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q-generalized (anti -) flexible algebras and bialgebras

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Abstract. In this work, we provide a q-generalization of flexible algebras and related bialgebraic structures, including center-symmetric (also called anti-flexible) algebras, and their bialgebras. Their basic properties are derived and discussed. Their connection with known algebraic structures, previously developed in the literature, is established. A q-Myung theorem is given. Main properties related to bimodules, matched pairs and dual bimodules as well as their algebraic consequences are investigated and analyzed. Besides, the equivalence between q-generalized flexible algebras, their Manin triple and bialgebras is established. Finally, various remarkable identities are established for the octonion algebra.

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§1. Introduction

Alternative algebras were introduced by Zorn [29] who established their fundamental identities, studied their nucleus by modifying the characteristics of the field, and investigated their Lie admissibility using the corresponding Jacobi identity. Furthermore, Zorn derived their power associativity conditions. Later, Schafer [26] gave a new formulation of these algebras in terms of left and right multiplication operators, and in terms of division algebras of degree two. He also provided the isotopes of these algebras. Santilli [27] introduced Lie admissible algebras and gave their basic properties. He extended his study to mutation algebras, examined their relation to associative algebras, Lie algebras, Jordan and special Jordan algebras, and established the passage from one type of algebra to another by using a hexahedron with oriented edges.

Radicals of flexible Lie admissible algebras were introduced, and some of their properties were established and discussed in [6]. Classes of flexible Lie admissible algebras were also investigated and discussed in [18]. Albert in [1] elaborated fundamental concepts, and studied the isotropy of nonassociative algebras. Simple and semi-simple algebras, and their characterization from nonassociative algebraic structures were developed and discussed in [2]. For more details, see a self-contained book by Schafer [25] addressing a nice compilation of basic properties of nonassociative algebras.

Contrarily to Lie algebras, and except for some classification based on the characteristics of closed fields (see [7] and references therein), a full classification of nonassociative algebras still remains a tremendous task. Some interesting properties and algebraic identities of anti-flexible structures were investigated and discussed in [8]. The properties of simple, semisimple and nearly semisimple anti-flexible algebras were also derived and analyzed in [22–24].

Among the nonassociative algebras, the alternative algebras, with an associator preserving certain symmetry by exchanging its elements, play a central role in both mathematics and physics as they possess interesting properties. A nice repertory of their applications in physics, including gauge theory and Yang-Mills gauge theory formulation from nonassociative algebras can be found in [20], (and references therein). A theory of nuclear boson-expansion for odd-fermion systems in the context of nonassociative algebras was also examined in [21]. A study on quark structure and octonion algebras was performed in [11]. Further, a generalization of the classical Hamiltonian dynamics to a three-dimensional phase space, generating equations of motion with two Hamiltonians and three canonical variables, was performed with an analog of Poisson bracket realized by means of the associator of nonassociative algebras [19].

Besides, flexible algebras were also investigated in terms of degree of algebras [14]. Other characterizations and applications of nonassociative algebras can be found in [17], [15] and [12] (and references therein).

Similarly to algebraic properties of quantum groups developed by Drinfeld [10], some nonassociative algebras possess interesting identities with applications in physics, and generate the so-called associative or classical Yang-Baxter equations [3, 9, 13], (and references therein). Furthermore, the bialgebras constructed from Jordan algebras [28] are related to the Lie bialgebras. The left-symmetric algebras, also called pre-Lie algebras [5], are known as Lie admissible algebras, and admit the left multiplication operator as a representation. They can also be used to produce symplectic Lie algebras, while their coboundary bialgebras lead to the identity known as S-equation, and generate para-Kähler Lie algebras. The case of associative algebras also furnished remarkable properties investigated by Aguiar [3] and Bai [4]. The center-symmetric algebras studied in [12] are also Lie admissible algebras.

The (anti-) flexible algebras were well investigated and their main properties well scrutinized in a series of works [12,14,15,17,18,22–24] (and references therein). Unfortunately, to our best knowledge of the literature, their characterization in terms of bialgebras, bimodules, matched pairs, Manin triples and their mutual link is still lacking. The present work aims at filling this gap in a global and unique way by combining the analysis of these algebras in an approach generalizing them, and where a specific algebra can be recovered as a particular case corresponding to a value of the introduced parameter q. It mainly addresses a q-generalization of flexible algebras and related bialgebraic structures, including center-symmetric (also called anti-flexible) algebras, and their bialgebras. Their basic properties are derived and discussed. Their connection with known algebraic structures existing in the literature is established. A q-version of the Myung theorem is given. Main properties related to bimodules, matched pairs and dual bimodules, and their algebraic consequences are investigated and analyzed. Besides, the Manin triple of q-generalized flexible algebras, and its link to q-generalized flexible bialgebras are built together with the equivalence with the matched pair of q-generalized flexible algebras. Finally, various remarkable identities are established for the octonion algebra.

§2. Basic properties of a q-generalized flexible algebra

In this section, a q-generalization of algebras encompassing flexible, anti-flexible and associative algebras is provided. Their relevant properties and link with known algebras are derived. Jordan identity and Lie admissibility condition are also established.

Definition 2.1. *Let $\mathcal{A}(q)$, where $q \in \{-1; 0; 1\} \subset \mathbb{K}$, ($\mathbb{R}$ or $\mathbb{C}$), be a finite dimensional vector space and "$\cdot$" a bilinear product on $\mathcal{A}(q)$. The couple $(\mathcal{A}(q), \cdot)$ is called a q-generalized flexible algebra if, for all $x, y, z \in \mathcal{A}(q)$, the following relation is satisfied:*

$$(x, y, z) = q(z, y, x), \tag{2.1}$$

or, equivalently,

$$\mu \circ (\mu \otimes \mathrm{id}) + q(\mu \circ \tau) \circ ((\mu \circ \tau) \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu) + q(\mu \circ \tau) \circ (\mathrm{id} \otimes (\mu \circ \tau)),$$

where $(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of the bilinear product "$\cdot$" on $\mathcal{A}(q)$; μ is defined by $\mu(x, y) = x \cdot y$; id is the identity map on $\mathcal{A}(q)$; and τ stands for the exchange map on $\mathcal{A}(q) \otimes \mathcal{A}(q)$ given by $\tau(x \otimes y) = y \otimes x$.

Remark 2.2. *We have:*

- *Equation* (2.1) *can be described by the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{A}(q)\otimes\mathcal{A}(q)\otimes\mathcal{A}(q) & \xrightarrow{\mu\otimes\mathrm{id}\,-\,\mathrm{id}\otimes\mu} & \mathcal{A}(q)\otimes\mathcal{A}(q) \\ {\scriptstyle \mathrm{id}\otimes(\mu\circ\tau)-(\mu\circ\tau)\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathcal{A}(q)\otimes\mathcal{A}(q) & \xrightarrow{\quad q\mu\circ\tau\quad} & \mathcal{A}(q) \end{array}$$

- *For* $q = 0$*, the algebra* $(\mathcal{A}(q), \cdot)$ *is reduced to an associative algebra;*
- *For* $q = -1$*,* $(\mathcal{A}(q), \cdot)$ *becomes a flexible algebra;*
- *For* $q = 1$*,* $(\mathcal{A}(q), \cdot)$ *turns to be a center-symmetric algebra [12], (also called anti-flexible algebra [15]).*

In the sequel, $(\mathcal{A}(q), \cdot)$ denotes a q-generalized flexible algebra over $\mathbb{K}$, ($\mathbb{R}$ or $\mathbb{C}$), with $q \in \{-1; 0; 1\}$. Besides, for notation simplification, we write xy instead of $x \cdot y$, for $x, y \in \mathcal{A}(q)$, i.e. the product "$\cdot$" is omitted when there is no confusion.

Definition 2.3. *Suppose* L *and* R *be left and right multiplication operators defined on* $\mathcal{A}(q)$ *as:*

$$L: \begin{array}{ll} \mathcal{A}(q) & \longrightarrow \mathfrak{gl}(\mathcal{A}(q)) \\ x & \longmapsto L_x : \begin{array}{lll} \mathcal{A}(q) & \longrightarrow & \mathcal{A}(q) \\ y & \longmapsto & L(x)y := x\cdot y \end{array} \end{array} \tag{2.2}$$

$$R: \begin{array}{ll} \mathcal{A}(q) & \longrightarrow \mathfrak{gl}(\mathcal{A}(q)) \\ x & \longmapsto R(x) : \begin{array}{lll} \mathcal{A}(q) & \longrightarrow & \mathcal{A}(q) \\ y & \longmapsto & R(x)y := y\cdot x. \end{array} \end{array} \tag{2.3}$$

Then, their associated dual maps are defined as follows:

$$L^*: \begin{array}{ll} \mathcal{A}(q) & \longrightarrow \mathfrak{gl}(\mathcal{A}(q)^*) \\ x & \longmapsto L^*(x) : \begin{array}{lll} \mathcal{A}(q)^* & \longrightarrow & \mathcal{A}(q)^* \\ a & \longmapsto & L^*(x)a : \begin{array}{lll} \mathcal{A}(q) & \longrightarrow & \mathbb{K} \\ x & \longmapsto & < L^*(x)a, y > \end{array} \end{array} \end{array}$$

$$R^*: \begin{array}{ll} \mathcal{A}(q) & \longrightarrow \mathfrak{gl}(\mathcal{A}(q)^*) \\ x & \longmapsto R^*(x) : \begin{array}{lll} \mathcal{A}(q)^* & \longrightarrow & \mathcal{A}(q)^* \\ a & \longmapsto & R^*(x)a : \begin{array}{lll} \mathcal{A}(q) & \longrightarrow & \mathbb{K} \\ x & \longmapsto & < R^*(x)a, y > \end{array} \end{array} \end{array}$$

where, for all $x, y \in \mathcal{A}(q)$ *and all* $a \in \mathcal{A}(q)^*$

$$< L^*(x)a, y > = < a, L(x)y >, \tag{2.4}$$

$$< R^*(x)a, y > = < a, R(x)y > . \tag{2.5}$$

Proposition 2.4. *Let L and R be the above defined left and right multiplication operators. The following relations are satisfied for all $x, y \in \mathcal{A}(q)$:*

$$L(xy) - L(x)L(y) = q(R(x)R(y) - R(yx)), \tag{2.6}$$

$$[R(x), L(y)] = q[R(y), L(x)], \tag{2.7}$$

$$R(x)R(y) - R(yx) = q(L(xy) - L(x)L(y)). \tag{2.8}$$

Proof. Let $\mathcal{A}(q)$ be a q-generalized flexible algebra over the field $\mathbb{K}$. We have for all $x, y, z \in \mathcal{A}(q)$, $(x, y, z) = q(z, y, x) \iff (xy)z - x(yz) = q(zy)x - qz(yx)$. Since $(x, y, z) = (L(xy) - L(x)L(y))(z) = [R(z), L(x)](y) = (R(z)R(y) - R_{(yz)})(x)$ and $q(z, y, x) = q(R(x)R(y) - R(yx))(z) = q[R(x), L(z)](y) = q(L(zy) - L(z)L(y))(x)$, the relations (2.6), (2.7) and (2.8) hold. □

Proposition 2.5. *Provided the sub-adjacent algebra $\mathcal{G}(\mathcal{A}(q)) := (\mathcal{A}(q), [., .])$, where the bilinear product $[., .]$ is the commutator associated to the product on $\mathcal{A}(q)$, we have, for all $x, y, z \in \mathcal{A}(q)$:*

$$\begin{aligned} J(x, y, z) &:= [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \\ &= (q - 1)\{(x, y, z) + (y, z, x) + (z, x, y)\}. \end{aligned} \tag{2.9}$$

Proof. It stems from a straightforward computation. □

Corollary 2.6.
1. *From the Proposition 2.5, for all $x, y, z \in \mathcal{A}(q)$, the relation*

$$S(x, y, z) := (x, y, z) + (y, z, x) + (z, x, y) = 0 \tag{2.10}$$

is a sufficient condition for $\mathcal{A}(q)$ to become a Lie admissible algebra, i.e. for $(\mathcal{A}(q), [., .])$ to be a Lie algebra. In the particular case where $q = 1$, we get a center-symmetric algebra which is Lie admissible as developed in [12].

2. *The q-generalized flexible algebra $\mathcal{A}(q)$ is Lie admissible if and only if, for all $x, y, z \in \mathcal{A}(q)$, we have: $(q-1)S(x, y, z) = 0$. In particular, any q-generalized flexible algebra defined on a field $\mathbb{K}_{q-1}$ of characteristic $q - 1$ is Lie admissible.*

Proposition 2.7. *The following relation is satisfied for all $x, y \in \mathcal{A}(q)$:*

$$\begin{aligned} &(L(xy) - L(x)L(y) + R(x)L(y) - L(y)R(x) + R(y)R(x) - R(xy)) \\ &= q(R(x)R(y) - R(yx) + R(y)L(x) - L(x)R(y) + L(yx) - L(y)L(x)), \end{aligned} \tag{2.11}$$

where L and R are representations of left and right multiplication operators, respectively.

Proof. Let us write the associator with the operators L and R. For all $x, y, z \in \mathcal{A}(q)$,

$$\begin{aligned}(x, y, z) &= (xy)z - x(yz) = (L(xy) - L(x)L(y))(z) \\ &= (R(z)L(x) - L(x)R(z))(y) = (R(z)R(y) - R(yz))(x).\end{aligned}$$

It follows that:

$$\begin{aligned}(x, y, z) + (y, z, x) + (z, x, y) &= (L(xy) - L(x)L(y) + R(x)L(y) \\ &- L(y)R(x) + R(y)R(x) - R(xy))(z) \\ &= q(z, y, x) + q(x, z, y) + q(y, x, z) \\ &= q(R(x)R(y) - R(yx) + R(y)L(x) \\ &- L(x)R(y) + L(yx) - L(y)L(x))(z).\end{aligned}$$

Therefore, for all $x, y \in \mathcal{A}(q)$:
$L(xy) - L(x)L(y) + R(x)L(y) - L(y)R(x) + R(y)R(x) - R(xy) =$
$q(R(x)R(y) - R(yx) + R(y)L(x) - L(x)R(y) + L(yx) - L(y)L(x))$. □

Remark 2.8. *The result* (2.11) *can also be derived from the Proposition 2.4 by summing the relations* (2.6), (2.7) *and* (2.8).

Theorem 2.9. *The following relation is satisfied:* $\forall x, y, z \in \mathcal{A}(q)$,

$$[xy - qyx, z] + [yz - qzy, x] + [zx - qxz, y] = 0, \tag{2.12}$$

where the bilinear product $[.,.]$ *is the commutator associated to the product "·" defined on* $\mathcal{A}(q)$.

Proof. By a direct computation. □

Remark 2.10. *By setting the parameter* $q = 1$, *we get the Jacobi identity from the relation* (2.12) *indicating that the underlying algebra is Lie admissible as shown in [12].*

We are therefore in right to set the following:

Definition 2.11. *Setting* $\mathcal{G}(\mathcal{A}(q)) := (\mathcal{A}(q), [.,.])$, *where* $\mathcal{A}(q)$ *is the underlying vector space associated to a q-generalized flexible algebra* $(\mathcal{A}(q), \cdot)$, *the equation* (2.12) *defines a q-generalized Jacobi identity.*

Theorem 2.12. *Consider a q-generalized flexible algebra* $(\mathcal{A}(q), \cdot)$.

1. *If for all* $x, y, z \in \mathcal{A}(q)$,

$$[z, x \cdot y] = \{z, x\}_q \cdot y - x \cdot \{y, z\}_q, \tag{2.13}$$

where $\{x, y\}_q := x \cdot y + qy \cdot x$, *then* $(\mathcal{A}(q), \cdot)$ *is a Lie-admissible algebra, i.e. for all* $x, y, z \in \mathcal{A}(q)$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \tag{2.14}$$

where $[x, y] = x \cdot y - y \cdot x$.

2. *The following relations are satisfied, for all* $x, y, z \in \mathcal{A}(q)$:

$$[z, x *_q y] = [z, x] *_q y + x *_q [z, y], \tag{2.15}$$

and

$$[x *_q y, z] + [y *_q z, x] + [z *_q x, y] = 0, \tag{2.16}$$

where $x *_q y := \frac{1}{2}(x \cdot y - qy \cdot x)$.

Remark 2.13. *From Theorem 2.12, we observe that:*

1. *For* $q = -1$, *the equation* (2.13) *turns out to be the derivation property for the commutator of a flexible algebra as postulated by the well known Myung Theorem [18], [20] (and references therein);*
2. *For* $q = 0$, *the equation* (2.13) *is trivial by using the associativity;*
3. *For* $q = 1$, *the equation* (2.13) *leads to the relation* $(x, y, z) + (y, z, x) + (z, x, y) = 0$, *which is the sufficient condition of the Lie admissibility of* $\mathcal{A}(q)$. *See the statement 1 of Corollary 2.6 for more details;*
4. *For* $q = 1$, *the equation* (2.15) *is equivalent to the Jacobi identity in a field of characteristics* 0, *what is the case for a center-symmetric (also called anti-flexible) algebra;*
5. *For* $q = 0$, *the equation* (2.15) *describes the derivation property of the commutator (or Lie bracket) of a Lie algebra induced by an associative algebra;*
6. *For* $q = -1$, *the flexibility condition* (2.15) *defines the derivation property of the Jordan product given as* $x \circ y := \frac{1}{2}(x \cdot y + y \cdot x)$, *see [20].*

§3. Bimodules and matched pairs of q-generalized flexible algebras

Definition 3.1. *The triple* (l, r, V), *where* V *is a finite dimensional vector space, and* $l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)$ *are two linear maps satisfying the following relations for all* $x, y \in \mathcal{A}(q)$:

$$l(xy) - l(x)l(y) = q(r(x)r(y) - r(yx)), \tag{3.1}$$

$$[r(x), l(y)] = q[r(y), l(x)], \tag{3.2}$$

$$r(x)r(y) - r(yx) = q(l(xy) - l(x)l(y)), \tag{3.3}$$

is called a bimodule of $\mathcal{A}(q)$*, also simply denoted by* (l, r)*.*

Proposition 3.2. *Let* $l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)$ *be two linear maps as above. The couple* (l, r) *is a bimodule of the q-generalized flexible algebra* $\mathcal{A}(q)$ *if and only if there exists a q-generalized flexible algebra structure "$*$" on the semi-direct vector space* $\mathcal{A}(q) \oplus V$ *given by*
$(x + u) * (y + v) := x \cdot y + l(x)v + r(y)u, \forall x, y \in \mathcal{A}(q),$ *and* $\forall u, v \in V$*.*

We denote such a q-generalized flexible algebra structure "$*$" on the semi-direct vector space $\mathcal{A}(q) \oplus V$ by $(\mathcal{A}(q) \oplus V, *)$ or simply $\mathcal{A}(q) \ltimes V$.

Proof. Let $x, y, z \in \mathcal{A}(q)$, and $u, v, w \in V$, where V is a finite dimensional vector space. Using the bilinear product defined by $(x + u) * (y + v) := x \cdot y + l(x)v + r(y)u$, where $l, r : \mathcal{A} \to \mathfrak{gl}(V)$ are linear maps, the associator of the bilinear product "$*$" can be written as, $\forall x, y, z \in \mathcal{A}(q)$, and $\forall u, v, w \in V$,

$$\begin{aligned} (x + u, y + v, z + w) &= (x, y, z) + (l(x \cdot y) - l(x)l(y))w \\ &+ (r(z)l(x) - l(x)r(z))v \\ &+ (r(z)r(y) - r(y \cdot z))u. \end{aligned} \tag{3.4}$$

Then,

$$\begin{aligned} q(z + w, y + v, x + u) &= q(z, y, x) + q(l(z \cdot y) - l(z)l(y))u \\ &+ q(r(x)l(z) - l(z)r(x))v \\ &+ q(r(x)r(y) - r(y \cdot x))w. \end{aligned} \tag{3.5}$$

The couple $(\mathcal{A}(q) \oplus V, *)$ is a q-generalized flexible algebra means the equality $(x+u, y+v, z+w) = q(z+w, y+v, x+u)$, which is equivalent to the relations (3.1), (3.2), (3.3). Therefore $(\mathcal{A}(q), \cdot)$ is a q-generalized flexible algebra. $\square$

Example 3.3. *According to the Proposition 2.4,* (L, R)*, where* R *and* L *are the representations of the right and left multiplication operators, respectively, is a bimodule of a q-generalized flexible algebra* $\mathcal{A}(q)$*. Indeed,* L *and* R *satisfy the equations* (3.1)*,* (3.2) *and* (3.3)*.*

Theorem 3.4. *Let* $(\mathcal{A}(q), \cdot)$ *and* $(\mathcal{B}(q), *)$ *be two q-generalized flexible algebras. Suppose there exist linear maps* $l_{\mathcal{A}}, r_{\mathcal{A}} : \mathcal{A}(q) \to \mathfrak{gl}(\mathcal{B}(q))$ *and* $l_{\mathcal{B}}, r_{\mathcal{B}} : \mathcal{B}(q) \to \mathfrak{gl}(\mathcal{A}(q))$ *such that* $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B}(q))$ *and* $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A}(q))$ *are bimodules of* $\mathcal{A}(q)$ *and* $\mathcal{B}(q)$*, respectively, satisfying the following relations:*

$$\begin{aligned} &(l_{\mathcal{B}}(a)x) \cdot y + l_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y - l_{\mathcal{B}}(a)(x \cdot y) = \\ &q(r_{\mathcal{B}}(a)(y \cdot x) - y \cdot (r_{\mathcal{B}}(a)x) - r_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y), \end{aligned} \tag{3.6}$$

$$r_{\mathcal{B}}(a)(x\cdot y)-x\cdot(r_{\mathcal{B}}(a)y)-r_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x= q((l_{\mathcal{B}}(a)y)\cdot x+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x-l_{\mathcal{B}}(a)(y\cdot x)), \tag{3.7}$$

$$(r_{\mathcal{B}}(a)x)\cdot y+l_{\mathcal{B}}(l_{\mathcal{A}}(x)a)y-x\cdot(l_{\mathcal{B}}(a)y)-r_{\mathcal{B}}(r_{\mathcal{A}}(y)a)x= q(((r_{\mathcal{B}}(a)y)\cdot x+l_{\mathcal{B}}(l_{\mathcal{A}}(y)a)x-y\cdot(l_{\mathcal{B}}(a)x)-r_{\mathcal{B}}(r_{\mathcal{A}}(x)a)y), \tag{3.8}$$

$$(l_{\mathcal{A}}(x)a)*b+l_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b-l_{\mathcal{A}}(x)(a*b)= q(r_{\mathcal{A}}(x)(b*a)-b*(r_{\mathcal{A}}(x)a)-r_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b), \tag{3.9}$$

$$r_{\mathcal{A}}(x)(a*b)-a*(r_{\mathcal{A}}(x)b)-r_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a= q((l_{\mathcal{A}}(x)b)*a+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a-l_{\mathcal{A}}(x)(b*a)), \tag{3.10}$$

$$(r_{\mathcal{A}}(x)a)*b+l_{\mathcal{A}}(l_{\mathcal{B}}(a)x)b-a*(l_{\mathcal{A}}(x)b)-r_{\mathcal{A}}(r_{\mathcal{B}}(b)x)a= q((r_{\mathcal{A}}(x)b)*a+l_{\mathcal{A}}(l_{\mathcal{B}}(b)x)a-b*(l_{\mathcal{A}}(x)a)-r_{\mathcal{A}}(r_{\mathcal{B}}(a)x)b), \tag{3.11}$$

$\forall x,y\in\mathcal{A}(q)$ and $\forall a,b\in\mathcal{B}(q)$. It follows that there is a q-generalized flexible algebra structure "$\star$" on the direct sum of vector spaces $\mathcal{A}(q)\oplus\mathcal{B}(q)$ given as: $(x+a)\star(y+b)=(x\cdot y+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a*b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a)$.

Proof. Let $(\mathcal{A}(q),\cdot)$ and $(\mathcal{B}(q),*)$ be two q-generalized flexible algebras, $l_{\mathcal{A}},r_{\mathcal{A}}:\mathcal{A}(q)\to\mathfrak{gl}(\mathcal{B}(q))$ and $l_{\mathcal{B}},r_{\mathcal{B}}:\mathcal{B}(q)\to\mathfrak{gl}(\mathcal{A}(q))$ be four linear maps satisfying the relations (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11). Consider the bilinear product "$\star$" defined on the vector space $\mathcal{A}(q)\oplus\mathcal{B}(q)$ as: $\forall x,y\in\mathcal{A}(q); a,b\in\mathcal{B}(q)$, $(x+a)\star(y+b)=(x\cdot y+l_{\mathcal{B}}(a)y+r_{\mathcal{B}}(b)x)+(a*b+l_{\mathcal{A}}(x)b+r_{\mathcal{A}}(y)a)$. We have:

$$\begin{aligned}
&(x+a,y+b,z+c)=\{(x+a)\star(y+b)\}\star(z+c)\\
-\;&(x+a)\star\{(y+b)\star(z+b)\}=(x,y,z)+\{l_{\mathcal{B}}(a*b)z-l_{\mathcal{B}}(a)(l_{\mathcal{B}}(b)z)\}\\
+\;&\{r_{\mathcal{B}}(c)(r_{\mathcal{B}}(b)x)-r_{\mathcal{B}}(b*c)x\}+\{r_{\mathcal{B}}(c)(l_{\mathcal{B}}(a)y)-l_{\mathcal{B}}(a)(r_{\mathcal{B}}(c)y)\}\\
+\;&\{(l_{\mathcal{B}}(a)y)\cdot z+l_{\mathcal{B}}(r_{\mathcal{A}}(y)a)z-l_{\mathcal{B}}(a)(y\cdot z)\}+\{r_{\mathcal{A}}(z)r_{\mathcal{A}}(y)a-r_{\mathcal{A}}(y\cdot z)a\}\\
+\;&(a,b,c)+\{l_{\mathcal{A}}(x\cdot y)c-l_{\mathcal{A}}(x)(l_{\mathcal{A}}(y)c)\}+\{r_{\mathcal{A}}(z)l_{\mathcal{A}}(x)b-l_{\mathcal{A}}(x)r_{\mathcal{A}}(z)b\}\\
+\;&\{l_{\mathcal{B}}(l_{\mathcal{A}}(x)b)z+(r_{\mathcal{B}}(b)x)\cdot z-r_{\mathcal{B}}(r_{\mathcal{A}}(z)b)x-x\cdot(l_{\mathcal{B}}(b)z)\}\\
+\;&\{(r_{\mathcal{A}}(y)a)*c+l_{\mathcal{A}}(l_{\mathcal{B}}(a)y)c-a*(l_{\mathcal{A}}(y)c)-r_{\mathcal{A}}(r_{\mathcal{B}}(c)y)a\}\\
+\;&\{r_{\mathcal{B}}(c)(x\cdot y)-x\cdot(r_{\mathcal{B}}(c)y)-r_{\mathcal{B}}(l_{\mathcal{A}}(y)c)x\}\\
+\;&\{(l_{\mathcal{A}}(x)b)*c+l_{\mathcal{A}}(r_{\mathcal{B}}(b)x)c-l_{\mathcal{A}}(x)(b*c)\}\\
+\;&\{r_{\mathcal{A}}(z)(a*b)-a*(r_{\mathcal{A}}(z)b)-r_{\mathcal{A}}(l_{\mathcal{B}}(b)z)a\}=(x,y,z)+(x,y,c)\\
+\;&(x,b,z)+(x,b,c)+(a,y,z)+(a,y,c)+(a,b,z)+(a,b,c).
\end{aligned}$$

Then, the q-generalized flexibility condition of the bilinear product "$\star$" is given by:

$$(x+a,y+b,z+c)=q(z+c,y+b,x+a),$$

or, equivalently, in terms of the equalities: $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B}(q))$ and $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A}(q))$ are bimodules of $\mathcal{A}(q)$ and $\mathcal{B}(q)$, respectively, and for all $x, y \in \mathcal{A}(q)$, $a, b \in \mathcal{B}(q)$,

$$\begin{array}{rcl}(x,y,a) & = & q(a,y,x),(x,a,y)=q(y,a,x),(x,a,b)=q(b,a,x),\\ (a,x,y) & = & q(y,x,a),(a,x,b)=q(b,x,a),(a,b,x)=q(x,b,a).\end{array}$$

These last relations are nothing but the equations (3.6), (3.7), (3.8), (3.9), (3.10), (3.11). Therefore, the bilinear product given, for all $x, y \in \mathcal{A}(q)$ and all $a, b \in \mathcal{B}(q)$, by $(x+a)\star(y+b) = (x\cdot y + l_{\mathcal{B}}(a)y + r_{\mathcal{B}}(b)x) + (a*b + l_{\mathcal{A}}(x)b + r_{\mathcal{A}}(y)a)$ on the direct sum of the underlying vector spaces $\mathcal{A}(q)$ and $\mathcal{B}(q)$ induces a q-generalized flexible algebra structure on $\mathcal{A}(q) \oplus \mathcal{B}(q)$. □

In this case, the obtained q-generalized flexible algebra $(\mathcal{A}(q) \oplus \mathcal{B}(q), \star)$ is denoted by $\mathcal{A}(q) \bowtie^{l_{\mathcal{A}}, r_{\mathcal{A}}}_{l_{\mathcal{B}}, r_{\mathcal{B}}} \mathcal{B}(q)$, or simply $\mathcal{A}(q) \bowtie \mathcal{B}(q)$.

Definition 3.5. *The sixtuple* $(\mathcal{A}(q), \mathcal{B}(q), l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}})$ *such that* $(l_{\mathcal{A}}, r_{\mathcal{A}}, \mathcal{B}(q))$ *and* $(l_{\mathcal{B}}, r_{\mathcal{B}}, \mathcal{A}(q))$ *are bimodules of* $\mathcal{A}(q)$ *and* $\mathcal{B}(q)$*, respectively, and the linear maps* $l_{\mathcal{A}}, r_{\mathcal{A}}, l_{\mathcal{B}}, r_{\mathcal{B}}$ *satisfy the relations* (3.6) - (3.11)*, is called matched pair of the q-generalized flexible algebras* $\mathcal{A}(q)$ *and* $\mathcal{B}(q)$*.*

Remark 3.6. *Theorem 3.4 is a q-generalization of main theorems, well known in the literature. Indeed,*

- *For* $q = 0$*, Theorem 3.4 is exactly the fundamental theorem for the matched pair of associative algebras. See [4] and references therein.*
- *For* $q = 1$*, Theorem 3.4 is reduced to the fundamental theorem for the matched pair of center-symmetric algebras formulated in [12].*
- *For* $q = -1$*, Theorem 3.4 becomes the fundamental theorem for the matched pair of flexible algebras.*

§4. Basic properties of the q-generalized flexible algebras

In this section, we construct and discuss the basic definitions and main properties of the q-generalized flexible algebras.

Definition 4.1. *Let* $l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)$ *be the two above mentioned linear maps. Their dual maps are defined as:*

$$l^* : \mathcal{A}(q) \longrightarrow \mathbb{K}; < l^*(x)v^*, u >=< v^*, l(x)u >, \tag{4.1}$$

$$r^* : \mathcal{A}(q) \longrightarrow \mathbb{K}; < r^*(x)v^*, u >=< v^*, r(x)u >, \tag{4.2}$$

where $V^* = Hom(V, \mathbb{K})$*,* $\mathfrak{gl}(V^*)$ *is the linear group of* V^* *and* $< ., . >$ *is the natural pairing between* $\mathcal{A}(q)$ *and* $\mathcal{A}(q)^*$*.*

Theorem 4.2. *For any finite dimensional vector space V, suppose $l, r : \mathcal{A}(q) \to \mathfrak{gl}(V)$ are two linear maps such that l^* and r^* are their respective dual maps. Then, the following propositions are equivalent:*

1. *(l, r, V) is a bimodule of the q-generalized flexible algebra $\mathcal{A}(q)$,*
2. *(r^*, l^*, V^*) is a bimodule of the q-generalized flexible algebra $\mathcal{A}(q)$.*

Remark 4.3. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra and (l, r, V) be its bimodule.*

- *For $q = 0$, (r^*, l^*, V^*) is a bimodule of the associative algebra $\mathcal{A}(q = 0)$. See [3, 4] for more details.*
- *For $q = 1$, (r^*, l^*, V^*) is a bimodule of the center-symmetric algebra $\mathcal{A}(q = 1)$. See [12] for more details.*
- *For $q = -1$, (r^*, l^*, V^*) is a bimodule of the flexible algebra $\mathcal{A}(q = -1)$.*

Therefore, the associative, center-symmetric and flexible algebras have the same dual bimodule.

Proposition 4.4. *The triple $(R^*_{\cdot}, L^*_{\cdot}, A(q)^*)$, where $A(q)^*$ is the dual space of $\mathcal{A}(q)$ given by $A(q)^* = Hom(\mathcal{A}(q), \mathbb{K})$, is a bimodule of $\mathcal{A}(q)$.*

Proof. By considering Definition 2.3, Proposition 2.4 and Proposition 4.2, we deduce that $(R^*_{\cdot}, L^*_{\cdot}, A(q)^*)$ is a bimodule of $\mathcal{A}(q)$. □

Theorem 4.5. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. Suppose that there is a q-generalized flexible algebra structure "$\circ$" on its dual space $\mathcal{A}(q)^* = Hom(\mathcal{A}(q), \mathbb{K})$. The sixtuple $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$ if, and only if, the linear maps $R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ}$ given by, $\forall x, y \in \mathcal{A}(q), \forall a, b \in \mathcal{A}(q)^*$,*

$$\begin{aligned} \langle R^*_{\cdot}(x)a, y\rangle &= \langle a, R_{\cdot}(x)y\rangle = \langle a, y \cdot x\rangle, \\ \langle R^*_{\circ}(a)x, b\rangle &= \langle x, R_{\circ}(a)b\rangle = \langle x, b \circ a\rangle, \\ \langle L^*_{\cdot}(x)a, y\rangle &= \langle a, L_{\cdot}(x)y\rangle = \langle a, x \cdot y\rangle, \\ \langle L^*_{\circ}(a)x, b\rangle &= \langle x, L_{\circ}(a)b\rangle = \langle x, a \circ b\rangle, \end{aligned}$$

where $\langle , \rangle$ is a natural pairing between $\mathcal{A}(q)$ and $\mathcal{A}(q)^*$, satisfy the following relations:

$$\begin{aligned} &(R^*_{\circ}(a)x) \cdot y + R^*_{\circ}(L^*_{\cdot}(x)a)y - R^*_{\circ}(a)(x \cdot y) = \\ &q(L^*_{\circ}(a)(y \cdot x) - y \cdot (L^*_{\circ}(a)x) - L^*_{\circ}(R^*_{\cdot}(x)a)y), \end{aligned} \tag{4.3}$$

$$L^*_\circ(a)(x\cdot y) - x\cdot(L^*_\circ(a)y) - L^*_\circ(R^*_\cdot(y)a)x = q((R^*_\circ(a)y)\cdot x + R^*_\circ(L^*_\cdot(y)a)x - R^*_\circ(a)(y\cdot x)), \tag{4.4}$$

$$(L^*_\circ(a)x)\cdot y + R^*_\circ(R^*_\cdot(x)a)y - x\cdot(R^*_\circ(a)y) - L^*_\circ(L^*_\cdot(y)a)x = q((L^*_\circ(a)y)\cdot x + R^*_\circ(R^*_\cdot(y)a)x - y\cdot(R^*_\circ(a)x) - L^*_\circ(L^*_\cdot(x)a)y). \tag{4.5}$$

Proof. Consider a q-generalized flexible algebra $(\mathcal{A}(q),\cdot)$ and assume that there is a q-generalized flexible algebra structure "$\circ$" on its dual space $(\mathcal{A}(q))^*$. Using Definition 2.3, Proposition 2.4, and Proposition 4.2, we deduct that $(R^*_\cdot, L^*_\cdot, \mathcal{A}(q)^*)$ is a bimodule of $(\mathcal{A}(q),\cdot)$, and $(R^*_\circ, L^*_\circ, \mathcal{A}(q))$ is a bimodule of $(\mathcal{A}(q)^*,\circ)$. Setting the correspondences $l_{\mathcal{A}} \to R^*_\cdot$, $r_{\mathcal{A}} \to L^*_\cdot$, $l_{\mathcal{B}} \to R^*_\circ$ and $r_{\mathcal{B}} \to L^*_\circ$, and using the relations (2.4) and (2.5), we establish the equivalences between (3.10) and (3.7), (3.9) and (3.6), and (3.11) and (3.8). Further more, we also obtain that:

$$(R^*_\circ(a)x)\cdot y + R^*_\circ(L^*_\cdot(x)a)y - R^*_\circ(a)(x\cdot y) = q(L^*_\circ(a)(y\cdot x) - y\cdot(L^*_\circ(a)x) - L^*_\circ(R^*_\cdot(x)a)y) \tag{4.6}$$

and

$$L^*_\cdot(R^*_\circ(a)x)b + b\circ(L^*_\cdot(x)a) - L^*_\cdot(x)(b\circ a) = q(R^*_\cdot(x)(a\circ b) - R^*_\cdot(L^*_\circ(a)x)b - (R^*_\cdot(x)a)\circ b) \tag{4.7}$$

are equivalent, and

$$L^*_\circ(a)(x\cdot y) - x\cdot(L^*_\circ(a)y) - L^*_\circ(R^*_\cdot(y)a)x = q((R^*_\circ(a)y)\cdot x + R^*_\circ(L^*_\cdot(y)a)x - R^*_\circ(a)(y\cdot x)) \tag{4.8}$$

is equivalent to

$$R^*_\cdot(y)(a\circ b) - R^*_\cdot(L^*_\circ(a)y)b - (R^*_\cdot(y)a)\circ b = q(L^*_\cdot(R^*_\circ(a)y)b + b\circ(L^*_\cdot(y)a) - L^*_\cdot(y)(b\circ a)). \tag{4.9}$$

In addition, the relation (4.6) is exactly the same as (4.3), and (4.8) is exactly (4.4). Therefore, the sixtuple $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ)$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q),\cdot)$ and $(\mathcal{A}(q)^*,\circ)$ if and only if the linear maps $R^*_\cdot, L^*_\cdot, R^*_\circ, L^*_\circ$ satisfy the equations (4.3), (4.4) and (4.5). □

Remark 4.6. *Theorem 4.5 encompasses particular results known in the literature, namely:*

- *For* $q = 0$, *Theorem 4.5 is exactly reduced to the result obtained by Bai in [4], (see also references therein) giving the construction of the dual matched pair for the associative algebras.*

- *For $q = 1$, we recover the theorem relating the dual matched pair of center-symmetric algebras with the dual matched pair of Lie algebras, investigated in [12].*
- *For $q = -1$, Theorem 4.5 gives the dual matched pair of flexible algebras. This is a new result, given in this work for the first time, to our best knowledge of the literature.*

Corollary 4.7. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. Assume that there is a q-generalized flexible algebra structure "$\circ$" on the dual space $\mathcal{A}(q)^*$. There is a q-generalized flexible algebra structure "$\star$" on the vector space $\mathcal{A}(q) \oplus \mathcal{A}(q)^*$ given, for all $x, y \in \mathcal{A}(q)$ and for all $a, b \in \mathcal{A}(q)^*$, by:*

$$(x + a) \star (y + b) = (x \cdot y + R^*_{\circ}(a)y + L^*_{\circ}(b)x) + (a \circ b + R^*_{\cdot}(x)b + L^*_{\cdot}(y)a), \tag{4.10}$$

if, and only if, the sixtuple $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$.

Proof. According to Theorem 3.4 and Theorem 4.5, the linear product defined in the equation (4.10) satisfies the relation (2.1) if and only if $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$. □

Remark 4.8. *From Corollary 4.7, we conclude that both the flexible and anti-flexible algebras have the same matched pairs. The same result extends to associative algebras obtained for the parameter $q = 0$.*

Theorem 4.9. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. Suppose there is a q-generalized flexible algebra structure "$\circ$" on $\mathcal{A}(q)^*$ given by the linear map $\Delta^* : \mathcal{A}(q)^* \otimes \mathcal{A}(q)^* \to \mathcal{A}(q)^*$. Then, $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of q-generalized flexible algebras if and only if the linear map $\Delta : \mathcal{A}(q) \to \mathcal{A}(q) \otimes \mathcal{A}(q)$ satisfies the following relations:*

$$\tau(R_{\cdot}(y) \otimes \mathrm{id})\Delta(x) + \tau(\mathrm{id} \otimes L_{\cdot}(x))\Delta(y) - \tau\Delta(x \cdot y) = q(\Delta(y \cdot x) - (\mathrm{id} \otimes L_{\cdot}(y))\Delta(x) - (R_{\cdot}(x) \otimes \mathrm{id})\Delta(y)), \tag{4.11}$$

$$\Delta(x \cdot y) - (\mathrm{id} \otimes L_{\cdot}(x))\Delta(y) - (R_{\cdot}(y) \otimes \mathrm{id})\Delta(x) = q(\tau(R_{\cdot}(x) \otimes \mathrm{id})\Delta(y) + \tau(\mathrm{id} \otimes L_{\cdot}(y))\Delta(x) - \tau\Delta(y \cdot x)), \tag{4.12}$$

$$(\mathrm{id} \otimes R_{\cdot}(y))\Delta(x) + \tau(R_{\cdot}(x) \otimes \mathrm{id})\Delta(y) - \tau(L_{\cdot}(x) \otimes \mathrm{id})\Delta(y) - (L_{\cdot}(y) \otimes \mathrm{id})\Delta(x) = q((\mathrm{id} \otimes R_{\cdot}(x))\Delta(y) + \tau(R_{\cdot}(y) \otimes \mathrm{id})\Delta(x) - \tau(L_{\cdot}(y) \otimes \mathrm{id})\Delta(x) - (L_{\cdot}(x) \otimes \mathrm{id})\Delta(y)) \tag{4.13}$$

for all $x, y \in \mathcal{A}(q)$.

Proof. Considering the following relations for all $x, y \in \mathcal{A}(q)$ and for all $a, b \in \mathcal{A}(q)^*$:

$$\begin{aligned}
&\langle \tau(R_{\cdot}(y)\otimes \mathrm{id})\Delta(x), a\otimes b\rangle = \langle x, (R_{\cdot}^*(y)b)\circ a\rangle = \langle (R_{\circ}^*(a)x)\cdot y, b\rangle,\\
&\langle \tau(\mathrm{id}\otimes L_{\cdot}(y))\Delta(x), a\otimes b\rangle = \langle x, b\circ (L_{\cdot}^*(y)a)\rangle = \langle R_{\circ}^*((L_{\cdot}^*(y)a))x, b\rangle,\\
&\langle (\mathrm{id}\otimes L_{\cdot}(y))\Delta(x), a\otimes b\rangle = \langle x, a\circ (L_{\cdot}^*(y)b)\rangle = \langle y\cdot (L_{\circ}^*(a)x), b\rangle,\\
&\langle (R_{\cdot}(y)\otimes \mathrm{id})\Delta(x), a\otimes b\rangle = \langle x, (R_{\cdot}^*(y)a)\circ b\rangle = \langle L_{\circ}^*(R_{\cdot}^*(y)a)x, b\rangle,\\
&\langle \tau(L_{\cdot}(y)\otimes \mathrm{id})\Delta(x), a\otimes b\rangle = \langle x, (L_{\cdot}^*(y)b)\circ a\rangle = \langle y\cdot (R_{\circ}^*(a)x), b\rangle,\\
&\langle (L_{\cdot}(y)\otimes \mathrm{id})\Delta(x), a\otimes b\rangle = \langle x, (L_{\cdot}^*(y)a)\circ b\rangle = \langle L_{\circ}^*(L_{\cdot}^*(y)a)x, b\rangle,
\end{aligned}$$

we straightforwardly establish the equivalences between (4.11) and (4.3), (4.12) and (4.4), and (4.13) and (4.5). According to Theorem 4.5, the proof is achieved. □

Definition 4.10. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. A q-generalized flexible bialgebra structure on $\mathcal{A}(q)$ is a linear map $\Delta : \mathcal{A}(q) \to \mathcal{A}(q) \otimes \mathcal{A}(q)$ such that:*

- *$\Delta^* : \mathcal{A}(q)^* \otimes \mathcal{A}(q)^* \to \mathcal{A}(q)^*$ defines a q-generalized flexible algebra structure on $\mathcal{A}(q)^*$;*
- *Δ satisfies equations (4.11), (4.12) and (4.13).*

§5. Manin triple of q-generalized flexible algebras and bialgebras

We start with the following definitions, consistent with analogous formulations for the Lie algebras [16]:

Definition 5.1. *The triple $(\mathcal{A}(q), \mathcal{A}_1(q), \mathcal{A}_2(q))$, where:*

- *$\mathcal{A}(q)$ is a q-generalized flexible algebra together with a nondegenerate, invariant and symmetric bilinear form $\mathfrak{B}$, and*
- *$\mathcal{A}_1(q)$ and $\mathcal{A}_2(q)$ are two Lagrangian sub-q-generalized flexible algebras of $\mathcal{A}(q)$ i.e., for all $x, y \in \mathcal{A}_1(q)$, $a, b \in \mathcal{A}_2(q)$, $\mathfrak{B}(x, y) = 0 = \mathfrak{B}(a, b)$, such that $\mathcal{A}(q) = \mathcal{A}_1(q) \oplus \mathcal{A}_2(q)$,*

is a Manin triple of the q-generalized flexible algebra $\mathcal{A}(q)$.

Definition 5.2. *Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. Suppose that there is a q-generalized flexible algebra structure "$\circ$" on its dual space $\mathcal{A}(q)^*$. A standard Manin triple of the q-generalized flexible algebras $\mathcal{A}(q)$ and $\mathcal{A}(q)^*$ associated to a symmetric, nondegenerate, invariant bilinear form $\mathfrak{B}$ defined on the vector space $\mathcal{A}(q) \oplus \mathcal{A}(q)^*$ by:*

$$\mathfrak{B}(x + a, y + b) = < x, b > + < y, a >, \tag{5.1}$$

for all $x, y \in \mathcal{A}(q)$ and all $a, b \in \mathcal{A}(q)^*$, where the bilinear product $< .,. >$ is the natural pairing between the vector spaces $\mathcal{A}(q)$ and $\mathcal{A}(q)^*$, is a triple $(\mathcal{A}(q) \oplus \mathcal{A}(q)^*, \mathcal{A}(q), \mathcal{A}(q)^*)$ such that the bilinear product "$\star$" defined for all $x, y \in \mathcal{A}(q)$ and all $a, b \in \mathcal{A}(q)^*$ by:

$$(x + a) \star (y + b) = (x \cdot y + R^*_{\circ}(a)y + L^*_{\circ}(b)x) + (a \circ b + R^*_{\cdot}(x)b + L^*_{\cdot}(y)a)$$

realizes a q-generalized flexible algebra structure on $\mathcal{A}(q) \oplus \mathcal{A}(q)^*$.

Theorem 5.3. *Suppose there is a q-generalized flexible algebra structure "$\circ$" on the dual space $\mathcal{A}(q)^*$. Then, the sixtuple $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$ if, and only if, $(\mathcal{A}(q) \oplus \mathcal{A}(q)^*, \mathcal{A}(q), \mathcal{A}(q)^*)$ is a standard Manin triple of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$.*

Proof. Let $(\mathcal{A}(q), \cdot)$ be a q-generalized flexible algebra. Assume that there is a q-generalized flexible algebra structure "$\circ$" on its dual vector space $\mathcal{A}(q)^*$. From Corollary 4.7, the vector space $\mathcal{A}(q) \oplus \mathcal{A}(q)^*$ has a q-generalized flexible algebra structure here denoted by "$\star$" given, for all $x, y \in \mathcal{A}(q)$ and all $a, b \in \mathcal{A}(q)^*$, by:

$$(x + a) \star (y + b) = (x \cdot y + R^*_{\circ}(a)y + L^*_{\circ}(b)x) + (a \circ b + R^*_{\cdot}(x)b + L^*_{\cdot}(y)a).$$

In fact, by its definitions, the triple $(L_{\cdot}, R_{\cdot}, A(q))$ is a bimodule of $(\mathcal{A}(q), \cdot)$ and $(L_{\circ}, R_{\circ}, A(q)^*)$ is a bimodule of $(\mathcal{A}(q)^*, \circ)$. According to Proposition 4.4, $(R^*_{\cdot}, L^*_{\cdot}, A(q)^*)$ is a bimodule of $(\mathcal{A}(q), \cdot)$, and $(R^*_{\circ}, L^*_{\circ}, A(q))$ is a bimodule of $(\mathcal{A}(q)^*, \circ)$. Then, the sixtuple $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ is a matched pair of the q-generalized flexible algebras $(\mathcal{A}(q), \cdot)$ and $(\mathcal{A}(q)^*, \circ)$.

Besides, we have for all $x, y, z \in \mathcal{A}(q)$ and for all $a, b, c \in \mathcal{A}(q)^*$,

$$\begin{aligned} \mathfrak{B}((x + a) \star (y + b), (z + c)) &= < x \cdot y, c > + < y, c \circ a > \\ &+ < x, b \circ c > + < z, a \circ b > \\ &+ < z \cdot x, b > + < y \cdot z, a >, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \mathfrak{B}((x + a), (y + b) \star (z + c)) &= < x, b \circ c > + < x \cdot y, c > \\ &+ < z \cdot x, b > + < y \cdot z, a > \\ &+ < z, a \circ b > + < y, c \circ a > . \end{aligned} \tag{5.3}$$

Therefore, from the relations (5.2) and (5.3), we have the required result. □

Theorem 5.4. *Suppose that there is a q-generalized flexible algebra structure "$\circ$" on the dual space $\mathcal{A}(q)^*$ provided by the dual map of the linear map defined as $\Delta : \mathcal{A}(q) \to \mathcal{A}(q) \otimes \mathcal{A}(q)$. The following propositions are equivalent:*

1. $(\mathcal{A}(q)\oplus\mathcal{A}(q)^*, \mathcal{A}(q), \mathcal{A}(q)^*)$ *is a standard Manin triple of the q-generalized flexible algebras* $\mathcal{A}(q)$ *and* $\mathcal{A}(q)^*$ *with the nondegenerate symmetric bilinear form* ω *defined on* $\mathcal{A}(q) \oplus \mathcal{A}(q)^*$*, for all* $x, y \in \mathcal{A}(q)$ *and all* $a, b \in \mathcal{A}(q)^*$ *by:* $\omega(x + a, y + b) :=< x, b > + < y, a >$*, where* $< ., . >$ *is the natural pairing between* $\mathcal{A}(q)$ *and* $\mathcal{A}(q)^*$*.*
2. *The sixtuple* $(\mathcal{A}(q), \mathcal{A}(q)^*, R^*_{\cdot}, L^*_{\cdot}, R^*_{\circ}, L^*_{\circ})$ *is a matched pair of the q-generalized flexible algebras* $(\mathcal{A}(q), \cdot)$ *and* $(\mathcal{A}(q)^*, \circ)$*.*
3. $(A(q), \cdot, \Delta)$ *is a q-generalized flexible bialgebra.*

Proof. By considering the Theorem 5.3, we deduct (1) $\Longleftrightarrow$ (2). Using the Theorem 4.9, we have the equivalence (2) $\Longleftrightarrow$ (3). □

§6. Application to octonion algebra

Definition 6.1. *An octonion algebra* $\mathcal{O}$ *is an eight dimensional vector space spanned by elements* $\{e_0, e_1, \cdots, e_7\}$ *satisfying the following relations:* $\forall i, j, k = 1, \cdots, 7$,

$$e_0^2 = e_0, e_ie_0 = e_i = e_0e_i, e_ie_j = -\delta_{ij}e_0 + c_{ijk}e_k, \tag{6.1}$$

where the fully antisymmetric structure constants c_{ijk} *are taken to be* 1 *for the combination of indexes:*

$$(ijk) \in \{(124), (137), (156), (235), (267), (346), (457)\},$$

and otherwise 0*. This bilinear product is given in Table 6.1:*

$\curvearrowright$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	$-e_0$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	$-e_0$	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	$-e_0$	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-e_0$	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-e_0$	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-e_0$

Table 6.1: Multiplication table of octonion algebra.

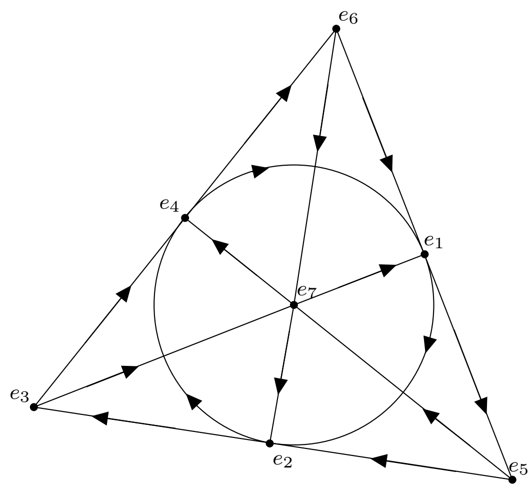

Figure 1: Realization of octonion algebra.

The associator of the octonion algebra $\mathcal{O} = Span\{e_0, e_1, \cdots, e_7\}$ defined as:

(6.2) $e_{ijk} := (e_i, e_j, e_k) = (e_ie_j)e_k - e_i(e_je_k), \forall i, j, k \in \{0, 1, 2, \cdots, 7\}$

obeys the following relations: $\forall i, j, k \in \{1, 2, \cdots, 7\}$,

$$\begin{aligned}(e_0, e_i, e_j) &= (e_i, e_0, e_j) = (e_i, e_j, e_0) = (e_i, e_i, e_j) \\ &= (e_i, e_j, e_i) = (e_i, e_j, e_i) = 0\end{aligned} \tag{6.3}$$

$$\begin{aligned}(e_i, e_j, e_k) &= \sum_{m=1}^{7}(c_{ijm}\delta_{mk} - c_{jkm}\delta_{im})e_0 \\ &+ \sum_{n=1}^{7}\sum_{m=1}^{7}(c_{ijm}c_{mkn} - c_{jkm}c_{imn})e_n.\end{aligned} \tag{6.4}$$

The associator (e_i, e_j, e_k) is also denoted by $e_{ij}e_k$.

Proposition 6.2. *Let $\mathcal{O}$ be an octonion algebra with basis $\{e_0, e_1, \cdots, e_7\}$. We have:*

1. *The 4 dimensional sub-algebras, spanned by the elements $\{e_0, e_i, e_j, e_k\}$, where the index $(ijk) \in \{(124), (137), (156), (235), (267), (346), (457)\}$, are associative, i.e. their associator vanishes. The associator (e_i, e_j, e_k) containing repeated indexes, or elements zero, also vanishes. Besides, the vector space $\{e_0, e_i, e_j, e_k\}$ is not stable under the bilinear product defined in* (6.1).

$\curvearrowright$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_{00}	0	0	0	0	0	0	0	0
e_{01}	0	0	0	0	0	0	0	0
e_{11}	0	0	0	0	0	0	0	0
e_{12}	0	0	0	$-2e_6$	0	$2e_7$	$2e_3$	$-2e_5$
e_{22}	0	0	0	0	0	0	0	0
e_{23}	0	$-2e_6$	0	0	$-2e_7$	0	$2e_1$	$2e_4$
e_{33}	0	0	0	0	0	0	0	0
e_{34}	0	$2e_5$	$-2e_7$	0	0	$-2e_1$	0	$2e_2$
e_{44}	0	0	0	0	0	0	0	0
e_{45}	0	$2e_3$	$2e_6$	$-2e_1$	0	0	$-2e_2$	0
e_{55}	0	0	0	0	0	0	0	0
e_{56}	0	0	$2e_4$	$2e_7$	$-2e_2$	0	0	$-2e_3$
e_{66}	0	0	0	0	0	0	0	0
e_{67}	0	$-2e_4$	0	$2e_5$	$2e_1$	$-2e_3$	0	0
e_{77}	0	0	0	0	0	0	0	0

Table 6.2: Table of associator composition for the octonion algebra.

2. *Other associators (e_i, e_j, e_k) do not vanish and are still fully skew - symmetric for the permutation of indexes (ijk) if and only if they belong to the set* $\{(123), (125), (126), (127), (234), (236), (237), (341), (342), (345), (347), (451), (452), (453), (456), (562), (563), (564), (567), (671), (673), (674), (675)\}$.

Proof.

1. By direct calculation, we have $(e_m, e_n, e_p) = 0$ if and only if (mnp) belongs to $\{(124), (137), (156), (235), (267), (346), (457)\}$ or at least one of elements of the triple (mnp) is repeated, for all $m, n, p \in \{0, 1, \cdots, 7\}$. In only this latter case, the space spanned by e_m, e_m, e_p does not possess the stability under octonion multiplication, and therefore it is not a subalgebra of $\mathcal{O}$.

2. A direct computation of the associator (e_i, e_j, e_k) and the permutation for $(ijk) \in \{(123), (125), (126), (127), (234), (236), (237), (341), (342), (345), (347), (451), (452), (453), (456), (562), (563), (564), (567), (671), (673), (674), (675)\}$ give the result. □

Definition 6.3. *A bimodule of the octonion algebra $\mathcal{O}$ is the triple (l, r, V), where V is a finite dimensional vector space, equipped with the basis* $\{u_1, u_2, \cdots, u_n\}$

with $n \geqslant 7$, *and* $l, r : \mathcal{O} \to \mathfrak{gl}(V)$ *are two linear maps satisfying:*

$$l(e_0) = \mathrm{id} = r(e_0), \tag{6.5}$$

$$l(e_i) = -r(e_i), \quad \forall i, j, k = 0, 1, \cdots, 7, \tag{6.6}$$

for all $(ijk) \in \{(124), (137), (156), (235), (267), (346), (457)\}$ *and its cyclic permutation,*

$$l(e_i)u_j + r(e_j)u_i = u_k, \tag{6.7}$$

and, for all non cyclic permutation of $(ijk) \in \{(124), (137), (156), (235), (267), (346), (457)\}$,

$$l(e_i)u_j + r(e_j)u_i = -u_k. \tag{6.8}$$

Proposition 6.4. *Let* $\mathcal{O}$ *be an octonion algebra, and* $l, r : \mathcal{O} \to \mathfrak{gl}(V)$ *be two linear maps. The triple* (l, r, V) *is a bimodule of the octonion algebra* $\mathcal{O}$ *if, and only if, there exists an octonion algebra structure "$*$" on the semi-direct vector space* $\mathcal{O} \oplus V$ *given by* , $\forall e_i, e_j \in \mathcal{O}, \forall u, v \in V, i, j = 0, 1, \cdots, 7$,

$$(e_i + u) * (e_j + v) := e_i e_j + l(e_i)v + r(e_j)u. \tag{6.9}$$

Proof. The associator corresponding to the bilinear product (6.9) is given, for all $i, j, k \in \{0, 1, \cdots, 7\}$ and for all $u, v, w \in V$, by

$$\begin{aligned}(e_i + u, e_j + v, e_k + w) &= (e_i, e_j, e_k) + (r(e_k)r(e_j) - r(e_j e_k))u \\ &+ (r(e_k)l(e_i) - l(e_i)r(e_k))v + (l(e_i e_j) - l(e_i)l(e_j))w.\end{aligned}$$

It satisfies the conditions on the Table 6.2: if and only if the linear maps l, r obey the relations (6.5) - (6.8). □

Theorem 6.5. *For an octonion algebra* $\mathcal{O}$ *spanned by* $\{e_0, e_1, \cdots, e_7\}$, *the following relations are equivalent:*

1.
$$[e_k, e_i e_j] = [e_k, e_i]e_j + e_i[e_k, e_j], \tag{6.10}$$

2.
$$c_{ijm}c_{kml} = c_{kim}c_{mjl} + c_{kjm}c_{iml}, \tag{6.11}$$

where the reals c_{ijk} *are defined in the equation* (6.1), *for all* $i, j, k \in \{0, 1, \cdots, 7\}$.

Theorem 6.5 is known as Myung Theorem. For more details, see [18, 20].

Proposition 6.6. *Let $\mathcal{O}$ be an octonion algebra with basis* $\{e_0, e_1, \cdots, e_7\}$. *Assuming that (l, r, V) is a bimodule of $\mathcal{O}$, the following relation is satisfied:*

$$2\delta_{ij} + c_{ijk}r(e_k) + 2r(e_j)r(e_i) = 0, \tag{6.12}$$

or, equivalently,

$$2\delta_{ij} - c_{ijk}l(e_k) + 2l(e_j)l(e_i) = 0, \tag{6.13}$$

$\forall i, j, k = 0, 1, \cdots, 7$, *where the c_{ijk}'s are defined in the equation* (6.1).

Proof. The identity (6.12) becomes straightforward by combining the relations (6.6) and (6.8), and the equivalence is guaranteed by the identity (6.6). □

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