SUT Journal of Mathematics
Vol. 56, No. 2 (2020), 171–177

Lifting Brauer-friendly modules

Nobukatsu Watanabe

(Received July 22, 2020)

Abstract. Let $\mathcal{O}$ be a complete discrete valuation ring with a residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, G a finite group, and b a block of kG with lift $\hat{b}$. In this paper, we show that any indecomposable Brauer-friendly kGb-module satisfying certain condition is liftable to an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module.

AMS 2010 *Mathematics Subject Classification.* 20C20.

Key words and phrases. Brauer-friendly module, liftability.

§1. Introduction

Let p be a prime number, $\mathcal{O}$ a complete discrete valuation ring with a residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, and G a finite group. Throughout this paper, RG-modules mean finitely generated RG-lattices, for $R \in \{\mathcal{O}, k\}$. A kG-module M is said to be *liftable* if there exists an $\mathcal{O}G$-module $\widehat{M}$ such that $k \otimes_{\mathcal{O}} \widehat{M} \cong M$. In the modular representation theory of finite groups, it is important to find a class of liftable modules. A few classes of liftable modules are known. For example, any p-permutation kG-module is liftable, in particular, any projective kG-module is liftable. Moreover, they lift to a p-permutation $\mathcal{O}G$-module and a projective $\mathcal{O}G$-module, respectively. In addition, any endo-permutation kG-module is liftable to an endo-permutation $\mathcal{O}G$-module. More details on these examples may be found in [3, 1. Introduction]. In [5], J.-M. Urfer introduced endo-p-permutation modules, which are generalizations of endo-permutation modules. In [3], Lassueur and Thévenaz proved that any endo-p-permutation kG-module is liftable to an endo-p-permutation $\mathcal{O}G$-module. By [5, Theorem 1.5], any indecomposable endo-p-permutation RG-module has a G-stable endo-permutation source. Hence, in [3, Remark 4.3], Lassueur and Thévenaz raised the question of whether or not kG-modules with an endo-permutation source which is not necessarily G-stable are liftable. In

[1], E. Biland introduced Brauer-friendly modules, which are generalizations of endo-p-permutation modules. Any indecomposable Brauer-friendly module has an endo-permutation source, which is not necessarily G-stable.

Let b be a block of kG with lift $\hat{b}$. In this study, we show that any indecomposable Brauer-friendly kGb-module satisfying certain condition is liftable to an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module.

In Section 2, we recall the definition of subpairs, fusion systems, and Brauer morphisms. Also, we review the definition of Brauer-friendly modules that E. Biland defined in [1]. In Section 3, we write lemmas to prove the main theorem. In section 4, we prove the main theorem.

§2. Notation

Throughout this paper, we use the following notations and terminologies. Let p be a prime number, $\mathcal{O}$ a complete discrete valuation ring with a residue field k of characteristic p, and set $\mathfrak{p} = J(\mathcal{O})$ and $R \in \{\mathcal{O}, k\}$. Throughout this paper, blocks mean block idempotents. We fix a finite group G and a block b of RG. For any $x \in \mathcal{O}G$, we denote by $\overline{x}$ its image by the natural map $\mathcal{O}G \twoheadrightarrow kG$. By the lifting theorem of idempotents, for a primitive idempotent $i \in kG$, there exists a primitive idempotent $\hat{i} \in \mathcal{O}G$ such that $\overline{\hat{i}} = i$. We only use the symbol $\hat{-}$ to satisfy the property, for primitive idempotents. For any G-set X and any subgroup H of G, we set $X^H = \{x \in X \mid h \cdot x = x, h \in H\}$. For any indecomposable RG-module M, we denote by $\mathrm{vtx}(M)$ a vertex of M. For any two RG-modules M and N, we write $M \mid N$ if M is isomorphic to a direct summand of N. For any RG-module M and any subgroup H of G, the relative trace map $\mathrm{Tr}_H^G : M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) = \sum_{x \in G/H} x \cdot m$. For any RG-module M and any p-subgroup P of G, the Brauer construction of M with respect to P is the $k\overline{N}_G(P)$-module defined by $\mathrm{Br}_P(M) = M^P/(\sum_{Q<P} \mathrm{Tr}_Q^P(M^Q) + J(R)M^P)$. We denote by $\mathrm{br}_P^M : M^P \to \mathrm{Br}_P(M)$ the natural map and we call this map the Brauer morphism of M with respect to P. We set $\mathrm{br}_P = \mathrm{br}_P^{RG}$.

We recall the definition of subpairs. A subpair of G is a pair (P, b_P) consisting of a p-subgroup P of G and a block b_P of $\mathcal{O}C_G(P)$. We call the subpair (P, b_P) a (G, b)-subpair if $\overline{b}_P \mathrm{br}_P(b) \neq 0$. For (G, b)-subpair (P, b_P), the block b_P is also a block of $\mathcal{O}H$ for a subgroup H such that $C_G(P) \leq H \leq N_G(P, b_P)$. The set of (G, b)-subpairs is a poset, and the group G acts on the set by conjugation. The analogous definitions are defined with k instead of $\mathcal{O}$.

We recall the definitions of Brauer categories and fusion systems. The Brauer category $\mathbf{Br}(G, b)$ is defined as follows: the objects of $\mathbf{Br}(G, b)$ are the (G, b)-subpairs and for any two objects (P, b_P), (Q, b_Q), the morphism set $\mathrm{Hom}_{\mathbf{Br}(G,b)}((P, b_P), (Q, b_Q))$ is the set of all group homomorphisms $\phi : P \to Q$

such that there exists $g \in G$ satisfying ${}^g(P, b_P) \leq (Q, b_Q)$ and $\phi(x) = {}^g x$ for any $x \in P$. Let (P, b_P) be a (G, b)-subpair. The fusion system $\mathcal{F}_{(P,b_P)}(G, b)$ is defined as follows: the objects of $\mathcal{F}_{(P,b_P)}(G, b)$ are the subgroup of P, and for any two objects Q and R, the morphism set $\mathrm{Hom}_{\mathcal{F}_{(P,b_P)}(G,b)}(Q, R)$ is the set of all group homomorphisms $\phi : Q \to R$ such that there exists $g \in G$ satisfying ${}^g(Q, b_Q) \leq (R, b_R)$ for $(Q, b_Q), (R, b_R) \leq (P, b_P)$ and $\phi(x) = {}^g x$ for any $x \in Q$.

We review the definition of vertex subpairs and source triples from [1]. Let M be an indecomposable RGb-module. A (G, b)-subpair (P, b_P) is called a vertex subpair of M if $M \mid bRGb_P \otimes_{RP} V$ and $P \leq_G \mathrm{vtx}(M)$ for some indecomposable RP-module V. For such V, it is called a source of M with respect to the vertex subpair (P, b_P). A triple (P, b_P, V) is called a source triple of M if V is a source of M with respect to the vertex subpair (P, b_P). If M has a source triple (P, b_P, V), then a vertex of M is P and a source of M is V, from [1, Lemma 1].

We recall the definition of fusion-stable endo-permutation modules. Let M be an RG-module. We call M an *endo-(p-)permutation RG-module* if $\mathrm{End}_R(M)$ is a (p-)permutation RG-module.

Definition 2.1. Let (P, b_P) be a (G, b)-subpair, V an endo-permutation RP-module, and set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G, b)$. We say that V is *$\mathcal{F}$-stable* if the endo-permutation $\mathcal{O}Q$-modules $\mathrm{Res}^P_Q(V)$ and $\mathrm{Res}_{\phi_{g^{-1}}}(V) \cong \mathrm{Res}^{{}^gP}_Q({}^gV)$ are compatible for any subgroup Q of P and any $\phi_{g^{-1}} \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. We call the triple (P, b_P, V) a *fusion-stable endo-permutation source triple* if V is an $\mathcal{F}$-stable indecomposable endo-permutation RP-module with vertex P.

We review the definitions of Brauer-friendly modules defined in [1].

Definition 2.2 ([1, Definition 6])**.** Let (P_1, b_{P_1}, V_1) and (P_2, b_{P_2}, V_2) be fusion-stable endo-permutation source triples in (G, b). We say that (P_1, b_{P_1}, V_1) and (P_2, b_{P_2}, V_2) are *compatible* if the endo-permutation RQ-modules $\mathrm{Res}_{\phi_1}(V_1)$ and $\mathrm{Res}_{\phi_2}(V_2)$ are compatible for any (G, b)-subpair (Q, b_Q) and any $\phi_i \in \mathrm{Hom}_{\mathrm{Br}(G,b)}((Q, b_Q), (P_i, b_{P_i}))$ for $i \in \{1, 2\}$.

Definition 2.3 ([1, Definition 8])**.** Let M be an RGb-module which admits the decomposition $M = \bigoplus_{1\leq i\leq n} M_i$ of M, where each M_i is an indecomposable RGb-module with source triple (P_i, b_{P_i}, V_i). We say that the RGb-module M is *Brauer-friendly* if (P_i, b_{P_i}, V_i) is a fusion-stable endo-permutation source triple for any $i \in \{1, \ldots, n\}$, and, (P_i, b_{P_i}, V_i) and (P_j, b_{P_j}, V_j) are compatible for any $i, j \in \{1, \ldots, n\}$.

Remark 2.4. Indecomposable endo-p-permutation RGb-modules are indecomposable Brauer-friendly RGb-modules (see [1, The sentences under Definition 8]).

§3. Lemmas

The following lemma can be proved in the same way as the proof of [4, Proposition 3.2 (i)].

Lemma 3.1. Let G be a finite group, b a block of $\mathcal{O}G$ with a defect group D, i a source idempotent of the block b, and P a subgroup of D. Set $A = i\mathcal{O}Gi$ and $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G,b)$, where b_P is the unique block of $\mathcal{O}C_G(P)$ such that $\bar{b}_P \mathrm{br}_P(i) \neq 0$. Let V be an $\mathcal{F}$-stable endo-permutation $\mathcal{O}P$-module having an indecomposable direct summand with vertex P. Set $U = A \otimes_{\mathcal{O}P} V$. Then, as an $\mathcal{O}P$-module, U is an endo-permutation module, and U has a direct summand isomorphic to V.

The following lemma can be proved in the similar way as the proof of [2, Lemma 8.3].

Lemma 3.2. Let G be a finite group, b a block of $\mathcal{O}G$ with a defect group D, i a source idempotent of the block b, and P a subgroup of D. Set $\mathcal{F} = \mathcal{F}_{(P,b_P)}(G,b)$, where b_P is the unique block of $\mathcal{O}C_G(P)$ such that $\bar{b}_P \mathrm{br}_P(i) \neq 0$. Let V be an indecomposable $\mathcal{F}$-stable endo-permutation $\mathcal{O}P$-module with vertex P. Set $X = OGi \otimes_{\mathcal{O}P} V$. The canonical algebra homomorphism

$$\mathrm{End}_{\mathcal{O}G}(X) \to \mathrm{End}_{kG}(k \otimes_{\mathcal{O}} X)$$

is surjective. In particular, for any indecomposable direct summand M of $k \otimes_{\mathcal{O}} X$, there is an indecomposable direct summand $\widehat{M}$ of X such that $k \otimes_{\mathcal{O}} \widehat{M} \cong M$.

Proof. In the proof of [2, Lemma 8.3], we use Lemma 3.1 instead of [4, Proposition 4.1]. □

To prove the main theorem, we need the following lemma.

Lemma 3.3 ([2, Lemma 8.4])**.** Let P be a finite p-group and $\mathcal{F}$ a saturated fusion system on P. The canonical map $\mathcal{D}_{\mathcal{O}}(P,\mathcal{F}) \to \mathcal{D}_k(P,\mathcal{F})$ is surjective.

The following lemmas are over k version of [1, Lemma 3 (i), (ii)] and can be proved in the similar way as the proof of [1, Lemma 3 (i), (ii)].

Lemma 3.4 ([1, Lemma 3 (i)])**.** Let M be an indecomposable kGb-module with a source triple (P, b_P, V). There exists a primitive idempotent i of the algebra $(kGb)^P$ such that $b_P \mathrm{br}_P(i) \neq 0$ and that M is isomorphic to a direct summand of the kGb-module $kGi \otimes_{kP} V$.

Lemma 3.5 ([1, Lemma 3 (ii)])**.** Let M be an indecomposable kGb-module with a source triple (P, b_P, V). There exists a defect group D of the block b such that $P \leq D$ and there exists a primitive idempotent j of the algebra $(kGb)^D$ such that $\mathrm{br}_D(j) \neq 0$ and $b_P \mathrm{br}_P(j) \neq 0$, and that M is isomorphic to a direct summand of the kGb-module $kGj \otimes_{kP} V$.

§4. Main theorem

The following theorem is the main theorem of this paper.

Theorem 4.1. Let G be a finite group, b a block of kG with a defect group D, and M an indecomposable Brauer-friendly kGb-module with a source triple (P, b_P, S). Suppose that $\mathcal{F} := \mathcal{F}_{(P,b_P)}(G, b)$ is saturated. Then there exists an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module $\widehat{M}$ with source triple $(P, \widehat{b}_P, \widehat{S})$ such that $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$ and $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$.

Proof. By Lemma 3.3, there exists an indecomposable endo-permutation $\mathcal{O}P$-module $\widehat{S}$ such that $\widehat{S} \in \mathcal{D}_{\mathcal{O}}(P, \mathcal{F})$ and $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$. Let $bb_P = i_1 + \cdots + i_n$ be a decomposition of bb_P into mutually orthogonal primitive idempotents in the algebra $(kGb)^P$. By Lemma 3.4 and its proof, for some primitive idempotent $i_\ell \in (kGb)^P$, we have $b_P \mathrm{br}_P(i_\ell) \neq 0$ and a following relation

$$M \mid kGi_\ell \otimes_{kP} S \mid \bigoplus_{1 \leq j \leq n} (kGi_j \otimes_{kP} S) = bkGb_P \otimes_{kP} S.$$

Set $i = i_\ell$. Also, by Lemma 3.5 and its proof, after suitable retake of a defect group, there exists a source idempotent j in $(kGb)^D$ such that $M \mid kGj \otimes_{kP} S$ and $b_P \mathrm{br}_P(j) \neq 0$, and there exists a decomposition $j = {}^x i + z_1 + \cdots + z_m$ of j into mutually orthogonal primitive idempotents in the algebra $(kGb)^P$, for some $x \in ((kGb)^P)^\times$. By the lifting theorem of idempotents, there exists a decomposition $\hat{b}\hat{b}_P = \hat{i}_1 + \cdots + \hat{i}_n$ of $\hat{b}\hat{b}_P$ into mutually orthogonal primitive idempotents in the algebra $(\mathcal{O}G\hat{b})^P$. Also, by the lifting theorem of idempotents, there exists a source idempotent $\hat{j}$ in $(\mathcal{O}G\hat{b})^D$ and a decomposition $\hat{j} = \widehat{{}^x i} + \hat{z}_1 + \cdots + \hat{z}_m$ of $\hat{j}$ into mutually orthogonal primitive idempotents in the algebra $(\mathcal{O}G\hat{b})^P$. By the lifting theorem of idempotents, $\hat{i} := \hat{i}_\ell$ and $\widehat{{}^x i}$ are conjugate. Hence we get an isomorphism of $\mathcal{O}Gb - \mathcal{O}P$-bimodule

$$\mathcal{O}G\hat{i} \cong \mathcal{O}G\widehat{{}^x i}.$$

Therefor we have

$$\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} \cong (\mathcal{O}G\widehat{{}^x i} \otimes_{\mathcal{O}P} \widehat{S}) \mid \mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}. \tag{4.1}$$

By $\bar{\hat{j}} = j$ and $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$, we get

$$M \mid kGj \otimes_{kP} S \cong k \otimes_{\mathcal{O}} \mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}.$$

Then by Lemma 3.2, there exists a direct summand $\widehat{M}$ of $\mathcal{O}G\hat{j} \otimes_{\mathcal{O}P} \widehat{S}$ such that

$$k \otimes_{\mathcal{O}} \widehat{M} \cong \widehat{M}/\mathfrak{p}\widehat{M} \cong M.$$

In the following, we show that the module $\widehat{M}$ has a source triple $(P, \hat{b}_P, \widehat{S})$. Let

$$\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} = \bigoplus_{1 \leq i \leq m} \widehat{M}'_i,$$

be a decomposition of $\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S}$ as a direct sum of indecomposable $\mathcal{O}G\hat{b}$-modules. Then we have an isomorphism

$$\bigoplus_{1 \leq i \leq m} (k \otimes_{\mathcal{O}} \widehat{M}'_i) \cong k \otimes_{\mathcal{O}} (\mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S}) \cong kGi \otimes_{kP} S.$$

Here by Lemma 3.2 and the lifting theorem of idempotents, if $k \otimes_{\mathcal{O}} \widehat{M}'_i \neq 0$, then $k \otimes_{\mathcal{O}} \widehat{M}'_i$ is indecomposable. Therefor by the Krull-Schmidt theorem, there exists some j such that

$$k \otimes_{\mathcal{O}} \widehat{M}'_j \cong M.$$

Therefor by (4.1) and the lifting theorem of idempotents, we get

$$\widehat{M} \cong \widehat{M}'_j \mid \mathcal{O}G\hat{i} \otimes_{\mathcal{O}P} \widehat{S} \mid \hat{b}\mathcal{O}G\hat{b}_P \otimes_{\mathcal{O}P} \widehat{S}.$$

By this relation, $\widehat{M}$ is relative P-projective. Also, if for some $Q \lneq_G P$, $\widehat{M}$ is a direct summand of $\mathrm{Ind}_Q^G(\mathrm{Res}_Q^G(\widehat{M}))$, then M is a direct summand of $\mathrm{Ind}_Q^G(\mathrm{Res}_Q^G(M))$, *i.e.* M is relative Q-projective. This is a contradiction. Hence, we get $\mathrm{vtx}(\widehat{M}) = P$. Since $(P, \hat{b}_P, \widehat{S})$ is a fusion-stable endo-permutation source triple, $\widehat{M}$ is an indecomposable Brauer-friendly $\mathcal{O}G\hat{b}$-module with a source triple $(P, \hat{b}_P, \widehat{S})$ such that $\widehat{S}/\mathfrak{p}\widehat{S} \cong S$ and $\widehat{M}/\mathfrak{p}\widehat{M} \cong M$. □

Remark 4.2.

1. In general, the lifts given by Theorem 4.1 are not necessarily unique.
2. Our proof of Theorem 4.1 depends on the classification of endo-permutation modules.

Remark 4.3. Let G, H be finite groups and b, c blocks of kG, kH with a defect group D, respectively. In [2], Kessar and Linckelmann proved that any indecomposable $kGb - kHc$-bimodule with a fusion-stable endo-permutation kD-source which induce a Morita equivalence (or a stable equivalence of Morita type) between kGb and kHc is liftable. Moreover, it lifts to an indecomposable $\mathcal{O}G\hat{b}$$-$$\mathcal{O}H\hat{c}$-bimodule with a fusion-stable endo-permutation $\mathcal{O}D$-source which induce a Morita equivalence (or a stable equivalence of Morita type) between $\mathcal{O}G\hat{b}$ and $\mathcal{O}\hat{c}$, under the assumption that k is a splitting field for all subgroups of $G \times H$.

Acknowledgement

The author would like to thank Naoko Kunugi for her careful reading and useful comments in an earlier version of the paper. The author also would like to thank the referee for her or his useful comments.

References

[1] E. Biland, *Brauer-friendly modules and slash functors*, J. Pure Appl. Algebra **218** (2014), 2319–2336.

[2] R. Kessar and M. Linckelmann, *Descent of equivalences and character bijections*, in: Geometric and topological aspects of the representation theory of finite groups, Springer Proc. Math. Stat., 242, Springer, (2018), 181–212.

[3] C. Lassueur and J. Thévenaz, *Lifting endo-p-permutation modules*, Arch. Math. (Basel) **110** (2018), 205–212.

[4] M. Linckelmann, *On stable equivalences with endopermutation source*, J. Algebra **434** (2015), 27–45.

[5] J.-M. Urfer, *Endo-p-permutation modules*, J. Algebra **316** (2007), 206–223.

Nobukatsu Watanabe
Department of Mathematics, Tokyo University of Science
Kagurazaka 1–3, Shinjuku, Tokyo 162–8601, Japan
E-mail: 1118704@ed.tus.ac.jp