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A two-dimensional index for marginal homogeneity in ordinal square contingency tables

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Abstract. For square contingency tables with ordered categories, indexes that represent the degree of departure from the marginal homogeneity (MH) model have been proposed. There are two types of directionalities of departure from MH. The existing indexes, however, can analyze either the degree of departure from MH or the directionality but not both. To address this issue, this study proposes a two-dimensional index, which combines the existing indexes, can simultaneously analyze both the degree and directionality of departure from MH. This study also evaluates the usefulness of the proposed index for visually comparing degrees of departure from MH in several square contingency tables using confidence region.

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§1. Introduction

Consider an $r \times r$ square contingency table with the same ordered row and column classifications. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$), and let X and Y denote the row and column variables, respectively.

In square contingency tables, typical models with symmetry structure are the symmetry (S) model (Bowker, 1948), the quasi-symmetry (QS) model (Caussinus, 1965), and the marginal homogeneity (MH) model (Stuart, 1955).

The S model is defined by

$$p_{ij} = \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji}$. For measuring the degree of departure from the S model, Tomizawa et al. (2001) and Tahata et al. (2010) proposed indexes γ and ψ, respectively (although the details are omitted). The index γ can judge whether or not the S model holds, but cannot distinguish the directionality of departure from S. The index ψ can distinguish the directionality of departure from S, but cannot judge whether or not the S model holds. Ando et al. (2017) proposed a two-dimensional index $\boldsymbol{\Psi} = (\gamma, \psi)^t$ in order to simultaneously analyze the degree and directionality of departure from S. Note that the symbol " t " denotes the transpose.

The QS model is defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $\psi_{ij} = \psi_{ji}$. For measuring the degree of departure from the QS model, Ando (2021) proposed a two-dimensional index that can simultaneously analyze the degree and directionality of departure from QS (although the details are omitted).

This study focuses on the MH model. The MH model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \ldots, r),$$

where $p_{i\cdot} = \sum_{k=1}^{r} p_{ik}$ and $p_{\cdot i} = \sum_{k=1}^{r} p_{ki}$.

Let

$$G_{1(i)} = \sum_{s=1}^{i} \sum_{t=i+1}^{r} p_{st} \quad [= \Pr(X \leq i, Y \geq i+1)]$$

and

$$G_{2(i)} = \sum_{s=i+1}^{r} \sum_{t=1}^{i} p_{st} \quad [= \Pr(X \geq i+1, Y \leq i)],$$

for $i = 1, \ldots, r-1$. Let

$$F_i^X = \Pr(X \leq i) \quad \text{and} \quad F_i^Y = \Pr(Y \leq i),$$

for $i = 1, \ldots, r-1$. By considering the difference between the cumulative marginal probabilities, $F_i^X - F_i^Y$ for $i = 1, \ldots, r-1$, the MH model is also expressed as

$$G_{1(i)} = G_{2(i)} \quad (i = 1, \ldots, r-1).$$

When the MH model does not hold, we are interested in measuring the degree of departure from MH. For square contingency tables with nominal categories, Tomizawa and Makii (2001) proposed an index that can measure the degree of departure from MH based on the marginal probabilities $\{p_{i\cdot}, p_{\cdot i}\}$. When we want to use the information with respect to the order of listing the categories, this index would not be suitable. Because this index is invariant under arbitrary similar permutations of row and column categories and does not depend on the order of listing the categories. For square contingency tables with ordered categories, Tomizawa et al. (2003) proposed an index Γ that can measure the degree of departure from MH based on the cumulative probabilities $\{G_{1(i)}, G_{2(i)}\}$. The index Γ is not invariant under arbitrary similar permutations of row and column categories and depends on the order of listing the categories. The index Γ has characteristics that (i) Γ lies between 0 and 1; (ii) $\Gamma = 0$ if and only if the MH model holds; and (iii) $\Gamma = 1$ if and only if the degree of departure from MH is the maximum, in the sense that $G_{1(i)} = 0$ (then $G_{2(i)} \neq 0$) for all $i = 1, \ldots, r-1$ (say, maximum lower-marginal inhomogeneity) or $G_{2(i)} = 0$ (then $G_{1(i)} \neq 0$) for all $i = 1, \ldots, r-1$ (say, maximum upper-marginal inhomogeneity). The index Γ cannot distinguish the directionality of departure from MH (i.e., maximum lower-marginal inhomogeneity or maximum upper-marginal inhomogeneity), see the property (iii).

Yamamoto et al. (2011) proposed an index Ψ that can distinguish the directionality of departure from MH. The index Ψ has characteristics that (I) Ψ lies between -1 and 1; (II) $\Psi = -1$ if and only if there is a structure of maximum upper-marginal inhomogeneity; (III) $\Psi = 1$ if and only if there is a structure of maximum lower-marginal inhomogeneity; and (IV) if the MH model holds then $\Psi = 0$, but the converse does not hold. Yamamoto et al. (2011) defined the structure ($\Psi = 0$) as an average marginal homogeneity model. From the property (IV), we see that the index Ψ cannot directly measure the degree of departure from MH (although the index Ψ can distinguish the directionality of departure from MH).

In a similar manner to the two-dimensional index $\boldsymbol{\Psi} = (\gamma, \psi)^t$, we propose a two-dimensional index $\boldsymbol{\Phi} = (\Gamma, \Psi)^t$ that can simultaneously analyze the degree and directionality of departure from MH. The proposed two-dimensional index $\boldsymbol{\Phi}$ would be useful for visually comparing degrees of departure from MH in several tables.

The rest of this paper is organized as follows. In Section 2, the proposed two-dimensional index $\boldsymbol{\Phi} = (\Gamma, \Psi)^t$ is defined. In Section 3, an approximate confidence region for the proposed two-dimensional index is derived. In Section 4, numerical examples show the utility of the proposed two-dimensional index. In Section 5, results obtained by applied the proposed two-dimensional index to real data are presented. We close with concluding remarks in Section 6.

§2. Two-dimensional index for marginal homogeneity

Let

$$\Delta = \sum_{i=1}^{r-1} \left(G_{1(i)} + G_{2(i)} \right), \quad G^*_{1(i)} = \frac{G_{1(i)}}{\Delta}, \quad G^*_{2(i)} = \frac{G_{2(i)}}{\Delta} \quad \text{and}$$

$$Q_i^* = \frac{1}{2}\left(G_{1(i)}^* + G_{2(i)}^*\right),$$

for $i = 1, \ldots, r-1$. Tomizawa et al. (2003) proposed the index $\Gamma^{(\lambda)}$ for $\lambda > -1$ based on the power-divergence between $\{G_{1(i)}^*, G_{2(i)}^*\}$ and $\{Q_i^*, Q_i^*\}$. The index $\Gamma^{(0)}$ (i.e., Γ) indicates the minimum Kullback-Leibler information between $\{G_{1(i)}^*, G_{2(i)}^*\}$ and probability distribution with the structure of the MH model. When $\lambda \neq 0$, however, the index $\Gamma^{(\lambda)}$ cannot be expressed as the minimum power-divergence. Therefore, this study uses the index Γ based on the minimum Kullback-Leibler information.

Assume that $\left\{G_{1(i)} + G_{2(i)} \neq 0\right\}$. We propose the two-dimensional index $\mathbf{\Phi} = (\Gamma, \Psi)^t$ that can simultaneously analyze the degree and directionality of departure from MH.

$$\mathbf{\Phi} = \begin{pmatrix} \Gamma \\ \Psi \end{pmatrix}; \text{ the } 2 \times 1 \text{ vector},$$

where

$$\Gamma = \frac{1}{\log 2}\sum_{i=1}^{r-1}\left[G_{1(i)}^* \log\left(\frac{G_{1(i)}^*}{Q_i^*}\right) + G_{2(i)}^* \log\left(\frac{G_{2(i)}^*}{Q_i^*}\right)\right]$$

and

$$\Psi = \frac{4}{\pi}\sum_{i=1}^{r-1}(G_{1(i)}^* + G_{2(i)}^*)\left(\theta_i - \frac{\pi}{4}\right)$$

with

$$\theta_i = \cos^{-1}\left(\frac{G_{1(i)}}{\sqrt{(G_{1(i)})^2 + (G_{2(i)})^2}}\right).$$

The index Ψ is given by Yamamoto et al. (2011). From characteristics of the indexes Γ and Ψ, we obtain the following proposition.

Proposition 2.1. *The proposed two-dimensional index* $\mathbf{\Phi}$ *satisfies the following properties.*

(1) $\mathbf{\Phi} = (0, 0)^t$ *if and only if the MH model holds;*

(2) $\mathbf{\Phi} = (1, -1)^t$ *if and only if there is a structure of maximum upper-marginal inhomogeneity;*

(3) $\mathbf{\Phi} = (1, 1)^t$ *if and only if there is a structure of maximum lower-marginal inhomogeneity.*

From Proposition 2.1, we see that the two-dimensional index $\mathbf{\Phi}$ can simultaneously analyze the degree and directionality of departure from MH.

§3. Approximate confidence region for two-dimensional index

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). Assume that a multinomial distribution applies to the $r \times r$ table. Let $\hat{p}_{ij} = n_{ij}/N$ ($i = 1, \ldots, r; j = 1, \ldots, r$) and $N = \sum\sum n_{ij}$. We estimate $\boldsymbol{\Phi}$ by $\widehat{\boldsymbol{\Phi}} = (\hat{\Gamma}, \hat{\Psi})^t$, where $\hat{\Gamma}$ and $\hat{\Psi}$ are given by Γ and Ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, respectively. Using the delta method (see, e.g., Agresti, 2013, p. 591), we obtain the following theorem.

Theorem 3.1. *Assume that $0 < \Gamma < 1$ and $-1 < \Psi < 1$. Then $\sqrt{N}(\widehat{\boldsymbol{\Phi}} - \boldsymbol{\Phi})$ asymptotically (as $N \to \infty$) has a bivariate normal distribution with zero mean and covariance matrix $\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}\,]$,*

$$\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}\,] \;=\; \frac{1}{\Delta^2}\sum_{s=1}^{r-1}\sum_{t=s+1}^{r}\begin{pmatrix} p_{st}\big(w_{st}\big)^2 + p_{ts}\big(w_{ts}\big)^2 & p_{st}w_{st}a_{st} + p_{ts}w_{ts}a_{ts} \\ p_{st}w_{st}a_{st} + p_{ts}w_{ts}a_{ts} & p_{st}\big(a_{st}\big)^2 + p_{ts}\big(a_{ts}\big)^2 \end{pmatrix},$$

where

$$\begin{aligned}
w_{st} &= \frac{1}{\log 2}\left[\sum_{i=s}^{t-1}\log\left(\frac{G_{1(i)}}{G_{1(i)} + G_{2(i)}}\right)\right] - (t-s)(\Gamma - 1), \\
w_{ts} &= \frac{1}{\log 2}\left[\sum_{i=s}^{t-1}\log\left(\frac{G_{2(i)}}{G_{1(i)} + G_{2(i)}}\right)\right] - (t-s)(\Gamma - 1), \\
a_{st} &= \frac{4}{\pi}\sum_{i=s}^{t-1}\left[\theta_i - \frac{G_{2(i)}(G_{1(i)} + G_{2(i)})}{(G_{1(i)})^2 + (G_{2(i)})^2}\right] - (t-s)(\Psi + 1) \quad \textit{and} \\
a_{ts} &= \frac{4}{\pi}\sum_{i=s}^{t-1}\left[\theta_i + \frac{G_{1(i)}(G_{1(i)} + G_{2(i)})}{(G_{1(i)})^2 + (G_{2(i)})^2}\right] - (t-s)(\Psi + 1).
\end{aligned}$$

Proof. Let

$$\begin{aligned}
\boldsymbol{n} &= (n_{11}, \ldots, n_{1r}, n_{21}, \ldots, n_{2r}, \ldots, n_{r1}, \ldots, n_{rr})^t;\ \text{the } r^2 \times 1 \text{ vector}, \\
\boldsymbol{p} &= (p_{11}, \ldots, p_{1r}, p_{22}, \ldots, p_{2r}, \ldots, p_{r1}, \ldots, p_{rr})^t;\ \text{the } r^2 \times 1 \text{ vector} \quad \text{and} \\
\hat{\boldsymbol{p}} &= (\hat{p}_{11}, \ldots, \hat{p}_{1r}, \hat{p}_{22}, \ldots, \hat{p}_{2r}, \ldots, \hat{p}_{r1}, \ldots, \hat{p}_{rr})^t;\ \text{the } r^2 \times 1 \text{ vector}.
\end{aligned}$$

From the central limit theorem, $\sqrt{N}(\hat{\boldsymbol{p}} - \boldsymbol{p})$ asymptotically (as $N \to \infty$) has a normal distribution with zero mean and covariance matrix $\mathbf{diag}(\boldsymbol{p}) - \boldsymbol{p}\boldsymbol{p}^t$, where $\mathbf{diag}(\boldsymbol{p})$ is a diagonal matrix of which elements of $\boldsymbol{p}$. Since the indexes $\boldsymbol{\Phi}$, Γ, and Ψ are functions of $\boldsymbol{p}$, we denote them by $\boldsymbol{\Phi}(\boldsymbol{p})$, $\Gamma(\boldsymbol{p})$, and $\Psi(\boldsymbol{p})$ in the proof. Let

$$\frac{\partial\boldsymbol{\Phi}(\boldsymbol{p})}{\partial\boldsymbol{p}^t} = \begin{pmatrix} \partial\Gamma(\boldsymbol{p})/\partial\boldsymbol{p}^t \\ \partial\Psi(\boldsymbol{p})/\partial\boldsymbol{p}^t \end{pmatrix}$$

denote the $2 \times r^2$ matrix for which the $(1, s)$ element is $\partial\Gamma(\boldsymbol{p})/\partial p_s$ and the $(2, s)$ element is $\partial\Psi(\boldsymbol{p})/\partial p_s$, where p_s is the sth element of $\boldsymbol{p}$. Note that the gradient vectors $\partial\Gamma(\boldsymbol{p})/\partial \boldsymbol{p}^t$ and $\partial\Psi(\boldsymbol{p})/\partial \boldsymbol{p}^t$ are not zero vector. Using the delta method (see, Agresti, 2013, p. 591), $\sqrt{N}(\widehat{\boldsymbol{\Phi}} - \boldsymbol{\Phi})$ asymptotically (as $N \to \infty$) has a bivariate normal distribution with the zero mean and the covariance matrix $\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}\,]$, where

$$\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}\,] = \left(\frac{\partial\boldsymbol{\Phi}(\boldsymbol{p})}{\partial\boldsymbol{p}^t}\right)\left(\mathbf{diag}(\boldsymbol{p}) - \boldsymbol{p}\boldsymbol{p}^t\right)\left(\frac{\partial\boldsymbol{\Phi}(\boldsymbol{p})}{\partial\boldsymbol{p}^t}\right)^t.$$

□

We note that the asymptotic normal distribution of $\sqrt{N}(\widehat{\boldsymbol{\Phi}} - \boldsymbol{\Phi})$ is not applicable when $\Gamma = 0$ and $\Gamma = 1$ because the variance of $\hat{\Gamma}$ equals zero, and also not applicable when $\Psi = -1$ and $\Psi = 1$ because the variance of $\hat{\Psi}$ equals zero.

Let $\widehat{\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}}\,]$ denote $\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}\,]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. From Theorem 3.1, an approximate $100(1-\alpha)$ percent confidence region for $\boldsymbol{\Phi}$ is obtained by

$$N\left(\widehat{\boldsymbol{\Phi}} - \boldsymbol{\Phi}\right)^t\left(\widehat{\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}}\,]\right)^{-1}\left(\widehat{\boldsymbol{\Phi}} - \boldsymbol{\Phi}\right) \leq \chi^2_{(1-\alpha;2)},$$

where $\chi^2_{(1-\alpha;2)}$ is the $1-\alpha$ quantile of the chi-square distribution with two degrees of freedom.

Consider comparing degrees of departure from MH between independent Tables A and B (with sample sizes N_A and N_B, respectively). For Tables A and B, $\boldsymbol{\Phi}$ are denoted by $\boldsymbol{\Phi}_A$ and $\boldsymbol{\Phi}_B$, respectively. We also denote their sample version by $\widehat{\boldsymbol{\Phi}}_A$ and $\widehat{\boldsymbol{\Phi}}_B$, respectively. When N_A and N_B are large, $\widehat{\boldsymbol{\Phi}}_A - \widehat{\boldsymbol{\Phi}}_B - (\boldsymbol{\Phi}_A - \boldsymbol{\Phi}_B)$ approximately has a bivariate normal distribution with zero mean and covariance matrix $\boldsymbol{V}$, where

$$\boldsymbol{V} = \frac{1}{N_A}\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_A\,] + \frac{1}{N_B}\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_B\,].$$

Therefore, an approximate $100(1-\alpha)$ percent confidence region for $\boldsymbol{\Phi}_A - \boldsymbol{\Phi}_B$ is obtained by

$$\left(\widehat{\boldsymbol{\Phi}}_A - \widehat{\boldsymbol{\Phi}}_B - (\boldsymbol{\Phi}_A - \boldsymbol{\Phi}_B)\right)^t\widehat{\boldsymbol{V}}^{-1}\left(\widehat{\boldsymbol{\Phi}}_A - \widehat{\boldsymbol{\Phi}}_B - (\boldsymbol{\Phi}_A - \boldsymbol{\Phi}_B)\right) \leq \chi^2_{(1-\alpha;2)},$$

where $\widehat{\boldsymbol{V}}$ is given by $\boldsymbol{V}$ with $\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_A\,]$ and $\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_B\,]$ replaced by $\widehat{\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_A}\,]$ and $\widehat{\boldsymbol{\Sigma}[\,\widehat{\boldsymbol{\Phi}}_B}\,]$, respectively.

§4. Numerical examples

We point out that the two-dimensional index $\mathbf{\Phi}$ is useful for visually comparing degrees of departure from MH. For instance, we consider the artificial data in Tables 1a, 1b, and 1c.

From Tables 2a, 2b, and 2c, we see that all of the values $\widehat{\Gamma}$ for Tables 1a, 1b, and 1c are equal, but the data in Tables 1a, 1b, and 1c have different structures of marginal inhomogeneity. Also, the value of $\widehat{\Psi}$ for Table 1c equals to zero, but the data in Table 1c does not have the structure of marginal homogeneity. From Figure 1, we see that the proposed two-dimensional index $\mathbf{\Phi}$ can simultaneously analyze the degree and directionality of departure from MH. Thus, we see that (i) the data in Table 1a tends to have the structure of lower-marginal inhomogeneity; (ii) the data in Table 1b tends to have the structure of upper-marginal inhomogeneity; and (iii) the data in Table 1c tends to have the structure of average marginal homogeneity.

§5. Real data analysis

We use the cross-classification of father's and son's occupational social class data in Japan. We take the data sets from Hashimoto (1999, p. 151) and summarized in Table 3. The data sets were examined in 1955 and 1995. Note that status (1) is Capitalist, (2) is New-Middle, (3) is Labor, (4) is Self-Support and (5) is Peasantry. Thus, for the data sets in Tables 3a and 3b, the structure of upper-marginal inhomogeneity shows that son's social class would be lower than father's one, and the structure of lower-marginal inhomogeneity shows that son's social class would be higher than father's one.

First, we measure degrees of departure from MH in Tables 3a and 3b using the confidence region for $\mathbf{\Phi}$. For Tables 3a and 3b, estimates of $\mathbf{\Phi}$ are

$$\widehat{\mathbf{\Phi}}_{3a} = \begin{pmatrix} 0.130 \\ 0.330 \end{pmatrix} \quad \text{and} \quad \widehat{\mathbf{\Phi}}_{3b} = \begin{pmatrix} 0.382 \\ 0.681 \end{pmatrix},$$

respectively, and estimates of $\mathbf{\Sigma}[\,\widehat{\mathbf{\Phi}}\,]$ are

$$\widehat{\mathbf{\Sigma}[\,\mathbf{\Phi}_{3a}\,]} = \begin{pmatrix} 0.493 & 0.934 \\ 0.934 & 2.931 \end{pmatrix} \quad \text{and} \quad \widehat{\mathbf{\Sigma}[\,\mathbf{\Phi}_{3b}\,]} = \begin{pmatrix} 0.955 & 0.873 \\ 0.873 & 1.050 \end{pmatrix},$$

respectively. From Figure 2, we can see that the data sets in both Tables 3a and 3b tend to have the structure of lower-marginal inhomogeneity rather than the structure of upper-marginal inhomogeneity. Thus, we see that son's social class would be higher than father's one in both 1955 and 1995.

Next, we compare degrees of departure from MH in Tables 3a and 3b using the confidence region for $\mathbf{\Phi}_{3a} - \mathbf{\Phi}_{3b}$. In Table 3, we note that (i) if the confidence region for $\mathbf{\Phi}_{3a} - \mathbf{\Phi}_{3b}$ is included within the first quadrant, we can see that the degree of departure from MH in Table 3a is greater than that in Table 3b, and the degree of rise of son's social class compared to father's one in 1955 is greater than that in 1995; (ii) if the confidence region is included within the second quadrant, we can see that the degree of departure from MH in Table 3a is greater than that in Table 3b, and degree of rise of son's social class compared to father's one in 1955 is smaller than that in 1995; (iii) if the confidence region is included within the third quadrant, we can see that the degree of departure from MH in Table 3a is smaller than that in Table 3b, and degree of rise of son's social class compared to father's one in 1955 is smaller than that in 1995, and (iv) if the confidence region is included within the fourth quadrant, we can see that the degree of departure from MH in Table 3a is smaller than that in Table 3b, and degree of rise of son's social class compared to father's one in 1955 is greater than that in 1995. From Figure 3, the confidence region for $\mathbf{\Phi}_{3a} - \mathbf{\Phi}_{3b}$ is included within the third quadrant. Thus, we see that the degree of departure from MH in Table 3a is smaller than that in Table 3b, and degree of rise of son's social class for father's in 1955 is smaller than that in 1995.

§6. Concluding remarks

For the analysis of square contingency tables having ordered categories, the index Γ was proposed to measure the degree of departure from MH based on the cumulative probabilities $\{G_{1(i)}, G_{2(i)}\}$. The index Γ can judge whether or not the MH model holds, but cannot distinguish the directionality of departure from MH (i.e., maximum lower-marginal inhomogeneity or maximum upper-marginal inhomogeneity). Thereafter, the index Ψ was proposed to distinguish the directionality of departure from MH based on the cumulative probabilities $\{G_{1(i)}, G_{2(i)}\}$. The index Ψ can distinguish the directionality of departure from MH, but cannot judge whether or not the MH model holds. This study proposed the two-dimensional index $\mathbf{\Phi} = (\Gamma, \Psi)^t$ in order to simultaneously analyze the degree and directionality of departure from MH. We derived the approximate confidence region for the two-dimensional index $\mathbf{\Phi}$. The utility of the two-dimensional index $\mathbf{\Phi}$ has been verified by numerical examples and real data analysis. The proposed two-dimensional index $\mathbf{\Phi}$ may be useful for visually comparing degrees of departure from MH in several tables.

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References

[1] Agresti, A. (2013). *Categorical Data Analysis*, 3rd edition. Wiley, Hoboken.

[2] Ando, S. (2021). A bivariate index vector to measure departure from quasi-symmetry for ordinal square contingency tables. *Austrian Journal of Statistics*, **50**, 115–126.

[3] Ando, S., Tahata, K. and Tomizawa, S. (2017). Visualized measure vector of departure from symmetry for square contingency tables. *Statistics in Biopharmaceutical Research*, **9**, 212–224.

[4] Bowker, A. H. (1948). A test for symmetry in contingency tables. *Journal of the American Statistical Association*, **43**, 572–574.

[5] Caussinus, H. (1965). Contribution à l'analyse statistique des tableaux de corrélation. *Annales de la Faculté des Sciences de l'Université de Toulouse*, **29**, 77–183.

[6] Hashimoto, K. (1999). *Gendai Nippon no Kaikyuu Kouzou (Class Structure in Modern Japan :Theory, Method and Quantitative Analysis)*. Toshindo Press, Tokyo (in Japanese).

[7] Stuart, A. (1955). A test for homogeneity of the marginal distributions in a two-way classification. *Biometrika*, **42**, 412–416.

[8] Tahata, K., Miyazawa, K. and Tomizawa, S. (2010). Measure of departure from average cumulative symmetry for square contingency tables with ordered categories. *American Journal of Biostatistics*, **1**, 62–66.

[9] Tomizawa, S. and Makii, T. (2001). Generalized measures of departure from marginal homogeneity for contingency tables with nominal categories. *Journal of Statistical Research*, **35**, 1–24.

[10] Tomizawa, S., Miyamoto, N. and Ashihara, N. (2003). Measure of departure from marginal homogeneity for square contingency tables having ordered categories. *Behaviormetrika*, **30**, 173–193.

[11] Tomizawa, S., Miyamoto, N. and Hatanaka, Y. (2001). Measure of asymmetry for square contingency tables having ordered categories. *Australian and New Zealand Journal of Statistics*, **43**, 335–349.

[12] Yamamoto, K., Ando, S. and Tomizawa, S. (2011). A measure of departure from average marginal homogeneity for square contingency tables with ordered categories. *Revstat*, **9**, 115–126.

Table 1: Artificial three datasets having a difference structure of marginal inhomogeneity.

	(1)	(2)	(3)	(4)	Total
(a) Lower-marginal inhomogeneity					
(1)	100	50	100	100	350
(2)	1000	100	100	100	1300
(3)	100	100	100	50	350
(4)	100	100	1000	100	1300
Total	1300	350	1300	350	3300
(b) Upper-marginal inhomogeneity					
(1)	100	1000	100	100	1300
(2)	50	100	100	100	350
(3)	100	100	100	1000	1300
(4)	100	100	50	100	350
Total	350	1300	350	1300	3300
(c) Average marginal homogeneity					
(1)	100	1000	100	100	1300
(2)	50	100	100	100	350
(3)	100	100	100	50	350
(4)	100	100	1000	100	1300
Total	350	1300	1300	350	3300

Table 2: Estimates of Γ and Ψ, approximate standard errors for $\widehat{\Gamma}$ and $\widehat{\Psi}$, and approximate 95% confidence intervals for Γ and Ψ, applied to Tables 1a, 1b and 1c.

	Estimated index	Standard error	Confidence interval
(a) For Table 1a			
Γ	0.264	0.015	(0.234, 0.294)
Ψ	0.579	0.020	(0.539, 0.618)
(b) For Table 1b			
Γ	0.264	0.015	(0.234, 0.294)
Ψ	−0.579	0.020	(−0.618, −0.539)
(c) For Table 1c			
Γ	0.264	0.013	(0.239, 0.288)
Ψ	0.000	0.021	(−0.042, 0.042)

Table 3: Cross-classification of Japanese father's and his son's social class; taken from Hashimoto (1999, p. 151). Note that social class (1) is Capitalist, (2) is New-Middle, (3) is Labor, (4) is Self-Support and (5) is Peasantry.

Father's	Son's class					
class	(1)	(2)	(3)	(4)	(5)	Total
(a) Examined in 1955						
(1)	39	39	39	57	23	197
(2)	12	78	23	23	37	173
(3)	6	16	78	23	20	143
(4)	18	80	79	126	31	334
(5)	28	106	136	122	628	1020
Total	103	319	355	351	739	1867
(b) Examined in 1995						
(1)	68	48	36	23	1	176
(2)	33	191	102	33	3	362
(3)	25	147	229	34	2	437
(4)	48	119	146	129	5	447
(5)	40	126	192	82	88	528
Total	214	631	705	301	99	1950

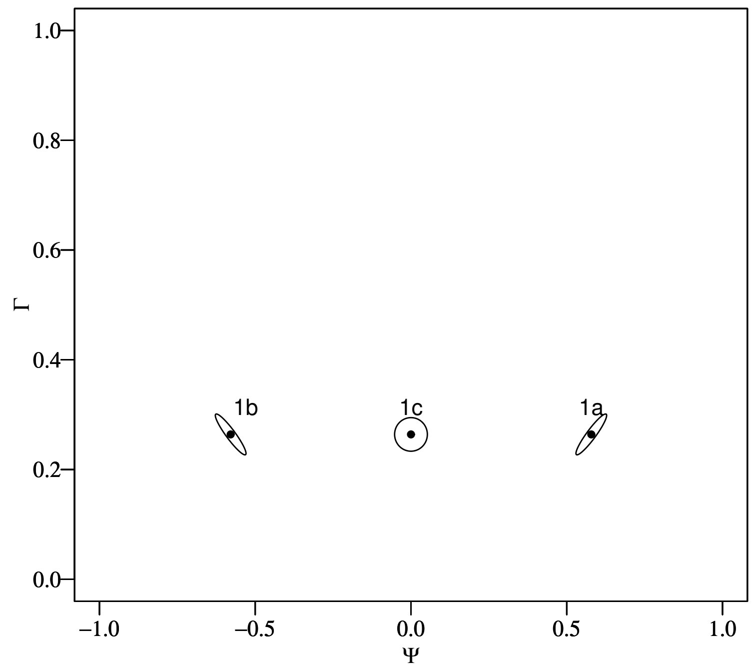

Figure 1: Approximate 95% confidence region for Tables 1a, 1b and 1c.

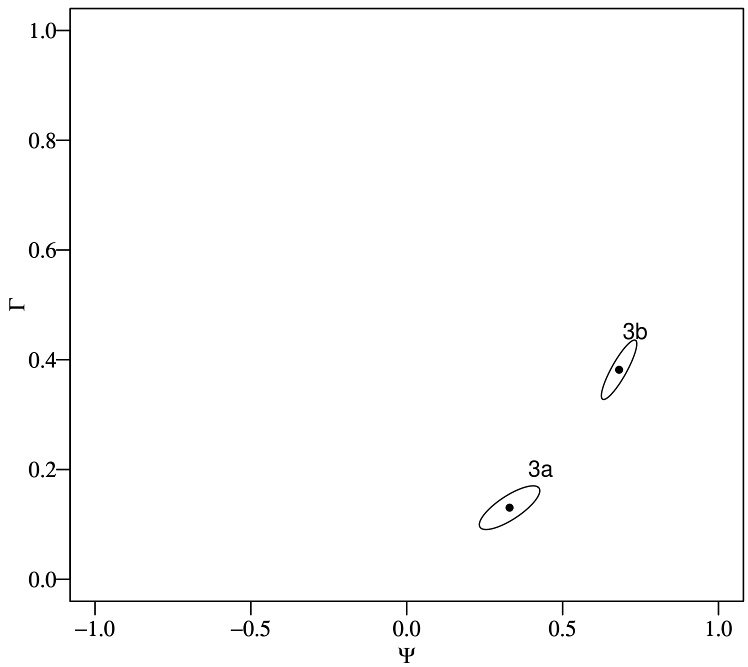

Figure 2: Approximate 95% confidence regions for $\mathbf{\Phi} = (\Gamma, \Psi)^t$, applied to Tables 3a and 3b.

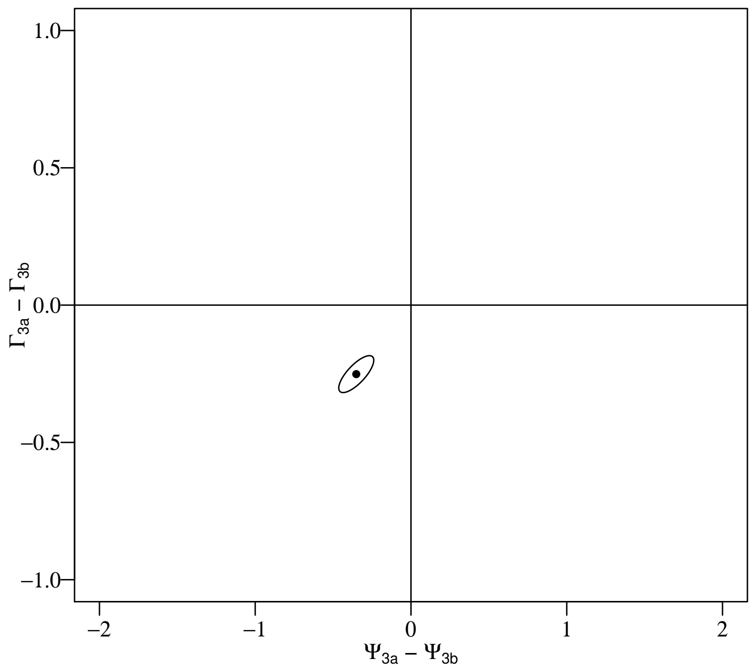

Figure 3: Approximate 95% confidence region for $\mathbf{\Phi}_{3a} - \mathbf{\Phi}_{3b}$.

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