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Certain curvature properties of $N(k)$-quasi Einstein manifolds

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Abstract. In this paper, we consider $N(k)$-quasi Einstein manifolds satisfying the curvature conditions $\bar{P}(\xi, X) \cdot W_2 = 0$ and $\bar{P}(\xi, X) \cdot H = 0$, where $\bar{P}$, W_2 and H are the pseudo projective, W_2 and conharmonic curvature tensors respectively. We study pseudo projectively symmetric $N(k)$-quasi Einstein manifolds and show that there does not exist a pseudo projectively semisymmetric $N(k)$-quasi Einstein manifold. Also, we construct some examples to support the existence of such manifolds..

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§1. Introduction

An n-dimensional Riemannian or semi-Riemannian manifold $M(n > 2)$ is said to be an Einstein manifold if it satisfies

$$S = \frac{r}{n} g, \tag{1.1}$$

where S and r are the Ricci tensor and the scalar curvature respectively. Equation (1.1) is called the Einstein metric condition [1]. Einstein manifolds are important in the study of Riemannian geometry and in general theory of relativity. Also, Einstein manifolds form a natural subclass of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [1].

The notion of a quasi Einstein manifold was introduced during the study of exact solutions to the Einstein field equations and consideration of quasi-umbilical hypersurfaces [3]. A non-flat Riemannian manifold (M, g) is called a quasi Einstein manifold its Ricci tensor S satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.2}$$

$\forall X, Y \in T(M)$ where a and b are smooth functions, $b \neq 0$ called the associated scalars, η is a non zero 1-form defined by

$$g(X,\xi) = \eta(X), \quad g(\xi,\xi) = 1, \tag{1.3}$$

called the associated 1-form and the unit vector field ξ is called the generator of the manifold. Chaki [2], Guha [13], De and Ghosh [7, 8] and several other geometers continued the study of quasi Einstein manifolds. Özgür also studied generalized quasi Einstein manifolds [17] and super quasi Einstein manifolds [18]. The k-nullity distribution of a Riemannian manifold M is defined as

$$N(k) : p \to N_p(k) = \{Z \in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}$$

for some smooth function k [24]. If the generator ξ in a quasi Einstein manifold M belongs to some k-nullity distribution, then M is called an $N(k)$-quasi Einstein manifold [19].

In an n-dimensional $N(k)$-quasi Einstein manifold, k is not arbitrary and is given by [19]

$$k = \frac{a+b}{n-1}. \tag{1.4}$$

Also, we have [19]

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{1.5}$$

$$R(X,\xi)Y = k[\eta(Y)X - g(X,Y)\xi] = -R(\xi,X)Y, \tag{1.6}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]. \tag{1.7}$$

In 2011, Hosseinzadeh and Taleshian [14] considered $N(k)$-quasi Einstein manifolds satisfying the curvature conditions $C(\xi,X)\cdot S = 0$, $\tilde{C}(\xi,X)\cdot S = 0$, $\bar{P}(\xi,X)\cdot C = 0$, $P(\xi,X)\cdot \tilde{C} = 0$ and $\bar{P}(\xi,X)\cdot \tilde{C} = 0$, where $C, \tilde{C}, P$ and $\bar{P}$ denote the conformal, quasi conformal, projective and pseudo projective curvature tensor respectively. Hui and Lemence [15] studied $N(k)$-quasi Einstein manifolds admitting W_2-curvature tensor and generalized Ricci recurrent $N(k)$-quasi Einstein manifolds. Chaubey [4] studied $N(k)$-quasi Einstein manifolds satisfying the curvature conditions $P(\xi,X)\cdot W^* = 0$, $P(\xi,X)\cdot \tilde{C} = 0$

and pseudo Ricci symmetric quasi Einstein manifolds, where W^* is the m-projective curvature tensor and $\tilde{C}$ is the concircular curvature tensor. $N(k)$-quasi Einstein manifolds have been studied by several authors such as Tripathi and Kim [25], De and Mallick [9], Yildiz et al. [26] and others.

This paper is organized as: After the preliminaries in Section 2, we construct some examples of $N(k)$-quasi Einstein manifolds in Section 3. In Section 4, we study certain properties of the pseudo projective curvature tensor in $N(k)$-quasi Einstein manifolds. Also, it is shown that there is no pseudo projective-symmetric $N(k)$-quasi Einstein manifold. Finally, we obtain a condition for an $N(k)$-quasi Einstein manifold to be conharmonically pseudosymmetric.

§2. Preliminaries

From (1.2) and (1.3), we have

$$r = an + b, \qquad QX = aX + b\eta(X)\xi, \tag{2.1}$$

$$S(X,\xi) = k(n-1)\eta(X), \tag{2.2}$$

where r is the scalar curvature and Q is the Ricci operator.

The pseudo projective curvature tensor [21], the conharmonic curvature tensor [10] and the W_2-curvature tensor [20] are defined as

$$\begin{aligned}\bar{P}(X,Y)Z = {} & \alpha R(X,Y)Z + \beta\big[S(Y,Z)X - S(X,Z)Y\big] \\ & - \frac{r}{n}\Big[\frac{\alpha}{(n-1)} + \beta\Big]\Big[g(Y,Z)X - g(X,Z)Y\Big]\end{aligned} \tag{2.3}$$

where α and β are constants such that $\alpha, \beta \neq 0$,

$$\begin{aligned}H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}\Big[& S(X,Z)Y - S(Y,Z)X \\ & + g(Y,Z)QX - g(X,Z)QY\Big],\end{aligned} \tag{2.4}$$

$$W_2(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\Big[g(Y,Z)QX - g(X,Z)QY\Big], \tag{2.5}$$

for arbitrary vector fields X, Y, Z.

Using equations (1.5), (1.6), (1.7), (2.1) and (2.2) in (2.3), (2.4), (2.5) we have

$$\bar{P}(X,Y)\xi = \Big[\frac{\beta(n-1)+\alpha)}{b}\Big]\Big\{\eta(Y)X - \eta(X)Y\Big\}, \tag{2.6}$$

$$\bar{P}(\xi, X)Y = \frac{(\alpha-\beta)}{n}\Big\{g(X,Y)\xi - \eta(Y)X\Big\} + \beta b\big\{\eta(Y)\eta(X) - \eta(Y)X\big\}, \tag{2.7}$$

$$\eta(\bar{P}(X,Y)Z) = \frac{(\alpha-\beta)}{n}\Big\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big\}, \tag{2.8}$$

$$H(X,Y)\xi = \frac{(na+b)}{(n-1)(n-2)}\Big\{\eta(Y)X - \eta(X)Y\Big\}, \tag{2.9}$$

$$H(\xi,X)Y = \frac{(na+b)}{(n-1)(n-2)}\Big\{g(X,Y)\xi - \eta(Y)X\Big\}, \tag{2.10}$$

$$\eta(H(X,Y)Z) = \frac{(na+b)}{(n-1)(n-2)}\Big\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\Big\}, \tag{2.11}$$

$$\eta(W_2(X,Y)Z) = 0. \tag{2.12}$$

§3. Examples of $N(k)$-quasi Einstein manifolds

Example 1. Consider a pseudo projectively flat quasi Einstein manifold. Then, from (2.3), we have

$$\alpha R(X,Y)Z = -\beta\big[S(Y,Z)X - S(X,Z)Y\big] + \frac{r}{n}\Big[\frac{\alpha}{n-1} + \beta\Big]\Big[g(Y,Z)X - g(X,Z)Y\Big] \tag{3.1}$$

Using (1.1) in (3.1), we get

$$\alpha R(X,Y)Z = -\Big\{\beta a - \frac{r}{n}\Big(\frac{\alpha}{n-1} + \beta\Big)\Big\}\Big[g(Y,Z)X - g(X,Z)Y\Big] - \beta b\big[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\big]. \tag{3.2}$$

Replacing Z by ξ in (3.2) we have,

$$R(X,Y)\xi = \Big[\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big]\big[\eta(Y)X - \eta(X)Y\big],$$

which shows that ξ belongs to the $\Big(\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big)$-nullity distribution. Therefore, we can state:

Theorem 3.1. *A pseudo projectively flat quasi Einstein manifold is an* $N\Big(\frac{r}{n(n-1)} - \frac{\beta b}{n}\big(\frac{n-1}{n}\big)\Big)$*-quasi Einstein manifold.*

Example 2. Consider a quasi Einstein manifold which is conharmonically flat. Then by equation (2.4), we have

$$R(X,Y)Z = \frac{1}{(n-2)}\Big[S(X,Z)Y - S(Y,Z)X + g(Y,Z)QX - g(X,Z)QY\Big]. \tag{3.3}$$

Using (1.1) and (2.1) in (3.3), we have

$$R(X,Y)Z = \frac{1}{(n-2)}\Big[2a\big\{g(Y,Z)X - g(X,Z)Y\big\} + b\big\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi) - g(X,Z)\eta(Y)\xi\big\}\Big]. \tag{3.4}$$

Substituting $Z = \xi$, equation (3.4) reduces to

$$R(X,Y)\xi = \Big(\frac{2a+b}{n-2}\Big)\big[\eta(Y)X - \eta(X)Y\big],$$

showing that the manifold is an $N\Big(\dfrac{2a+b}{n-2}\Big)$-quasi Einstein manifold. Thus, we have the following theorem:

Theorem 3.2. *A conharmonically flat quasi Einstein manifold is an* $N\Big(\dfrac{2a+b}{n-2}\Big)$*-quasi Einstein manifold.*

Example 3. Consider the 4-dimensional Riemannian space $(\mathbb{R}^4, g)$ endowed with the metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{3.5}$$

$(i, j = 1, 2, 3, 4)$ where $q = \frac{e^{x^1}}{k^2}$ and k is a non zero constant. Then, $(\mathbb{R}^4, g)$ is an $N\Big(\frac{q}{(1+2q)^3}\Big)$-quasi Einstein manifold [11].

Example 4. Consider a Riemannian metric g on $\mathbb{R}^4 = \{(x^1, x^2, x^3, x^4) : x^i \in \mathbb{R}\}$ defined by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{3.6}$$

$(i, j = 1, 2, 3, 4)$. Then, $(\mathbb{R}^4, g)$ is an $N\Big(\frac{[1-18(x^4)^4]}{9(x^4)^2}\Big)$-quasi Einstein manifold [6].

Example 5: A special Para-Sasakian manifold with vanishing D-concircular curvature tensor is an $N(k)$-quasi Einstein manifold [6].

Example 6: A four-dimensional conformally flat perfect fluid (M^4, g) is an $N\Big(\frac{1}{3}\big\{\frac{1}{2}r + f(T) + 8\pi\rho + 2(\rho+p)f'(T)\big\}\Big)$-quasi Einstein manifold, where ρ is the energy density and p is the pressure [5].

Example 7: An n-dimensional conformally flat Kenmotsu manifold equipped with Ricci soliton is an $N(k)$-quasi Einstein manifold [5].

Example 8: Every m-projectively flat quasi Einstein manifold is an $N\big(\frac{2a+b}{2(n-1)}\big)$-quasi Einstein manifold [4].

§4. Pseudo projective curvature tensor in an $N(k)$-quasi Einstein manifold

Theorem 4.1. *An n-dimensional $N(k)$-quasi Einstein manifold M satisfies the curvature condition $\bar{P}(\xi, X)\cdot W_2 = 0$ provided $\alpha = \beta$ or the manifold is W_2-flat.*

Proof. Suppose M satisfies the curvature condition $\bar{P}(\xi, X)\cdot W_2 = 0$. Then,

$$\begin{aligned} (4.1)\quad \bar{P}(\xi,X)W_2(U,V)Z \;&-\; W_2(\bar{P}(\xi,X)U,V)Z - W_2(U,\bar{P}(\xi,X)V)Z \\ &-\; W_2(U,V)\bar{P}(\xi,X)Z = 0, \end{aligned}$$

for all vector fields $U, V, Z, X \in M$.

Using (2.7) in (4.1), we have

$$\begin{aligned} (4.2)\qquad & b\Bigg[\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X)\xi - \eta(W_2(U,V)Z)X \\ & -g(X,U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\ & -g(X,V)W_2(U,\xi)Z + \eta(V)W_2(U,X)Z \\ & -g(X,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\Big\} \\ & +\beta\Big\{\eta(X)\eta(W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)X \\ & -\eta(X)\eta(U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\ & -\eta(X)\eta(V)W_2(U,\xi)Z + \eta(V)W_2(U,X)Z \\ & -\eta(X)\eta(Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\Big\}\Bigg] = 0. \end{aligned}$$

Since $b \neq 0$, equation (4.2) can be written as

$$
\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X)\xi - \eta(W_2(U,V)Z)X \\
&- g(X,U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\
&- g(X,V)W_2(U,\xi)Z + \eta(V)W_2(U,X)Z \\
&- g(X,Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\Big\} \\
&+ \beta\Big\{\eta(X)\eta(W_2(U,V)Z)\xi - \eta(W_2(U,V)Z)X \\
&- \eta(X)\eta(U)W_2(\xi,V)Z + \eta(U)W_2(X,V)Z \\
&- \eta(X)\eta(V)W_2(U,\xi)Z + \eta(V)W_2(U,X)Z \\
&- \eta(X)\eta(Z)W_2(U,V)\xi + \eta(Z)W_2(U,V)X\Big\} = 0.
\end{aligned} \tag{4.3}
$$

Taking inner product of (4.3) with respect to ξ, we have

$$
\begin{aligned}
&\frac{(\alpha-\beta)}{n}\Big\{W_2'(U,V,Z,X) - \eta(W_2(U,V)Z)\eta(X) \\
&- g(X,U)\eta(W_2(\xi,V)Z) + \eta(U)\eta(W_2(X,V)Z) \\
&- g(X,V)\eta(W_2(U,\xi)Z) + \eta(V)\eta(W_2(U,X)Z) \\
&- g(X,Z)\eta(W_2(U,V)\xi) + \eta(Z)\eta(W_2(U,V)X)\Big\} \\
&+ \beta\Big\{\eta(X)\eta(W_2(U,V)Z) - \eta(W_2(U,V)Z)\eta(X) \\
&- \eta(X)\eta(U)\eta(W_2(\xi,V)Z) + \eta(U)\eta(W_2(X,V)Z) \\
&- \eta(X)\eta(V)\eta(W_2(U,\xi)Z) + \eta(V)\eta(W_2(U,X)Z) \\
&- \eta(X)\eta(Z)\eta(W_2(U,V)\xi) + \eta(Z)\eta(W_2(U,V)X)\Big\} = 0.
\end{aligned} \tag{4.4}
$$

From (2.12) and (4.4), it follows that

$$\frac{(\alpha-\beta)}{n}W_2'(U,V,Z,X) = 0.$$

Since $n > 2$ this implies that

$$\alpha = \beta \quad \text{or} \quad W_2 = 0.$$

This completes the proof. □

Theorem 4.2. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the curvature condition $\bar{P}(\xi, X) \cdot H = 0$ if $\alpha = \beta$ or the manifold is conharmonically flat.*

Proof. Suppose M satisfies the curvature condition $\bar{P}(\xi, X) \cdot H = 0$. Then, we can write

$$\begin{aligned} \bar{P}(\xi, X)H(U, V)Z &- H(\bar{P}(\xi, X)U, V)Z - H(U, \bar{P}(\xi, X)V)Z \\ &- H(U, V)\bar{P}(\xi, X)Z = 0. \end{aligned} \tag{4.5}$$

Using (2.8) in (4.5), we have

$$\begin{aligned} & b\Bigg[\frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)\xi - \eta(H(U,V)Z)X \\ & - g(X,U)H(\xi,V)Z + \eta(U)H(X,V)Z \\ & - g(X,V)H(U,\xi)Z + \eta(V)H(U,X)Z \\ & - g(X,Z)H(U,V)\xi + \eta(Z)H(U,V)X\Big\} \\ & + \beta\Big\{\eta(X)\eta(H(U,V)Z)\xi - \eta(H(U,V)Z)X \\ & - \eta(X)\eta(U)H(\xi,V)Z + \eta(U)H(X,V)Z \\ & - \eta(X)\eta(V)H(U,\xi)Z + \eta(V)H(U,X)Z \\ & - \eta(X)\eta(Z)H(U,V)\xi + \eta(Z)H(U,V)X\Big\}\Bigg] = 0. \end{aligned} \tag{4.6}$$

Since $b \neq 0$, (4.6) can be written as

$$\begin{aligned} & \frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)\xi - \eta(H(U,V)Z)X \\ & - g(X,U)H(\xi,V)Z + \eta(U)H(X,V)Z \\ & - g(X,V)H(U,\xi)Z + \eta(V)H(U,X)Z \\ & - g(X,Z)H(U,V)\xi + \eta(Z)H(U,V)X\Big\} \\ & + \beta\Big\{\eta(X)\eta(H(U,V)Z)\xi - \eta(H(U,V)Z)X \\ & - \eta(X)\eta(U)H(\xi,V)Z + \eta(U)H(X,V)Z \\ & - \eta(X)\eta(V)H(U,\xi)Z + \eta(V)H(U,X)Z \\ & - \eta(X)\eta(Z)H(U,V)\xi + \eta(Z)H(U,V)X\Big\} = 0. \end{aligned} \tag{4.7}$$

Taking inner product of (4.7) with respect to ξ, we have

$$\begin{aligned}(4.8)\qquad &\frac{(\alpha-\beta)}{n}\Big\{H'(U,V,Z,X)-\eta(H(U,V)Z)\eta(X)\\ &-g(X,U)\eta(H(\xi,V)Z)+\eta(U)\eta(H(X,V)Z)\\ &-g(X,V)\eta(H(U,\xi)Z)+\eta(V)\eta(H(U,X)Z)\\ &-g(X,Z)\eta(H(U,V)\xi)+\eta(Z)\eta(H(U,V)X)\Big\}\\ &+\beta\Big\{\eta(X)\eta(H(U,V)Z)-\eta(H(U,V)Z)\eta(X)\\ &-\eta(X)\eta(U)\eta(H(\xi,V)Z)+\eta(U)\eta(H(X,V)Z)\\ &-\eta(X)\eta(V)\eta(H(U,\xi)Z)+\eta(V)\eta(H(U,X)Z)\\ &-\eta(X)\eta(Z)\eta(H(U,V)\xi)+\eta(Z)\eta(H(U,V)X)\Big\}=0.\end{aligned}$$

Using (2.10) in (4.8), we get

$$\begin{aligned}(4.9)\qquad &\frac{(\alpha-\beta)}{n}\Big[H'(U,V,Z,X)+\frac{(na+b)}{(n-1)(n-2)}\{g(X,U)g(V,Z)\\ &-g(X,V)g(U,Z)\}\Big]-\beta\frac{(na+b)}{(n-1)(n-2)}\Big[\eta(X)\eta(U)g(V,Z)\\ &-\eta(X)\eta(V)g(U,Z)\Big]=0.\end{aligned}$$

Making use of (2.4) in (4.9), we obtain

$$\begin{aligned}(4.10)\qquad &\frac{(\alpha-\beta)}{n}\Big[R'(U,V,Z,X)-\frac{1}{(n-2)}\{S(V,Z)g(X,U)\\ &-S(U,Z)g(X,V)+g(V,Z)S(X,U)-g(U,Z)S(X,V)\}\\ &+\frac{(na+b)}{(n-1)(n-2)}\{g(V,Z)g(X,U)-g(U,Z)g(X,V)\}\Big]\\ &+\beta\frac{(na+b)}{(n-1)(n-2)}\Big[\eta(V)\eta(Z)g(X,U)-\eta(U)\eta(Z)g(X,V)\Big]=0.\end{aligned}$$

Taking $U=X=e_i$ in (4.10) and summing over $i, 1\leq i\leq n$, we get

$$\beta\Big(\frac{na+b}{n-2}\Big)\eta(V)\eta(Z)=0.$$

Since $n>2$, $\beta\neq 0$ and $\eta\neq 0$, we have

$$na+b=0.$$

Using this in (4.9), it follows that

$$\frac{(\alpha-\beta)}{n}H'(U,V,Z,X)=0,$$
$$\Rightarrow \alpha=\beta=0 \quad \text{or} \quad H'(U,V,Z,X)=0,$$

which proves the theorem. □

Definition 4.1. *A Riemannian manifold is said to be semisymmetric [22, 23] if*

$$R\cdot R=0. \tag{4.11}$$

Consider an $N(k)$-quasi Einstein manifold which is pseudo projectively semisymmetric. Then,

$$(R(X,Y)\cdot \bar{P})(U,V)W=0,$$

$$R(X,Y)\bar{P}(U,V)W-\bar{P}(R(X,Y)U,V)W-\bar{P}(U,R(X,Y)V)W - \bar{P}(U,V)R(X,Y)W=0, \tag{4.12}$$

Taking inner product of (4.12) with respect to ξ, we have

$$g(R(X,Y)\bar{P}(U,V)W,\xi)-g(\bar{P}(R(X,Y)U,V)W,\xi)-g(\bar{P}(U,R(X,Y)V)W,\xi) - g(\bar{P}(U,V)R(X,Y)W,\xi)=0,$$

Substituting $X=\xi$, the above equation becomes

$$g(R(\xi,Y)\bar{P}(U,V)W,\xi)-g(\bar{P}(R(\xi,Y)U,V)W,\xi)-g(\bar{P}(U,R(\xi,Y)V)W,\xi) - g(\bar{P}(U,V)R(\xi,Y)W,\xi)=0.$$

Using (1.6), we have

$$k\Bigg[\bar{P}'(U,V,W,Y)-\frac{(\alpha-\beta)}{n}\Big\{g(V,W)g(U,Y) - g(U,W)g(V,Y)\Big\}\Bigg]=0. \tag{4.13}$$

Assuming $k\neq 0$ and making use of (2.3), (4.13) becomes

$$\alpha R'(U,V,W,Y)=-\beta[S(V,W)g(U,Y)-S(U,W)g(V,Y)] + \left\{\frac{r}{n}\Big(\frac{\alpha}{n-1}+\beta\Big)+\frac{(\alpha-\beta)b}{n}\right\}\big[g(V,W)g(U,Y) - g(U,W)g(V,Y)\big]. \tag{4.14}$$

Contracting (4.14) with respect to U and Y , we get

$$S(V,W)=\Big[\frac{r}{n(n-1)}+\frac{(\alpha-\beta)b}{n(\alpha+\beta(n-1))}\Big]g(V,W)$$

showing that the manifold is Einstein, which is not possible since M is an $N(k)$-quasi Einstein manifold. Thus, we can state:

Theorem 4.3. *There does not exist a pseudo projectively semisymmetric $N(k)$-quasi Einstein manifold.*

Definition 4.2. *A Riemannian manifold is called a symmetric manifold [12, 16] if*

$$(\nabla_X R)(Y,Z)V=0, \tag{4.15}$$

where ∇ is the operator of covariant differentiation with respect to the metric g.

Consider a pseudo projectively symmetric $N(k)$-quasi Einstein manifold. Then,

$$(\nabla_X \bar{P}')(Z,U,V,W)=0.$$

By virtue of (2.3), the above equation can be written as

$$\begin{aligned}\alpha(\nabla_X R')(Z,U,V,W)&+\beta[(\nabla_X S)((U,V)g(Z,W)\\ &-(\nabla_X S)(Z,V)g(U,W)]-\frac{dr(X)}{n}\Big(\frac{\alpha}{n-1}+\beta\Big)\big[g(U,V)g(Z,W)\\ &-g(Z,V)g(U,W)\big]=0.\end{aligned} \tag{4.16}$$

Contracting (4.16) with respect to Z and W, we obtain

$$(\nabla_X S)(U,V)=\frac{dr(X)}{n}g(U,V). \tag{4.17}$$

Using (1.2) in equation (4.8), we have

$$\begin{aligned}da(X)g(U,V)+db(X)\eta(U)\eta(V)&+b[(\nabla_X\eta)(U)\eta(V)\\ &+(\nabla_X\eta)(V)\eta(U)]=\frac{dr(X)}{n}g(U,V).\end{aligned} \tag{4.18}$$

Substituting $U=V=\xi$, (4.18) becomes

$$dr(X)=n[da(X)+db(X)]. \tag{4.19}$$

Also, taking covariant derivative of (2.1) with respect to X, we obtain

$$dr(X)=nda(X)+db(X). \tag{4.20}$$

From equations (4.19) and (4.20), it follows that

$$(n-1)db(X) = 0,$$

or

$$db(X) = 0.$$

which implies that b is constant. This leads to the following theorem:

Theorem 4.4. *An $N(k)$-quasi Einstein manifold is pseudo projectively symmetric provided the associated scalar b is non-zero constant.*

§5. Conharmonically pseudosymmetric $N(k)$-quasi Einstein manifolds

Definition 5.1. *An $N(k)$-quasi Einstein manifold is said to be conharmonically pseudosymmetric if the tensors*

$$\begin{aligned}(R(X,Y)\cdot H)(Z,W)U &= R(X,Y)H(Z,W)U - H(R(X,Y)Z,W)U \\ &- H(Z,R(X,Y)W)U - H(Z,W)R(X,Y)U\end{aligned}$$

and

$$\begin{aligned}Q(g,H)(Z,W,U,X,Y) &= (X\wedge_H Y)H(Z,W)U - H((X\wedge_H Y)Z,W)U \\ &- H(Z,(X\wedge_H Y)W)U - H(Z,W)(X\wedge_H Y)U\end{aligned}$$

are linearly dependent, i. e.,

$$(R(X,Y)\cdot H)(Z,W)U = L_H\{Q(g,H)(Z,W,U,X,Y)\} \tag{5.1}$$

for a smooth function $L_H \in A_H = \{x \in M : H \neq 0 \text{ at } x\}$, where X, Y, Z, W, U are arbitrary.

Consider an $N(k)$-quasi Einstein manifold which is conharmonically pseudosymmetric. Then, from equation (5.1) we have

$$\begin{aligned}&R(X,Y)H(Z,W)U - H(R(X,Y)Z,W)U \\ &- H(Z,R(X,Y)W)U - H(Z,W)R(X,Y)U \\ &= L_H\{(X\wedge_H Y)H(Z,W)U - H((X\wedge_H Y)Z,W)U \\ &- H(Z,(X\wedge_H Y)W)U - H(Z,W)(X\wedge_H Y)U\}.\end{aligned} \tag{5.2}$$

Putting $X = \xi$, (5.2) becomes

$$\begin{aligned}&R(\xi,Y)H(Z,W)U - H(R(\xi,Y)Z,W)U \\ &- H(Z,R(\xi,Y)W)U - H(Z,W)R(\xi,Y)U \\ &= L_H\{(\xi\wedge_H Y)H(Z,W)U - H((\xi\wedge_H Y)Z,W)U \\ &- H(Z,(\xi\wedge_H Y)W)U - H(Z,W)(\xi\wedge_H Y)U\}.\end{aligned} \tag{5.3}$$

Making use of equation (1.6) in (5.3) and

$$(X \wedge_H Y)Z = g(Y,Z)X - g(X,Z)Y,$$

we have

$$\begin{aligned}&(k - L_H)[H'(Z,W,U,Y)\xi - \eta(H(Z,W)U)Y - g(Y,Z)H(\xi,W)U\\&+ \eta(Z)H(Y,W)U - g(Y,W)H(Z,\xi)U + \eta(W)H(Z,Y)U\\&- g(Y,U)H(Z,W)\xi + \eta(U)H(Z,W)Y] = 0.\end{aligned}$$

Assuming $k \neq L_H$ and taking inner product of the above equation with respect to ξ, we have

$$\begin{aligned}H'(Z,W,U,Y) + \frac{na+b}{(n-1)(n-2)}\Big[&g(Z,Y)g(W,U)\\&- g(Z,U)g(W,Y)\Big] = 0.\end{aligned} \tag{5.4}$$

From equations (2.4) and (5.4), we obtain

$$\begin{aligned}R'(Z,W,U,Y) = {}& a_1[g(Z,Y)g(W,U) - g(Z,U)g(W,Y)]\\&+ a_2[\eta(W)\eta(U)g(Y,Z) - \eta(Z)\eta(U)g(W,Y)\\&+ \eta(Z)\eta(Y)g(W,U) - \eta(W)\eta(Y)g(Z,U)],\end{aligned}$$

where $a_1 = \dfrac{a}{n-1} - \dfrac{b}{(n-1)(n-2)}$ and $a_2 = \dfrac{b}{n-2}$. This leads to the following theorem:

Theorem 5.1. *An n-dimensional $N(k)$-quasi Einstein manifold which is conharmonically pseudosymmetric and $k \neq L_H$ is of quasi-constant curvature.*

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