Lambda(\xi)},$$

$$\mathcal{F}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\phi(x)\,dx,\ \mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\phi(\xi)d\xi,$$

$$\Lambda(\xi) = \frac{a}{3}\xi^3 + \frac{b}{5}\xi^5,$$

$$M = M(t,\eta) = e^{it(\eta\Lambda'(\eta)-\Lambda(\eta))\theta(\eta)},$$

$$\theta(x) = 1 \text{ for } x \geq 0;\ \theta(x) = 0 \text{ for } x < 0,$$

$$\mathcal{V}\phi = \sqrt{\frac{it\left|\Lambda''(\eta)\right|}{2\pi}}\int_0^\infty e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi,$$

$$\mathcal{Q}\phi = \sqrt{\frac{t}{2\pi i}}\int_{\mathbf{R}} e^{itS(\xi,\eta)}\phi(\eta)\left|\Lambda''(\eta)\right|^{\frac{1}{2}}d\eta,$$

$$A(t,\eta) = \sqrt{\frac{it}{2\pi}}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\int_0^\infty e^{-itS(\xi,\eta)}\chi\left(\xi\eta^{-1}\right)d\xi,$$

$$A^*(t,\xi) = \sqrt{\frac{t}{2\pi i}}\int_0^\infty e^{itS(\xi,\eta)}\left|\Lambda''(\eta)\right|^{\frac{1}{2}}\chi\left(\xi\eta^{-1}\right)d\eta,$$

$$\chi(z) \in \mathbf{C}^2(\mathbf{R}) : \chi(z) = 0 \text{ for } z \leq \frac{1}{3};$$

$$\chi(z) = 1 \text{ for } \frac{2}{3} \leq z \leq \frac{3}{2};\ \chi(z) = 0 \text{ for } z \geq 3,$$

$$S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta),$$

$$\left(\mathcal{B}\phi\right)(x) = \left|\Lambda''(\eta(x))\right|^{-\frac{1}{2}}\phi(\eta(x)),$$

$$\left(\mathcal{B}^{-1}\phi\right)(\eta) = \left|\Lambda''(\eta)\right|^{\frac{1}{2}}\phi\left(\frac{\eta}{|\eta|}\Lambda'(\eta)\right),$$

$$\eta(x) = \frac{x}{\sqrt{2b}\,|x|}\sqrt{\sqrt{4b\,|x| + a^2} - a},$$

$$\mathcal{A} = \frac{\overline{M}}{t\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}\partial_\eta\frac{M}{\left|\Lambda''(\eta)\right|^{\frac{1}{2}}} = \mathcal{A}_0 + i\eta\theta(\eta),$$

$$\mathcal{A}_0 = \frac{1}{t\left|\Lambda''(\eta)\right|^{\frac{1}{2}}}\partial_\eta \frac{1}{\left|\Lambda''(\eta)\right|^{\frac{1}{2}}},$$

$$\mathcal{J} = x - t\Lambda'(-i\partial_x),$$

$$\mathcal{P} = t\partial_t + \frac{1}{5}x\partial_x - \frac{2}{5}a\partial_a = t\mathcal{L} + \frac{1}{5}\mathcal{J}\partial_x + \frac{2a}{5}\mathcal{I},$$

$$\mathcal{I} = -\partial_a + \frac{1}{3}t\partial_x^3,$$

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}\left(\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{H}^{0,2}_\infty} + t^{-\gamma}\|\mathcal{J}u(t)\|_{\mathbf{H}^1}\right).$$

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