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$\mathcal{R}(p, q)$-multivariate discrete probability distributions

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Abstract. We construct the multivariate probability distributions (Pólya, inverse Pólya, hypergeometric and negative hypergeometric) from the generalized quantum deformed algebras. Moreover, we derive the corresponding bivariate probability distributions and determine their properties ($\mathcal{R}(p, q)$-factorial moments and covariance). Besides, we deduce particular cases of probability distributions from the quantum algebras known in the literature.

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§1. Introduction

Charalambos presented the q-deformed Vandermonde and Cauchy formulae. Moreover, the q-deformed univariate discrete probability distributions were investigated. Their properties and limiting distributions were derived [2].

Furthermore, the q-deformed multinomial coefficients was defined and their recurrence relations were deduced. Also, the q-deformed multinomial and negative q-deformed multinomial probability distributions of the first and second kind were presented [3].

The same author extended the multivariate q-deformed Vandermonde and Cauchy formulae. Also, the multivariate q-Polýa and inverse q-Polýa were constructed [4].

Let p and q be two positive real numbers such that $0 < q < p < 1$. We consider a meromorphic function $\mathcal{R}$ defined on $\mathbb{C} \times \mathbb{C}$ by [8]:

$$\mathcal{R}(u, v) = \sum_{s,t=-l}^{\infty} r_{st} u^s v^t, \tag{1.1}$$

with an eventual isolated singularity at the zero, where r_{st} are complex numbers, $l \in \mathbb{N} \cup \{0\}$, $\mathcal{R}(p^n, q^n) > 0, \forall n \in \mathbb{N}$, and $\mathcal{R}(1,1) = 0$ by definition. We denote by $\mathbb{D}_R$ the bidisk

$$\mathbb{D}_R = \left\{ w = (w_1, w_2) \in \mathbb{C}^2 : |w_j| < R_j \right\},$$

where R is the convergence radius of the series (1.1) defined by Hadamard formula as follows:

$$\limsup_{s+t \longrightarrow \infty} \sqrt[s+t]{|r_{st}| R_1^s \, R_2^t} = 1.$$

For the proof and more details see [12]. We denote by $\mathcal{O}(\mathbb{D}_R)$ the set of holomorphic functions defined on $\mathbb{D}_R$.

The $\mathcal{R}(p,q)$-deformed numbers is defined by [8]:

$$[x]_{\mathcal{R}(p,q)} := \mathcal{R}(p^x, q^x), \qquad x \in \mathbb{N}, \tag{1.2}$$

the $\mathcal{R}(p,q)$-deformed factorials and binomial coefficients are given as:

$$[x]!_{\mathcal{R}(p,q)} := \begin{cases} 1 \quad \text{for} \quad x = 0 \\ \mathcal{R}(p,q) \cdots \mathcal{R}(p^x, q^x) \quad \text{for} \quad x \geq 1, \end{cases}$$

and

$$\left[\begin{array}{c} x \\ y \end{array} \right]_{\mathcal{R}(p,q)} := \frac{[x]!_{\mathcal{R}(p,q)}}{[y]!_{\mathcal{R}(p,q)} [x-y]!_{\mathcal{R}(p,q)}}, \quad x, y = 0, 1, 2, \cdots ; \quad x \geq y.$$

We consider the following linear operators on $\mathcal{O}(\mathbb{D}_R)$ given by:

$$Q : \varphi \longmapsto Q\varphi(z) := \varphi(qz),$$
$$P : \varphi \longmapsto P\varphi(z) := \varphi(pz),$$

leading to define the $\mathcal{R}(p,q)$-deformed derivative:

$$\partial_{\mathcal{R},p,q} := \partial_{p,q} \frac{p-q}{P-Q} \mathcal{R}(P,Q) = \frac{p-q}{pP-qQ} \mathcal{R}(pP, qQ) \partial_{p,q},$$

where $\partial_{p,q}$ is the (p,q)-derivative:

$$\partial_{p,q} : \varphi \longmapsto \partial_{p,q}\varphi(z) := \frac{\varphi(pz) - \varphi(qz)}{z(p-q)}.$$

The quantum algebra associated with the $\mathcal{R}(p,q)$-deformation, denoted by $\mathcal{A}_{\mathcal{R}(p,q)}$ is generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations [7]:

$$AA^\dagger = [N+1]_{\mathcal{R}(p,q)}, \qquad A^\dagger A = [N]_{\mathcal{R}(p,q)}.$$

$$[N,\ A] = -A, \qquad\qquad \left[N,\ A^\dagger\right] = A^\dagger$$

with its realization on $\mathcal{O}(\mathbb{D}_R)$ given by:

$$A^\dagger := z, \qquad A := \partial_{\mathcal{R}(p,q)}, \qquad N := z\partial_z,$$

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$.

The $\mathcal{R}(p,q)$-deformed numbers (1.2) can be rewritten as follows [5]:

$$[x]_{\mathcal{R}(p,q)} = \frac{\tau_1^x - \tau_2^x}{\tau_1 - \tau_2}, \quad \tau_1 \neq \tau_2, \tag{1.3}$$

where $\left(\tau_i\right)_{i\in\{1,2\}}$ are functions depending of the parameters deformations p and q.

The $\mathcal{R}(p,q)$-deformed factorial of x of order r is defined by:

$$[x]_{r,\mathcal{R}(p,q)} := \prod_{i=1}^{r} [x - i + 1]_{\mathcal{R}(p,q)}, \quad \text{with} \quad [x]_{0,\mathcal{R}(p,q)} = 1. \tag{1.4}$$

The above relation (1.4) can be extended to the negative order as:

$$[x]_{-r,\mathcal{R}(p,q)} = \frac{1}{[x+r]_{r,\mathcal{R}(p,q)}}, \quad r \in \mathbb{N}.$$

Furthermore, the generalized Vandermonde, Cauchy formulae and univariate probability distributions induced from the $\mathcal{R}(p,q)$-deformed quantum algebras were investigated in [5].

Our aims are to construct the multinomial coefficient, multivariate Vandermonde, and Cauchy formulae, multivariate probability distributions, and properties associated to the $\mathcal{R}(p,q)$-deformed quantum algebras [7].

This paper is organized as follows: In section 2, we investigate the $\mathcal{R}(p,q)$-multinomial coefficient, the multivariate Vandermonde formula associated to the $\mathcal{R}(p,q)$-deformed quantum algebras. The inverse multivariate $\mathcal{R}(p,q)$-Vandermonde and negative multivariate $\mathcal{R}(p,q)$-Vandermonde formula are computed. Moreover, the $\mathcal{R}(p,q)$-deformed Cauchy formula is deduced. Section 3 is reserved to the construction of the multivariate probability distributions. We derive the case of bivariate distributions and compute their $\mathcal{R}(p,q)$-factorial moments and covariance. Relevant particular cases corresponding to the quantum algebras known in the literature are derived from the general formalism.

§2. $\mathcal{R}(p,q)$-multivariate Vandermonde and Cauchy formulae

In this section, we investigate the generalized multinomial coefficient, multivariate Vandermonde formula, and multivariate Cauchy formula associated to the $\mathcal{R}(p,q)$-deformed quantum algebras. Their recurrence relations are also derived.

Proposition 2.1. *The generalized multinomial coefficient*

$$\left[\begin{matrix} x \\ r_1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} = \frac{[x]_{r_1+r_2+\cdots+r_k,\mathcal{R}(p,q)}}{[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!} \tag{2.1}$$

satisfies the recursion relation:

$$\begin{aligned}\left[\begin{matrix} x \\ r_1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} &= \tau_1^{s_k}\left[\begin{matrix} x-1 \\ r_1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} \\ &+ \tau_2^{x-m_1}\left[\begin{matrix} x-1 \\ r_1-1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} \\ &+ \cdots + \tau_2^{x-m_k}\left[\begin{matrix} x-1 \\ r_1, r_2, \ldots, r_k-1 \end{matrix}\right]_{\mathcal{R}(p,q)}.\end{aligned} \tag{2.2}$$

Equivalently,

$$\begin{aligned}\left[\begin{matrix} x \\ r_1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} &= \tau_2^{s_k}\left[\begin{matrix} x-1 \\ r_1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} \\ &+ \tau_1^{x-m_1}\left[\begin{matrix} x-1 \\ r_1-1, r_2, \ldots, r_k \end{matrix}\right]_{\mathcal{R}(p,q)} \\ &+ \cdots + \tau_2^{s_{k-1}}\tau_1^{m_k-x}\left[\begin{matrix} x-1 \\ r_1, r_2, \ldots, r_k-1 \end{matrix}\right]_{\mathcal{R}(p,q)},\end{aligned} \tag{2.3}$$

where $r_j \in \mathbb{N}$ and $j \in \{1,2,\ldots,k\}$, with $m_j = \sum_{i=j}^{k} r_i$ and $s_j = \sum_{i=1}^{j} r_i$.

Proof. We have

$$[x]_{s_k,\mathcal{R}(p,q)} = [x]_{\mathcal{R}(p,q)}\,[x-1]_{s_k-1,\mathcal{R}(p,q)},$$

$$[x-1]_{s_k,\mathcal{R}(p,q)} = [x-1]_{s_k-1,\mathcal{R}(p,q)}\,[x-s_k]_{\mathcal{R}(p,q)}$$

and

$$[x]_{\mathcal{R}(p,q)} = \tau_1^{s_k}\,[x-s_k]_{\mathcal{R}(p,q)} + \tau_2^{x-s_k}\,[s_k]_{\mathcal{R}(p,q)}.$$

Then, the $\mathcal{R}(p,q)$-deformed factorials of x of order $s_k = \sum_{i=1}^{k} r_k$ satisfy the recursion relation:

$$[x]_{s_k,\mathcal{R}(p,q)} = \tau_1^{s_k}[x-1]_{s_k,\mathcal{R}(p,q)} + \sum_{j=1}^{k}\tau_2^{x-m_j}\,[r_j]_{\mathcal{R}(p,q)}[x-1]_{s_k-1,\mathcal{R}(p,q)}. \tag{2.4}$$

Multiplying both sides of equation (2.4) by $1/[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\ldots[r_k]_{\mathcal{R}(p,q)}!$ and using the $\mathcal{R}(p,q)$-deformed multinomial coefficient (2.1), we obtain the relation (2.2). Similarly, the $\mathcal{R}(p,q)$-deformed number can be expressed as follows:

$$[x]_{\mathcal{R}(p,q)} = \tau_2^{s_k}\,[x-s_k]_{\mathcal{R}(p,q)} + \sum_{j=1}^{k}\tau_1^{x-m_j}\,\tau_2^{s_{j-1}}\,[r_j]_{\mathcal{R}(p,q)}$$

and the $\mathcal{R}(p,q)$-deformed factorials of x of order s_k satisfy the recursion relation:

$$\begin{aligned}[x]_{s_k,\mathcal{R}(p,q)} =& \sum_{j=1}^{k}\tau_1^{x-m_j}\,\tau_2^{s_{j-1}}\,[r_j]_{\mathcal{R}(p,q)}\,[x-1]_{s_k-1,\mathcal{R}(p,q)}\\ &+\tau_2^{s_k}\,[x-1]_{s_k,\mathcal{R}(p,q)},\end{aligned}\tag{2.5}$$

with $s_0=0$.

Dividing both sides of the relation (2.5) by $[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!$ and using (2.1), the relation (2.3) is readily derived and the proof is achieved. □

From the relations

$$\begin{aligned}[x]_{\mathcal{R}(p^{-1},q^{-1})} &= (\tau_1\,\tau_2)^{1-x}\,[x]_{\mathcal{R}(p,q)},\\ [r]_{\mathcal{R}(p^{-1},q^{-1})}! &= (\tau_1\,\tau_2)^{-\binom{r}{2}}\,[r]_{\mathcal{R}(p,q)}!,\\ [x]_{r,\mathcal{R}(p^{-1},q^{-1})} &= (\tau_1\,\tau_2)^{-xr+\binom{r+1}{2}}\,[x]_{r,\mathcal{R}(p,q)}\end{aligned}$$

and

$$\begin{aligned}-xs_k+\binom{s_k+1}{2}+\sum_{j=1}^{k}\binom{r_j}{2} &= -\,xs_k+\sum_{j=1}^{k-1}r_j\,m_{j+1}\\ &\quad+\sum_{j=1}^{k}\left(\binom{r_j+1}{2}+\binom{r_j}{2}\right)\\ &= -\sum_{j=1}^{k}r_j(x-m_j) = -\sum_{j=1}^{k}r_j(x-s_j),\end{aligned}$$

we obtain other expressions of the generalized multinomial coefficient in the simpler form:

$$\begin{bmatrix}x\\ r_1,r_2,\cdots,r_k\end{bmatrix}_{\mathcal{R}(p^{-1},q^{-1})} = (\tau_1\tau_2)^{-\sum_{j=1}^{k}r_j(x-m_j)}\begin{bmatrix}x\\ r_1,r_2,\cdots,r_k\end{bmatrix}_{\mathcal{R}(p,q)}\tag{2.6}$$

and

$$\begin{bmatrix} x \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p^{-1},q^{-1})} = (\tau_1\tau_2)^{-\sum_{j=1}^{k} r_j(x-s_j)} \begin{bmatrix} x \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)}, \tag{2.7}$$

where $s_j = \sum_{i=1}^{j} r_i$, $m_j = \sum_{i=j}^{k} r_i$, $r_j \in \mathbb{N}$, $j \in \{1, 2, \cdots, k\}$ and $k \in \mathbb{N}$.

Other recurrence relations can be obtained by using the expression (2.6), respectively. Thus, we get:

$$\begin{aligned} \begin{bmatrix} x \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} &= \tau_2^{m_1} \begin{bmatrix} x-1 \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \tau_2^{m_3} \begin{bmatrix} x-1 \\ r_1, r_2-1, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \cdots + \tau_1^{x} \begin{bmatrix} x-1 \\ r_1, r_2, \cdots, r_k-1 \end{bmatrix}_{\mathcal{R}(p,q)} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} x \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} &= \tau_1^{x} \begin{bmatrix} x-1 \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \tau_2^{x-s_2} \begin{bmatrix} x-1 \\ r_1, r_2-1, \ldots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \cdots + \tau_2^{x-s_k} \begin{bmatrix} x-1 \\ r_1, r_2, \ldots, r_k-1 \end{bmatrix}_{\mathcal{R}(p,q)}. \end{aligned} \tag{2.8}$$

Theorem 2.2. *The generalized multivariate Vandermonde formula is given by the following relations:*

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{n,\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \begin{bmatrix} n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_1, \tau_2, n) \prod_{j=1}^{k+1} [x_j]_{r_j,\mathcal{R}(p,q)} \tag{2.9}$$

and

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{n,\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \begin{bmatrix} n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_2, \tau_1, n) \prod_{j=1}^{k+1} [x_j]_{r_j,\mathcal{R}(p,q)}, \tag{2.10}$$

where

$$\mathcal{V}(\tau_1, \tau_2) = \tau_1^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))} \tau_2^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)},$$

$s_j = \sum_{i=1}^{j} r_i$, $z_j = \sum_{i=j+1}^{k+1} x_i$, $j \in \{1, 2, \cdots, k\}$, $r_{k+1} = n - s_k$, *and* $\sum_{i=1}^{k} r_i \leq n$.

Proof. We consider the multiple sums:

$$s_n(x_1,\cdots,x_{k+1};\mathcal{R}(p,q)) = \sum \begin{bmatrix} n \\ r_1,\ldots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_1,\tau_2,n)\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)},$$

with $n\in\mathbb{N}$ and $s_1(x_1,x_2,\cdots,x_{k+1};\mathcal{R}(p,q)) = \left[\sum_{i=1}^{k+1} x_i\right]_{\mathcal{R}(p,q)}$. From the recurrence relation (2.8), with $x=n$, the above sequence can be rewritten in the following form:

(2.11)

$$\begin{aligned} &s_n(x_1,\ldots,x_{k+1};\mathcal{R}(p,q)) \\ &= \sum_{r_j=0}^{n} \begin{bmatrix} n-1 \\ r_1,r_2,\ldots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \tau_1^n\mathcal{V}(\tau_1,\tau_2,n)\times\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)} \\ &\quad+\sum_{r_j=0}^{n} \begin{bmatrix} n-1 \\ r_1-1,r_2,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_1,\tau_2,n)\tau_2^{n-s_1}\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)} \\ &\quad+\cdots+\sum_{r_j=0}^{n} \begin{bmatrix} n-1 \\ r_1,r_2,\cdots,r_k-1 \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_1,\tau_2,n)\tau_2^{n-s_k}\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)}. \end{aligned}$$

Interchanging r_j-1 by r_j in the $(j+1)$th multiple sum, for $j\in\{1,2,\ldots,k\}$, and after computation, we have:

$$\begin{aligned} &s_n(x_1,\ldots,x_{k+1};\mathcal{R}(p,q)) \\ &=\sum \begin{bmatrix} n-1 \\ r_1,r_2,\ldots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \tau_1^{\sum_{j=1}^{k}(x_2-r_2+1)}\tau_2^{\sum_{j=1}^{k}(x_j-r_j)}\mathcal{V}(\tau_1,\tau_2,n-1) \\ &\quad\times[x_1-r_1]_{\mathcal{R}(p,q)}\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)} \\ &\quad+\cdots+ \\ &\quad+\sum \begin{bmatrix} n-1 \\ r_1,r_2,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \tau_1^{\sum_{j=1}^{k}(x_{j+1}-r_{j+1}+1)}\tau_2^{\sum_{j=1}^{k-1}(x_j-r_j)}\mathcal{V}(\tau_1,\tau_2,n-1) \\ &\quad\times[x_k-r_k]_{\mathcal{R}(p,q)}\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)} \\ &\quad+\sum \begin{bmatrix} n-1 \\ r_1,r_2,\ldots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \mathcal{V}(\tau_1,\tau_2,n-1)\tau_1^{-\sum_{j=1}^{k} r_j}\tau_2^{\sum_{j=1}^{k}(x_j-r_j)} \\ &\quad\times[x_{k+1}-r_{k+1}+1]_{\mathcal{R}(p,q)}\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)}. \end{aligned}$$

By using the relation

$$\Big[\sum_{i=1}^{k+1}x_i-n+1\Big]_{\mathcal{R}(p,q)}=\tau_2^{\sum_{j=1}^{k}(x_j-r_j)}[x_{k+1}-r_{k+1}+1]_{\mathcal{R}(p,q)}+\sum_{i=1}^{k}\tau_1^{\sum_{j=1}^{i}(x_{j+1}-r_{j+1}+1)}\tau_2^{\sum_{j=1}^{i-1}(x_j-r_j)}[x_i-r_i]_{\mathcal{R}(p,q)},$$

the sequence $s_n(x_1,x_2,\ldots,x_{k+1};\mathcal{R}(p,q))$, $n\in\mathbb{N}$, is reduced to

$$s_n(x_1,\cdots,x_{k+1};\mathcal{R}(p,q))=\Big[\sum_{i=1}^{k+1}x_i-n+1\Big]_{\mathcal{R}(p,q)}s_{n-1}(x_1,\cdots,x_{k+1};\mathcal{R}(p,q)),$$

for $n\in\mathbb{N}$, with $s_1(x_1,x_2,\cdots,x_{k+1};\mathcal{R}(p,q))=\big[\sum_{i=1}^{k+1}x_i\big]_{\mathcal{R}(p,q)}$. Applying it successively, we have $s_n(x_1,x_2,\cdots,x_{k+1};\mathcal{R}(p,q))=\big[\sum_{i=1}^{k+1}x_i\big]_{n,\mathcal{R}(p,q)}$, and so (2.9) is deduced. The relation (2.10) can be deduced in the same way as the relation (2.9) by using the recurrence relation (2.3), with $x=n$, and the expression

$$\begin{aligned}\Big[\sum_{i=1}^{k+1}x_i-n+1\Big]_{\mathcal{R}(p,q)}=&[x_{k+1}-r_{k+1}+1]_{\mathcal{R}(p,q)}\\&+\tau_2^{(x_{k+1}-r_{k+1})+1}[x_k-r_k]_{\mathcal{R}(p,q)}\\&+\tau_2^{\sum_{j=2}^{k+1}(x_j-r_j)+1}[x_1-r_1]_{\mathcal{R}(p,q)}\\&+\cdots+\tau_2^{\sum_{j=3}^{k+1}(x_j-r_j)+1}[x_2-r_2]_{\mathcal{R}(p,q)}.\end{aligned}$$

□

Proposition 2.3. *The negative generalized multivariate Vandermonde formula is described by the following relations:*

$$\Big[\sum_{i=1}^{k+1}x_i\Big]_{-n,\mathcal{R}(p,q)}=\sum_{r_j=0}^{n}\begin{bmatrix}-n\\r_1,\cdots,r_k\end{bmatrix}_{\mathcal{R}(p,q)}\mathcal{V}(\tau_1,\tau_2,-n)\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)}\tag{2.12}$$

and

$$\Big[\sum_{i=1}^{k+1}x_i\Big]_{-n,\mathcal{R}(p,q)}=\sum_{r_j=0}^{n}\begin{bmatrix}-n\\r_1,\cdots,r_k\end{bmatrix}_{\mathcal{R}(p,q)}\mathcal{V}(\tau_2,\tau_1,-n)\prod_{j=1}^{k+1}[x_j]_{r_j,\mathcal{R}(p,q)},\tag{2.13}$$

where $s_j=\sum_{i=1}^{j}r_i$, $z_j=\sum_{i=j+1}^{k+1}x_i$, $j\in\{1,2,\cdots,k\}$, $r_{k+1}=n-s_k$, *and* $\sum_{i=1}^{k}r_i\le n$.

Proof. We use the same procedure as the proof of Theorem (2.2). □

Remark 2.4. By replacing x_j by $-x_j$, for $j \in \{1, 2, \ldots, k+1\}$, in the relations (2.9) and (2.10), and using the following expressions:

$$[-x]_{r,\mathcal{R}(p,q)} = (-1)^x(\tau_1\tau_2)^{-xr-\binom{r}{2}}[x+r-1]_{r,\mathcal{R}(p,q)}, \tag{2.14}$$

and

$$\binom{n}{2} = \sum_{j=1}^{k+1}\binom{r_j}{2} + \sum_{j=1}^{k} r_j(n-s_j), \tag{2.15}$$

the generalized multivariate Vandermonde formula can be expressed as:

$$\Big[\sum_{i=1}^{k+1} x_i + n - 1\Big]_{n,\mathcal{R}(p,q)} = \sum_{r_j=0}^{n}\begin{bmatrix} n \\ r_1, r_2, \cdots, r_k\end{bmatrix}_{\mathcal{R}(p,q)} \tau_1^{\sum_{j=1}^{k} x_j(n-s_j)} \times \tau_2^{\sum_{j=1}^{k} r_j z_j}\prod_{j=1}^{k+1}[x_j+r_j-1]_{r_j,\mathcal{R}(p,q)}, \tag{2.16}$$

and

$$\Big[\sum_{i=1}^{k+1} x_i + n - 1\Big]_{n,\mathcal{R}(p,q)} = \sum_{r_j=0}^{n}\begin{bmatrix} n \\ r_1, r_2, \ldots, r_k\end{bmatrix}_{\mathcal{R}(p,q)} \tau_1^{\sum_{j=1}^{k} r_j z_j} \times \tau_2^{\sum_{j=1}^{k} x_j(n-s_j)}\prod_{j=1}^{k+1}[x_j+r_j-1]_{r_j,\mathcal{R}(p,q)}, \tag{2.17}$$

where $s_j = \sum_{i=1}^{j} r_i$, $z_j = \sum_{i=j+1}^{k+1} x_i$, $j \in \{1, 2, \ldots, k\}$, and $r_{k+1} = n - s_k$, and $\sum_{i=1}^{k} r_i \leq n$.

Theorem 2.5. *The generalized multivariate inverse Vandermonde formula is presented by the following relations as follows:*

$$\frac{1}{[x_{k+1}]_{n,\mathcal{R}(p,q)}} = \sum_{r_j=0}^{n}\begin{bmatrix} n+s_k-1 \\ r_1, \ldots, r_k\end{bmatrix}_{\mathcal{R}(p,q)} \frac{\tau_1^{\sum_{j=1}^{k} r_j(z_j-s_k+s_j-n+1)}}{\tau_2^{\sum_{j=1}^{k}(-n-s_k+s_j)(x_j-r_j)}} \times \frac{\prod_{j=1}^{k}[x_j]_{r_j,\mathcal{R}(p,q)}}{[\sum_{i=1}^{k+1} x_i]_{n+s_k,\mathcal{R}(p,q)}}, \tag{2.18}$$

provided $|\big(\frac{\tau_2}{\tau_1}\big)^{-x_{k+1}}| < 1$, *and*

$$\frac{1}{[x_{k+1}]_{n,\mathcal{R}(p,q)}} = \sum_{r_j=0}^{n}\begin{bmatrix} n+s_k-1 \\ r_1, \cdots, r_k\end{bmatrix}_{\mathcal{R}(p,q)} \frac{\tau_1^{\sum_{j=1}^{k}(n+s_k-s_j)(x_j-r_j)}}{\tau_2^{\sum_{j=1}^{k} r_j(-z_j+s_k-s_j+n-1)}} \tag{2.19}$$

$$\times \frac{\prod_{j=1}^{k}[x_j]_{r_j,\mathcal{R}(p,q)}}{[\sum_{i=1}^{k+1} x_i]_{n+s_k,\mathcal{R}(p,q)}},$$

provided $|\big(\frac{\tau_2}{\tau_1}\big)^{x_{k+1}}| < 1$, *where* $s_j = \sum_{i=1}^{j} r_i$ *and* $z_j = \sum_{i=j+1}^{k+1} x_i$, *for* $j \in \{1, 2, \cdots, k\}$.

Proof. From the inverse $\mathcal{R}(p,q)$-Vandermonde formula [6], we get:

$$\frac{1}{[x_{k+1}]_{n,\mathcal{R}(p,q)}} = \sum_{r_k=0}^{\infty} \begin{bmatrix} n+r_k-1 \\ r_k \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\tau_1^{r_k(x_{k+1}-n+1)}}{\tau_2^{-n(x_k-r_k)}} \frac{[x_k]_{r_k,\mathcal{R}(p,q)}}{[x_k+x_{k+1}]_{n+r_k,\mathcal{R}(p,q)}}.$$

Similarly,

$$\frac{1}{[x_k+x_{k+1}]_{n+r_k,\mathcal{R}(p,q)}} = \sum_{r_{k-1}=0}^{\infty} \begin{bmatrix} n+r_k+r_{k-1}-1 \\ r_{k-1} \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times \tau_1^{r_{k-1}(x_k+x_{k+1}-n-r_k+1)} \frac{\tau_2^{(n+r_k)(x_{k-1}-r_{k-1})}[x_{k-1}]_{r_{k-1},\mathcal{R}(p,q)}}{[x_{k-1}+x_k+x_{k+1}]_{n+r_k+r_{k-1},\mathcal{R}(p,q)}}$$

and finally,

$$\frac{1}{\big[\sum_{j=2}^{k+1} x_j\big]_{n+s_k-s_1,\mathcal{R}(p,q)}} = \sum_{r_1=0}^{\infty} \begin{bmatrix} n+s_k-1 \\ r_1 \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times \tau_1^{r_1(x_2+x_3+\ldots+x_{k+1}-n-s_k+s_1+1)} \frac{\tau_2^{(n+s_k-s_1)(x_1-r_1)}[x_1]_{r_1,\mathcal{R}(p,q)}}{[x_1+x_2+\cdots+x_{k+1}]_{n+s_k,\mathcal{R}(p,q)}}.$$

Applying these k expansions, one after the other in the inner sum of each step, and using the relation:

$$\begin{bmatrix} n+s_k-1 \\ r_1, r_2, \ldots, r_k \end{bmatrix}_{\mathcal{R}(p,q)} = \begin{bmatrix} n+s_k-1 \\ r_1 \end{bmatrix}_{\mathcal{R}(p,q)} \begin{bmatrix} n+r_k-1 \\ r_k \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times \begin{bmatrix} n+r_k+r_{k-1}-1 \\ r_{k-1} \end{bmatrix}_{\mathcal{R}(p,q)},$$

the expansion (2.18) is obtained. The alternative expansion (2.19), is similarly deduced by using the following inverse $\mathcal{R}(p,q)$-Vandermonde expansions [6]:

$$\frac{1}{\big[\sum_{i=j+1}^{k+1} x_i\big]_{n+s_k-s_j,\mathcal{R}(p,q)}} = \sum_{r_j=0}^{\infty} \begin{bmatrix} n+s_k-s_{j-1}-1 \\ r_j \end{bmatrix}_{\mathcal{R}(p,q)}$$
$$\times \frac{\tau_2^{r_j(z_j-s_k+s_j-n+1)}[x_j]_{r_j,\mathcal{R}(p,q)}}{[x_j+\cdots+x_{k+1}]_{n+s_k-s_{j-1},\mathcal{R}(p,q)}},$$

for $j \in \{1, 2, \cdots, k\}$, with $s_0 = 0$. $\square$

Now, we investigate the generalized multivariate Cauchy formula and its related formulae.

Corollary 2.6. *The generalized multivariate Cauchy formula is furnished by:*

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i\\ n\end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \tau_1^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))}\tau_2^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)}\prod_{j=1}^{k+1}\begin{bmatrix}x_j\\ r_j\end{bmatrix}_{\mathcal{R}(p,q)}$$

and

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i\\ n\end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \tau_1^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)}\tau_2^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))}\prod_{j=1}^{k+1}\begin{bmatrix}x_j\\ r_j\end{bmatrix}_{\mathcal{R}(p,q)},$$

where $s_j = \sum_{i=1}^{j} r_i$, $z_j = \sum_{i=j+1}^{k+1} x_i$, $j \in \{1, 2, \cdots, k\}$, *and* $r_{k+1} = n - s_k$ *with* $\sum_{i=1}^{k} r_i \leq n$.

Remark 2.7. Several formulae can be deduced as follows:

(a) The generalized multivariate Cauchy formula can be rewritten as:

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i + n - 1\\ n\end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \tau_1^{\sum_{j=1}^{k} x_j(n-s_j)}\tau_2^{\sum_{j=1}^{k} r_j z_j} \times \prod_{j=1}^{k+1}\begin{bmatrix}x_j + r_j - 1\\ r_j\end{bmatrix}_{\mathcal{R}(p,q)}$$

and

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i + n - 1\\ n\end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \tau_1^{\sum_{j=1}^{k} r_j z_j}\tau_2^{\sum_{j=1}^{k} x_j(n-s_j)} \times \prod_{j=1}^{k+1}\begin{bmatrix}x_j + r_j - 1\\ r_j\end{bmatrix}_{\mathcal{R}(p,q)}.$$

(b) Other expressions of the generalized multivariate Cauchy formula:

$$\begin{bmatrix}r + k\\ n + k\end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=x_j}^{r} \tau_1^{\sum_{j=1}^{k}(x_j+1)(n-s_j-r+y_j)} \times \tau_2^{\sum_{j=1}^{k}(r_j-x_j)(n-y_j+k-j+1)}\prod_{j=1}^{k+1}\begin{bmatrix}r_j\\ x_j\end{bmatrix}_{\mathcal{R}(p,q)}$$

and

$$\begin{bmatrix} r+k \\ n+k \end{bmatrix}_{\mathcal{R}(p,q)} = \sum_{r_j=x_j}^{r} \tau_1^{\sum_{j=1}^{k}(r_j-x_j)(n-y_j+k-j+1)} \times \tau_2^{\sum_{j=1}^{k}(x_j+1)(n-s_j-r+y_j)} \prod_{j=1}^{k+1} \begin{bmatrix} r_j \\ x_j \end{bmatrix}_{\mathcal{R}(p,q)},$$

where $s_j = \sum_{i=1}^{j} r_i,\ y_j = \sum_{i=1}^{j} x_i,\ j \in \{1,2,\ldots,k\},\ r_{k+1} = r - s_k$, $x_{k+1} = r - y_k$, with $\sum_{i=1}^{k} r_i \leq r$.

Remark 2.8. Here, we determine the particular case of the generalized multivariate Vandermonde and Cauchy formulae corresponding to the quantum deformed algebras known in the literature.

(a) Setting $\mathcal{R}(x,y) = \frac{x-y}{p-q}$, we deduce the multivariate Vandermonde and Cauchy formulae associated to the quantum algebra [10]:

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{n,p,q} = \sum_{r_j=0}^{n} \begin{bmatrix} n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{p,q} \mathcal{V}(p\,q,n) \prod_{j=1}^{k+1} [x_j]_{r_j,p,q}$$

and

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{n,p,q} = \sum_{r_j=0}^{n} \begin{bmatrix} n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{p,q} \mathcal{V}(q\,p,n) \prod_{j=1}^{k+1} [x_j]_{r_j,p,q},$$

where $\mathcal{V}(p\,q,n) = p^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))} q^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)}$. Moreover, the negative multivariate (p,q)-Vandermonde formula is presented by:

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{-n,p,q} = \sum \begin{bmatrix} -n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{p,q} \mathcal{V}(pq,-n) \prod_{j=1}^{k+1} [x_j]_{r_j,p,q}$$

and

$$\Big[\sum_{i=1}^{k+1} x_i\Big]_{-n,p,q} = \sum \begin{bmatrix} -n \\ r_1, r_2, \cdots, r_k \end{bmatrix}_{p,q} \mathcal{V}(qp,-n) \prod_{j=1}^{k+1} [x_j]_{r_j,p,q},$$

where $\mathcal{V}(p\,q,-n) = p^{\sum_{j=1}^{k} r_j(z_j-(-n-s_j))}\, q^{\sum_{j=1}^{k}(-n-s_j)(x_j-r_j)}$. Also, the multivariate (p,q)-Vandermonde formula can be rewritten as:

$$\Big[\sum_{i=1}^{k+1} x_i + n - 1\Big]_{n,p,q} = \sum_{r_j=0}^{n} \begin{bmatrix} n \\ r_1, r_2, \ldots, r_k \end{bmatrix}_{p,q} p^{\sum_{j=1}^{k} x_j(n-s_j)}$$

$$\times\, q^{\sum_{j=1}^{k} r_j z_j}\prod_{j=1}^{k+1}[x_j+r_j-1]_{r_j,p,q},$$

and

$$\Big[\sum_{i=1}^{k+1} x_i+n-1\Big]_{n,p,q}=\sum_{r_j=0}^{n}\begin{bmatrix} n\\ r_1,r_2,\ldots,r_k\end{bmatrix}_{p,q} p^{\sum_{j=1}^{k} r_j z_j}$$
$$\times\, q^{\sum_{j=1}^{k} x_j(n-s_j)}\prod_{j=1}^{k+1}[x_j+r_j-1]_{r_j,p,q},$$

and the multivariate inverse (p,q)-Vandermonde formula as:

$$\frac{1}{[x_{k+1}]_{n,p,q}}=\sum_{r_j=0}^{n}\begin{bmatrix} n+s_k-1\\ r_1,\ldots,r_k\end{bmatrix}_{p,q}\frac{p^{\sum_{j=1}^{k} r_j(z_j-s_k+s_j-n+1)}}{q^{\sum_{j=1}^{k}(-n-s_k+s_j)(x_j-r_j)}}$$
$$\times\,\frac{\prod_{j=1}^{k}[x_j]_{r_j,p,q}}{[x_1+\cdots+x_{k+1}]_{n+s_k,p,q}},$$

and

$$\frac{1}{[x_{k+1}]_{n,p,q}}=\sum_{r_j=0}^{n}\begin{bmatrix} n+s_k-1\\ r_1,\ldots,r_k\end{bmatrix}_{p,q}\frac{p^{\sum_{j=1}^{k}(n+s_k-s_j)(x_j-r_j)}}{q^{\sum_{j=1}^{k} r_j(-z_j+s_k-s_j+n-1)}}$$
$$\times\,\frac{\prod_{j=1}^{k}[x_j]_{r_j,p,q}}{[x_1+\cdots+x_{k+1}]_{n+s_k,p,q}}.$$

Furthermore, the multivariate (p,q)-Cauchy formula is furnished by:

$$\begin{bmatrix} x_1+x_2+\cdots+x_{k+1}\\ n\end{bmatrix}_{p,q}=\sum_{r_j=0}^{n} p^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))}$$
$$\times\, q^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)}\prod_{j=1}^{k+1}\begin{bmatrix} x_j\\ r_j\end{bmatrix}_{p,q}$$

and

$$\begin{bmatrix} x_1+x_2+\cdots+x_{k+1}\\ n\end{bmatrix}_{p,q}=\sum_{r_j=0}^{n} p^{\sum_{j=1}^{k}(n-s_j)(x_j-r_j)}$$
$$\times\, q^{\sum_{j=1}^{k} r_j(z_j-(n-s_j))}\prod_{j=1}^{k+1}\begin{bmatrix} x_j\\ r_j\end{bmatrix}_{p,q}.$$

Moreover, the multivariate (p,q)-Cauchy formula can be rewritten as:

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i+n-1\\ n\end{bmatrix}_{p,q} = \sum_{r_j=0}^{n} p^{\sum_{j=1}^{k} x_j(n-s_j)} q^{\sum_{j=1}^{k} r_j z_j} \prod_{j=1}^{k+1} \begin{bmatrix}x_j+r_j-1\\ r_j\end{bmatrix}_{p,q}$$

and

$$\begin{bmatrix}\sum_{i=1}^{k+1} x_i+n-1\\ n\end{bmatrix}_{p,q} = \sum_{r_j=0}^{n} p^{\sum_{j=1}^{k} r_j z_j} q^{\sum_{j=1}^{k} x_j(n-s_j)} \prod_{j=1}^{k+1} \begin{bmatrix}x_j+r_j-1\\ r_j\end{bmatrix}_{p,q}.$$

Other expressions of the (p,q)-deformed multivariate Cauchy formulae are furnished by:

$$\begin{bmatrix}r+k\\ n+k\end{bmatrix}_{p,q} = \sum_{r_j=0}^{n} p^{\sum_{j=1}^{k}(x_j+1)(n-s_j-r+y_j)} \times q^{\sum_{j=1}^{k}(r_j-x_j)(n-y_j+k-j+1)} \prod_{j=1}^{k+1} \begin{bmatrix}r_j\\ x_j\end{bmatrix}_{p,q}$$

and

$$\begin{bmatrix}r+k\\ n+k\end{bmatrix}_{p,q} = \sum_{r_j=0}^{n} p^{\sum_{j=1}^{k}(r_j-x_j)(n-y_j+k-j+1)} \times q^{\sum_{j=1}^{k}(x_j+1)(n-s_j-r+y_j)} \prod_{j=1}^{k+1} \begin{bmatrix}r_j\\ x_j\end{bmatrix}_{p,q}.$$

(b) Putting $\mathcal{R}(x,y) = \frac{xy-1}{(q^{-1}-p)y}$, we derive the multivariate Vandermonde and Cauchy formulae related to the quantum algebra [9].

§3. Multivariate probability distributions from $\mathcal{R}(p,q)$-deformed quantum algebras

In this section, we construct some multivariate probability distributions (Pólya, inverse Pólya, hypergeometric and negative hypergeometric) from the generalized quantum algebras [7]. The corresponding bivariate probability distributions and properties are also investigated.

3.1. Multivariate $\mathcal{R}(p,q)$-Pólya distribution

We suppose that random $\mathcal{R}(p,q)$-drawings of balls are sequentially carried out, one after the other, from an urn initially containing r balls of $k+1$ different colors, with r_μ distinct balls of color c_μ, for $\mu = 1,2,\cdots,k+1$, according to the scheme below. After each $\mathcal{R}(p,q)$-drawings, the drawn ball is placed back in the urn together with m balls of the same color.

We denotes by Y_μ the number of balls of color c_μ drawn in n $\mathcal{R}(p,q)$-drawings in a multiple $\mathcal{R}(p,q)$-Pólya urn model, with conditional probability of drawing a ball of color c_μ at the ith $\mathcal{R}(p,q)$-drawing, given that $j_\mu - 1$ balls of color c_μ and a total of $i_{\mu-1}$ balls of colors $c_1, c_2, \cdots, c_{\mu-1}$ are drawn in the previous $i-1$ $\mathcal{R}(p,q)$-drawings, is given by

$$p_{i,j_\mu}(i_{\mu-1}) = \frac{\tau_2^{s_{\mu-1}+mi_{\mu-1}}[r_\mu + m(j_\mu - 1)]_{\mathcal{R}(p,q)}}{[r+m(i-1)]_{\mathcal{R}(p,q)}}, \tag{3.1}$$

where $j_\mu = 1,2,\cdots,i, i_\mu = 0,1,\cdots,i-1, i = 0,1,\cdots$, and $\mu \in \{1,2,\cdots,k+1\}$.

The probability distribution of the random vector $\underline{Y} = (Y_1,\ldots,Y_k)$ may be called multivariate $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, β, p, $\underline{\beta} = (\beta_1,\beta_2,\ldots,\beta_k)$, and q.

Theorem 3.1. *The mass function of the multivariate $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta} = (\beta_1,\beta_2,\ldots,\beta_k)$, β, p, and q, is presented as follows:*

$$P(\underline{Y} = \underline{y}) = \Psi_k(p,q) \begin{bmatrix} n \\ y_1,\ldots,y_k \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \frac{\prod_{j=1}^{k+1}[\beta_j]_{y_j,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n,\mathcal{R}(p^{-m},q^{-m})}}, \tag{3.2}$$

where

$$\Psi_k(p,q) = \tau_1^{-m\sum_{j=1}^k y_j(\beta_{j+1}-y_{j+1})}\tau_2^{-m\sum_{j=1}^k (n-x_j)(\beta_j-y_j)},$$

$y_j \in \{0,1,\cdots,n\}$, $j \in \{1,2,\ldots,k\}$, $\sum_{j=1}^k y_j \le n$, $y_{k+1} = n - \sum_{j=1}^k y_j$, $\beta_{k+1} = \beta - \sum_{j=1}^k \beta_j$, *and* $x_j = \sum_{i=1}^j y_i$.

Proof. For the proof, we use the total probability theorem. Moreover, the probabilities (3.2) sum to unity using the multivariate $\mathcal{R}(p,q)$-Vandermonde formula (2.9), the multivariate $\mathcal{R}(p,q)$-Cauchy formula (2.6). □

Remark 3.2. Another relations of the multivariate Pólya distribution from generalized quantum algebras are interested.

(i) The multivariate $\mathcal{R}(p,q)$-Pólya probability distribution (3.2) may be rewritten as:

$$P(\underline{Y} = \underline{y}) = \tau_1^{-m\sum_{j=1}^k y_j(\beta_{j+1}-x_{j+1})}\tau_2^{m\sum_{j=1}^k (n-x_j)(\beta_j-y_j)} \tag{3.3}$$

$$\times \frac{\prod_{j=1}^{k+1} \begin{bmatrix} \beta_j \\ y_j \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} \beta \\ n \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}.$$

Indeed, it is derived from (3.2) according to the relations

$$\begin{bmatrix} n \\ y_1, y_2, \cdots, y_k \end{bmatrix}_{\mathcal{R}(p,q)} = \frac{[n]_{\mathcal{R}(p,q)}!}{[y_1]_{\mathcal{R}(p,q)}![y_2]_{\mathcal{R}(p,q)}!\cdots[y_k]_{\mathcal{R}(p,q)}![y_{k+1}]_{\mathcal{R}(p,q)}!}$$

and

$$\begin{bmatrix} \beta_j \\ y_j \end{bmatrix}_{\mathcal{R}(p,q)} = \frac{[\beta_j]_{y_j,\mathcal{R}(p,q)}}{[y_j]_{\mathcal{R}(p,q)}!}.$$

(ii) Taking $k = 1$, we obtain the $\mathcal{R}(p,q)$-Pólya distribution given in ([5], Definition 3.7 in page 14).

We assume that the random vector $\underline{Y} = (Y_1, Y_2, \ldots, Y_k)$ obeys the multivariate $\mathcal{R}(p,q)$-Pólya probability distribution with parameters n, β, p, $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$, and q. Then:

Proposition 3.3. *For $\mu \in \{1, 2, \cdots, k\}$, then, the marginal distribution of the random vector $\underline{Y} = (Y_1, Y_2, \cdots, Y_\mu)$ is a $\mathcal{R}(p,q)$-Pólya distribution of order μ, with parameters n, $(\beta_1, \beta_2, \cdots, \beta_\mu)$, β, p, and q.*

Moreover, for $\mu \in \{1, 2, \cdots, k-t\}$ and $t \in \{1, 2, \cdots, k-1\}$, the conditional distribution of the random vector $(Y_{\mu+1}, Y_{\mu+2}, \ldots, Y_{\mu+t})$, given that $(Y_1, Y_2, \ldots, Y_\mu) = (y_1, y_2, \cdots, y_\mu)$ is a $\mathcal{R}(p,q)$-Pólya probability distribution of order t, with parameters n, $(\beta_{\mu+1}, \beta_{\mu+2}, \cdots, \beta_{\mu+t})$, $\beta - \beta_1 - \beta_2 - \cdots - \beta_\mu$, p, and q.

Proof. For the proof, we use the probability (3.3). Summing it for $y_j \in \{0, 1, \cdots, n - x_\mu\}$, $j \in \{\mu+1, \mu+2, \cdots, k\}$, with $y_{\mu+1} + y_{\mu+2} + \cdots + y_k \leq n - x_\mu$, and according to the relation (2.6), we have:

$$\begin{aligned} P(\underline{Y} = \underline{y} =) =& \Psi_k(p,q) \prod_{j=1}^{\mu} \begin{bmatrix} \beta_j \\ y_j \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \sum \Phi_k(p,q) \\ & \times \prod_{j=\mu+1}^{k} \begin{bmatrix} \beta_j \\ y_j \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \frac{\begin{bmatrix} \beta-\beta_1-\cdots-\beta_k \\ n-y_1-\cdots-y_k \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} \beta \\ n \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} \\ =& \Psi_k(p,q) \prod_{j=1}^{\mu} \begin{bmatrix} \beta_j \\ y_j \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \frac{\begin{bmatrix} \beta-\beta_1-\cdots-\beta_\mu \\ n-y_1-\cdots-y_\mu \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} \beta \\ n \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}, \end{aligned}$$

which is the probability function of a $\mathcal{R}(p,q)$-Pólya distribution of order μ, with parameters n, $\underline{\beta}=(\beta_1,\beta_2,\cdots,\beta_\mu)$, β, p, and q. Furthermore, according to the relation,

$$P(Y_\mu=y_\mu,\cdots,Y_{\mu+k-1}=y_{\mu+k-1}|Y_1=y_1,Y_2=y_2,\cdots,Y_{\mu-1}=y_{\mu-1}) = \frac{P\big(Y_1=y_1,Y_2=y_2,\ldots,Y_{\mu+k-1}=y_{\mu+k-1}\big)}{P(Y_1=y_1,Y_2=y_2,\ldots,Y_{\mu-1}=y_{\mu-1}\big)},$$

we obtain, the density function of the conditional distribution of the random vector $(Y_\mu,Y_{\mu+1},\cdots,Y_{\mu+k-1})$, given that $(Y_1,Y_2,\ldots,Y_{\mu-1})=(y_1,\cdots,y_{\mu-1})$:

$$P(Y_\mu=y_\mu,\ldots,Y_{\mu+k-1}=y_{\mu+k-1}|Y_1=y_1,Y_2=y_2,\ldots,Y_{\mu-1}=y_{\mu-1}) = \Psi_k(p,q)\frac{\prod_{j=\mu}^{\mu+k-1}\begin{bmatrix}\beta_j\\ y_j\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\begin{bmatrix}\beta-\beta_\mu-\cdots-\beta_{\mu+k-1}\\ n-y_\mu-\cdots-y_{\mu+k-1}\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix}\beta\\ n\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}},$$

which is the mass function of a $\mathcal{R}(p,q)$-Pólya distribution of order k, with parameters n, $(\beta_\mu,\beta_{\mu+1},\ldots,\beta_{\mu+k-1})$, $\beta-\beta_1-\beta_2-\cdots-\beta_{\mu-1}$, p, and q. □

3.1.1. Bivariate $\mathcal{R}(p,q)$-Pólya distribution

For $k=2$ in the multiple $\mathcal{R}(p,q)$-urn model, we obtain the following results: Then, we denote by $\underline{Y}=\big(Y_1,Y_2\big)$ the random vector. Also, $\underline{Y}$ follows the bivariate $\mathcal{R}(p,q)$-Pólya distribution with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q. The probability function is derived by the following relation:

$$P(\underline{Y}=\underline{y})=\Psi_2(p,q)\begin{bmatrix}n\\ y_1,y_2\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\frac{\prod_{j=1}^{3}[\beta_j]_{y_j,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n,\mathcal{R}(p^{-m},q^{-m})}},$$

equivalently,

$$P(\underline{Y}=\underline{y})=\Psi_2(p,q)\frac{\prod_{j=1}^{3}\begin{bmatrix}\beta_j\\ y_j\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix}\beta\\ n\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}},$$

where

$$\Psi_2(p,q)=\tau_1^{-m\sum_{j=1}^{2}y_j(\beta_{j+1}-y_{j+1})}\,\tau_2^{-m\sum_{j=1}^{2}(n-x_j)(\beta_j-y_j)},$$

$y_j\in\{0,1,\cdots,n\}$, $j\in\{1,2\}$, $y_1+y_2\leq n$, $y_3=n-y_1+y_2$, $\beta_3=\beta-\beta_1-\beta_2$, $x_1=y_1$, and $x_2=y_1+y_2$. The properties of the bivariate $\mathcal{R}(p,q)$-Pólya distribution are presented as:

Theorem 3.4. *The $\mathcal{R}(p,q)$-factorial moments of the bivariate $\mathcal{R}(p,q)$-Pólya probability distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2)$, β, p and q, are given by:*

$$E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}\big)=\frac{[n]_{i_1,\mathcal{R}(p^m,q^m)}[\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{i_1,\mathcal{R}(p^{-m},q^{-m})}},\ i_1\in\{0,1,\cdots,n\}, \tag{3.4}$$

$$E\big([Y_2]_{i_1,\mathcal{R}(p^m,q^m)}|Y_1=y_1\big)=\frac{[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)}[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}},\tag{3.5}$$

with $i_2\in\{0,1,\cdots,n-y_1\}$,

$$E\big([Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)=\frac{[n]_{i_2,\mathcal{R}(p^m,q^m)}[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\beta_1}[\beta]_{i_2,\mathcal{R}(p^{-m},q^{-m})}},\tag{3.6}$$

with $i_2\in\{0,1,\cdots,n\}$ *and*

$$\begin{aligned}E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)=&\frac{[n]_{i_1+i_2,\mathcal{R}(p^m,q^m)}}{\tau_2^{mi_2\beta_1}},\\&\times\frac{[\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}},\end{aligned}\tag{3.7}$$

where $i_1\in\{0,1,\cdots,n-i_2\}$ *and* $i_2\in\{0,1,\cdots,n\}$.

Proof. The random variable Y_1 follows a $\mathcal{R}(p,q)$-Pólya distribution, with mass function [5]:

$$P(Y_1=y_1)=\Psi_2(p,q)\begin{bmatrix}n\\y_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}[\beta_1]_{y_1,\mathcal{R}(p^{-m},q^{-m})}\frac{[\beta-\beta_1]_{n-y_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n,\mathcal{R}(p^{-m},q^{-m})}},$$

where $\Psi_2(p,q)=\tau_1^{-my_1(\beta-\beta_1+y_1-n)}\tau_2^{-m(n-y_1)(\beta-y_1)}$ and $y_1\in\{0,1,\cdots,n\}$.

Thus, from [5], the $\mathcal{R}(p,q)$-factorial moments of the random variable Y_1 are determined by (3.4). Moreover, the conditional distribution of the random variable Y_2, given that $Y_1=y_1$, is a $\mathcal{R}(p,q)$-Pólya distribution, with mass function

$$\begin{aligned}P(Y_2=y_2|Y_1=y_1)=&\frac{\tau_1^{-my_1(\beta-\beta_1+y_1-n)}}{\tau_2^{m(n-y_1)(\beta-y_1)}}\begin{bmatrix}n-y_1\\y_2\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\\&\times\frac{[\beta_2]_{y_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n-y_1-y_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n-y_1,\mathcal{R}(p^{-m},q^{-m})}},\end{aligned}$$

where $y_2\in\{0,1,\cdots,n-y_1\}$. Once, using [5], the conditional $\mathcal{R}(p,q)$-factorial moments of Y_2, given that $Y_1=y_1$, are furnished by (3.5).

Now, we compute the $\mathcal{R}(p,q)$-factorial moments $E\big([Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)$, $i_2\in\{0,1,\cdots,n\}$ by using the relation

$$\begin{aligned}E\big([Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)=&E\Big(E\big([Y_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}|Y_1\big)\Big)\\=&\frac{[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}E\big([n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big).\end{aligned}$$

Clearly we have

$$E\big([n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \sum_{y_1=0}^{n-i_2}[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)}\begin{bmatrix} n\\ y_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\Psi_2(p,q) \times \frac{[\beta_1]_{y_1,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1]_{n-y_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n,\mathcal{R}(p^{-m},q^{-m})}}$$

and from the relations

$$[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2(n-\beta_1)-m\binom{i_2+1}{2}} \times \big(\tau_1\tau_2\big)^{mi_2(\beta_1-y_1)}[n-y_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

$$[n]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2\,n-m\binom{i_2+1}{2}}[n]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

and

$$\frac{\begin{bmatrix} n\\ y_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} n-i_2\\ y_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} = \frac{[n]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[n-y_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}},$$

we obtain the following expression

$$E\big([n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \frac{[n]_{i_2,\mathcal{R}(p^m,q^m)}[n-\beta_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\alpha_1}[\beta]_{i_2,\mathcal{R}(p^{-m},q^{-m})}} \times \sum_{y_1=0}^{n-i_2}[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)}\Psi_2(p,q)\begin{bmatrix} n-i_2\\ y_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \times \frac{[\beta_1]_{y_1,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-i_2]_{n-i_2-y_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-i_2]_{n-i_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

Using the $\mathcal{R}(p,q)$-Vandermonde formula [5, 6], we have:

$$E\big([n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \frac{[n]_{i_2,\mathcal{R}(p^m,q^m)}[n-\beta_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\alpha_1}[\beta]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

Thus, the relation (3.6) follows.

Analogously, we calculate the joint $\mathcal{R}(p,q)$-factorial moments

$$E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big),$$

where $i_1\in\{0,1,\cdots,n\}$ and $i_2\in\{0,1,\cdots,n-i_1\}$. They can computed according to the relation

$$E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big) = E\Big(E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[Y_2]_{i_2,\mathcal{R}(p^m,q^m)}|Y_1\big)\Big)$$

$$= \frac{[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}} E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big).$$

Clearly we have

$$E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \sum_{y_1=0}^{n-i_2}[y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)} \times \begin{bmatrix} n \\ y_1 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}[\beta_1]_{y_1,\mathcal{R}(p^{-m},q^{-m})}\Psi_2(p,q)\frac{[\beta-\beta_1]_{n-y_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n,\mathcal{R}(p^{-m},q^{-m})}}$$

and using the relations

$$[y_1]_{i_1,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_1\,y_1-m\binom{i_1+1}{2}}[y_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})},$$

$$[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2(n-\beta_1)-m\binom{i_2+1}{2}} \times \big(\tau_1\tau_2\big)^{mi_2(\beta_1-y_1)}[n-y_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

$$[n]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2\,n-m\binom{i_2+1}{2}}[n]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

$$\frac{\begin{bmatrix} n-i_1-i_2 \\ y_1-i_1 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} n \\ y_1 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} = \frac{[y_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}[n-y_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[n]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}},$$

and

$$[n]_{i_1+i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{m(i_1+i_2)n-\binom{i_1+i_2+1}{2}}[n]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}$$

we have

$$\begin{aligned} E\big(\hat{\gamma}\big) :=& E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) \\ =& \frac{[n]_{i_1+i_2,\mathcal{R}(p^m,q^m)}[\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\alpha_1}}\frac{[n-\beta_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}} \\ &\times \sum_{y_1=0}^{n-i_2}\Psi_2(p,q)\begin{bmatrix} n-i_2-i_1 \\ y_1-i_1 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}[n-y_1]_{i_2,\mathcal{R}(p^m,q^m)} \\ &\times \frac{[\beta-\beta_1-i_2]_{n-i_2-y_1,\mathcal{R}(p^{-m},q^{-m})}[\beta_1-i_1]_{y_1-i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-i_2-i_1]_{n-i_2-i_1,\mathcal{R}(p^{-m},q^{-m})}}. \end{aligned}$$

From the $\mathcal{R}(p,q)$-Vandermonde formula [5, 6], we get:

$$E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \frac{[n]_{i_1+i_2,\mathcal{R}(p^m,q^m)}[\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\alpha_1}}$$

$$\times \frac{[n-\beta_1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

Thus,

$$\begin{aligned}E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[Y_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)\\ &= \frac{[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}E\big([Y_1]_{i_1,\mathcal{R}(p^m,q^m)}[n-Y_1]_{i_2,\mathcal{R}(p^m,q^m)}\big)\\ &= \frac{[n]_{i_1+i_2,\mathcal{R}(p^m,q^m)}[\beta_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{mi_2\alpha_1}}\frac{[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}}.\end{aligned}$$

and the relation (3.7) holds. □

Corollary 3.5. *The $\mathcal{R}(p,q)$-covariance of $[Y_1]_{\mathcal{R}(p^m,q^m)}$ and $[Y_2]_{\mathcal{R}(p^m,q^m)}$ is given by the relation:*

$$\begin{aligned}Cov([Y_1]_{\mathcal{R}(p^m,q^m)},[Y_2]_{\mathcal{R}(p^m,q^m)}) =& \frac{[n]_{\mathcal{R}(p^m,q^m)}[\beta_1]_{\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{m\beta_1}}\\ &\times \frac{[\beta_2]_{\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{\mathcal{R}(p^{-m},q^{-m})}}\Delta(n,m,\beta),\end{aligned}$$

where

$$\Delta(n,m,\beta) = \frac{[n-1]_{\mathcal{R}(p^m,q^m)}}{[\beta-1]_{\mathcal{R}(p^{-m},q^{-m})}} - \frac{[n]_{\mathcal{R}(p^m,q^m)}}{[\beta_1]_{\mathcal{R}(p^{-m},q^{-m})}}$$

and $\underline{Y} = \big(Y_1,Y_2\big)$ a random vector verifying the bivariate $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta} = (\beta_1,\beta_2), \beta$, p and q.

Proof. By definition, we get:

$$\begin{aligned}Cov\big([Y_1]_{\mathcal{R}(p^m,q^m)},[Y_2]_{\mathcal{R}(p^m,q^m)}\big) &= E\big([Y_1]_{\mathcal{R}(p^m,q^m)}[Y_2]_{\mathcal{R}(p^m,q^m)}\big)\\ &- E\big([Y_1]_{\mathcal{R}(p^m,q^m)}\big)E\big([Y_2]_{\mathcal{R}(p^m,q^m)}\big).\end{aligned}$$

Taking $i_1 = i_2 = 1$, in the relations (3.4), (3.6), and (3.7), we obtain:

$$\begin{aligned}E\big([Y_1]_{\mathcal{R}(p^m,q^m)}[Y_2]_{\mathcal{R}(p^m,q^m)}\big) &= \frac{[n]_{2,\mathcal{R}(p^m,q^m)}[\beta_1]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{\mathcal{R}(p^{-m},q^{-m})}}{\tau_2^{m\beta_1}\,[\beta]_{2,\mathcal{R}(p^{-m},q^{-m})}}\\ &= \frac{[n]_{\mathcal{R}(p^m,q^m)}[n-1]_{\mathcal{R}(p^m,q^m)}}{\tau_2^{m\beta_1}[\beta]_{\mathcal{R}(p^{-m},q^{-m})}}\frac{[\beta_1]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{\mathcal{R}(p^{-m},q^{-m})}}{[\beta-1]_{\mathcal{R}(p^{-m},q^{-m})}}\end{aligned}$$

and

$$E\big([Y_1]_{\mathcal{R}(p^m,q^m)}\big)E\big([Y_2]_{\mathcal{R}(p^m,q^m)}\big)$$

$$= [n]^2_{\mathcal{R}(p^m,q^m)}[\beta_1]_{\mathcal{R}(p^{-m},q^{-m})}\frac{[\beta_2]_{\mathcal{R}(p^{-m},q^{-m})}\tau_2^{-m\beta_1}}{[\beta]^2_{\mathcal{R}(p^{-m},q^{-m})}}.$$

After computation, the proof is achieved. □

Remark 3.6. The multivariate Pólya probability distribution and properties generated by quantum algebras are presented in the sequel:

The mass function of the multivariate (p,q)-Pólya distribution, with parameters n, β, p, $(\beta_1,\beta_2,\ldots,\beta_k)$, and q, is presented as follows:

$$P(\underline{Y}=\underline{y}) = \Psi_k(p,q)\begin{bmatrix} n \\ y_1,\ldots,y_k\end{bmatrix}_{p^{-m},q^{-m}}\frac{\prod_{j=1}^{k+1}[\beta_j]_{y_j,p^{-m},q^{-m}}}{[\beta]_{n,p^{-m},q^{-m}}},$$

or

$$P(\underline{Y}=\underline{y}) = \Psi_k(p,q)\frac{\prod_{j=1}^{k+1}\begin{bmatrix}\beta_j\\ y_j\end{bmatrix}_{p^{-m},q^{-m}}}{\begin{bmatrix}\beta\\ n\end{bmatrix}_{p^{-m},q^{-m}}},$$

where

$$\Psi_k(p,q) = p^{-m\sum_{j=1}^k y_j(\beta_{j+1}-y_{j+1})}q^{-m\sum_{j=1}^k (n-x_j)(\beta_j-y_j)},$$

$y_j\in\{0,1,\ldots,n\}$, $j\in\{1,2,\ldots,k\}$, $\sum_{j=1}^k y_j\leq n$, $y_{k+1}=n-\sum_{j=1}^k y_j$, $\beta_{k+1}=\beta-\sum_{j=1}^k\beta_j$, and$x_j=\sum_{i=1}^j y_i$.

The probability distribution of the random vector $\underline{Y}=\big(Y_1,Y_2\big)$ is called the bivariate (p,q)-Pólya distribution with parameters n, $\underline{\beta}=(\beta_1,\beta_2)$, β, p and q. Also, it's probability function is given by the following relation:

$$P(Y_1=y_1,Y_2=y_2) = \Psi_2(p,q)\begin{bmatrix} n \\ y_1,y_2\end{bmatrix}_{p^{-m},q^{-m}}\frac{\prod_{j=1}^{3}[\beta_j]_{y_j,p^{-m},q^{-m}}}{[\beta]_{n,p^{-m},q^{-m}}},$$

where

$$\Psi_2(p,q) = p^{-m\sum_{j=1}^2 y_j(\beta_{j+1}-y_{j+1})}q^{-m\sum_{j=1}^2 (n-x_j)(\beta_j-y_j)},$$

$y_j\in\{0,1,\ldots,n\}$, $j\in\{1,2\}$, $y_1+y_2\leq n$, $y_3=n-y_1+y_2$, $\beta_3=\beta-\beta_1-\beta_2$, $x_1=y_1$, and $x_2=y_1+y_2$. Besides, its (p,q)-factorial moments are given by:

$$E\big([Y_1]_{i_1,p^m,q^m}\big) = \frac{[n]_{i_1,p^m,q^m}[\beta_1]_{i_1,p^{-m},q^{-m}}}{[\beta]_{i_1,p^{-m},q^{-m}}},\ i_1\in\{0,1,\cdots,n\},$$

$$E\big([Y_2]_{i_1,p^m,q^m}|Y_1=y_1\big) = \frac{[n-y_1]_{i_2,p^m,q^m}[\beta_2]_{i_2,p^{-m},q^{-m}}}{[\beta-\beta_1]_{i_2,p^{-m},q^{-m}}},$$

with $i_2\in\{0,1,\cdots,n-y_1\}$,

$$E\big([Y_2]_{i_2,p^m,q^m}\big) = \frac{[n]_{i_2,p^m,q^m}[\beta_2]_{i_2,p^{-m},q^{-m}}q^{-mi_2\beta_1}}{[\beta]_{i_2,p^{-m},q^{-m}}},$$

with $i_2 \in \{0, 1, \cdots, n\}$ and

$$E\big([Y_1]_{i_1,p^m,q^m}[Y_2]_{i_2,p^m,q^m}\big) = \frac{[n]_{i_1+i_2,p^m,q^m}[\beta_1]_{i_1,p^{-m},q^{-m}}[\beta_2]_{i_2,p^{-m},q^{-m}}}{q^{mi_2\beta_1}[\beta]_{i_1+i_2,p^{-m},q^{-m})}},$$

where $i_1 \in \{0, 1, \cdots, n - i_2\}$ and $i_2 \in \{0, 1, \cdots, n\}$. Moreover, the (p,q)-covariance of $[Y_1]_{p^m,q^m}$ and $[Y_2]_{p^m,q^m}$ is given by:

$$Cov\big([Y_1]_{p^m,q^m}, [Y_2]_{p^m,q^m}\big) = \frac{[n]_{p^m,q^m}[\beta_1]_{p^{-m},q^{-m}}[\beta_2]_{p^{-m},q^{-m}}}{q^{m\beta_1}[\beta]_{p^{-m},q^{-m}}}\Delta(n,m,\beta),$$

where

$$\Delta(n,m,\beta) = \frac{[n-1]_{p^m,q^m}}{[\beta-1]_{p^{-m},q^{-m}}} - \frac{[n]_{p^m,q^m}}{[\beta_1]_{p^{-m},q^{-m}}}.$$

3.2. Multivariate inverse $\mathcal{R}(p,q)$-Pólya distribution

Let W_ν be the number of balls of color c_ν drawn until the nth ball of color c_{k+1} is drawn in a multiple $\mathcal{R}(p,q)$-Pólya urn model, with the conditional probability of drawing a ball of color c_ν at the ith $\mathcal{R}(p,q)$-drawing, given that $j_\nu - 1$ balls of color c_ν and a total of $i_{\nu-1}$ balls of colors $c_1, c_2, \ldots, c_{\nu-1}$ are drawn in the previous $i-1$ $\mathcal{R}(p,q)$-drawings, given by (3.1), for $\nu \in \{1, 2, \cdots, k\}$. The distribution of the random vector $\underline{W} = (W_1, W_2, \cdots, W_k)$ can be called multivariate inverse $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_k)$, β, p, and q.

Theorem 3.7. *The probability function of the multivariate inverse $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta} = (\beta_1, \beta_2, \cdots, \beta_k)$, β, p, and q, is given by:*

$$P(\underline{W} = \underline{w}) = F_k(p,q)\begin{bmatrix} n + w_k - 1 \\ w_1, w_2, \cdots, w_k \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \times \frac{\prod_{j=1}^{k}[\beta_j]_{w_j,\mathcal{R}(p^{-m},q^{-m})}[\beta_{k+1}]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n+w_k,\mathcal{R}(p^{-m},q^{-m})}}, \tag{3.8}$$

where

$$F_k(p,q) = \tau_1^{-m\sum_{j=1}^{k} w_j(\beta_{j+1}-n+1)}\tau_2^{-m\sum_{j=1}^{k}(n+w_k-w_j)(\beta_j-w_j)}$$

for $w_j \in \mathbb{N} \cup \{0\}$, $j \in \{1, 2, \cdots, k\}$, $\beta_{k+1} = \beta - \sum_{j=1}^{k}\beta_j$, and $w_j = \sum_{i=1}^{j} w_i$, for $j \in \{1, 2, \cdots, k\}$.

Proof. The probability function of the k-variate inverse $\mathcal{R}(p,q)$-Pólya distribution is connected to the probability function k-variate $\mathcal{R}(p,q)$-Pólya distribution. Specifically,

$$P(W_1=w_1,W_2=w_2,\cdots,W_k=w_k)=p_{n+u_k-1}(w_1,w_2,\cdots,w_k)p_{n+u_k,n},$$

where $p_{n+u_k-1}(w_1,w_2,\cdots,w_k)$ is the probability of drawing w_ν balls of color c_ν, for all $\nu\in\{1,2,\cdots,k\}$, and $n-1$ balls of color c_{k+1} in $n+w_k-1$ $\mathcal{R}(p,q)$-drawings and

$$p_{n+w_k,n}=\frac{q^{-m(\beta_k-w_k)}[a_{k+1}-n+1]_{\mathcal{R}(p^{-m},q^{-m})}}{[a-n-w_k+1]_{\mathcal{R}(p^{-m},q^{-m})}}$$

is the conditional probability of drawing a ball of color c_{k+1} at the $(n+w_k)$th $\mathcal{R}(p,q)$-drawing, given that $n-1$ balls of color c_{k+1} and a total of w_k balls of colors $c_1,c_2,\ldots,c_k$ are drawn in the previous $n+w_k-1$ $\mathcal{R}(p,q)$-drawings. Thus using (3.2), expression (3.8) is deduced. Note that the multivariate inverse $\mathcal{R}(p,q)$-Vandermonde formula (2.18) guarantees that the probabilities (3.8) sum to unity. □

3.2.1. Bivariate inverse $\mathcal{R}(p,q)$-Pólya distribution

The mass function of the bivariate inverse $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q, is derived as:

$$\begin{aligned}(3.9)\quad & P(W_1=w_1,W_2=w_2)\\ &=F_2(p,q)\begin{bmatrix}n+w_2-1\\ w_1,w_2\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\frac{\prod_{j=1}^{2}[\beta_j]_{w_j,\mathcal{R}(p^{-m},q^{-m})}[\beta_3]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}},\end{aligned}$$

where

$$F_2(p,q)=\tau_1^{-m\sum_{j=1}^{2}w_j(\beta_{j+1}-n+1)}\tau_2^{-m\sum_{j=1}^{2}(n+w_2-w_j)(\beta_j-w_j)}$$

for $w_j\in\mathbb{N}\cup\{0\}$, $j\in\{1,2\}$, $\beta_3=\beta-\sum_{j=1}^{2}\beta_j$, and $w_j=\sum_{i=1}^{j}w_i$ for $j\in\{1,2\}$.

Theorem 3.8. *For $i_1\in\mathbb{N}\cup\{0\}$ and $i_2\in\mathbb{N}\cup\{0\}$, the $\mathcal{R}(p,q)$-factorial moments of the bivariate inverse $\mathcal{R}(p,q)$-Pólya probability distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q, are given by:*

$$(3.10)\qquad E\big([W_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)=\frac{[n+i_2-1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-\beta_2+i_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}},$$

(3.11)

$$E\big([W_1]_{i_1,\mathcal{R}(p^m,q^m)}|W_2=w_2\big)=\frac{[n+w_2+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1+i_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)},$$

(3.12)

$$E\Big(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\Big) = [n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}\frac{[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)}}{\Theta(n,m,\beta)},$$

and

(3.13)

$$E\left(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}[W_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\right) = [n+i_1+i_2-1]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})} \times \frac{[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-\beta_2+i_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}\frac{[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}}{\Theta(n,m,\beta)},$$

where

$$\Theta(n,m,\beta) = [\beta-\beta_1+i_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^m,q^m)}.$$

Proof. The marginal distribution of the random variable is an inverse $\mathcal{R}(p,q)$-Pólya distribution, with probability function:

$$P(W_1=w_1) = F_2(p,q)\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \times \frac{[\beta_2]_{w_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

The $\mathcal{R}(p,q)$- factorial moments $E\big([W_2]_{i_2,\mathcal{R}(p^m,q^m)}\big)$, $i_2\in\mathbb{N}\cup\{0\}$ are given by:

(3.14)

$$E\big([W_2]_{i_2,\mathcal{R}(p^m,q^m)}\big) = \sum_{w_2=i_2}^{\infty}[w_2]_{i_2,\mathcal{R}(p^m,q^m)}\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \times \frac{F_2(p,q)[\beta_2]_{w_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

By using the relations

$$\frac{\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} n+w_2-1 \\ w_2-i_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} = \frac{[n+i_2-1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}{[w_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}},$$

$$[w_2]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2\,w_2-m\binom{i_2+1}{2}}\,[w_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

$$[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)} = \big(\tau_1\tau_2\big)^{mi_2\,\beta_2-m\binom{i_2+1}{2}}\,[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})},$$

and

$$-mn(\beta_2-w_2)+mi_2\,w_2-mi_2\,\beta_2 = -m(n+i_2)(\beta_2-w_2),$$

the relation (3.14) can be rewritten as:

$$\begin{aligned} E\big([W_2]_{i_2,\mathcal{R}(p^m,q^m)}\big) =&[n+i_2-1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}\\ &\times [\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}\sum_{w_2=i_2}^{\infty}F_2(p,q)\begin{bmatrix} n+w_2-1\\ w_2-i_2\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\\ &\times \frac{[\beta_2-i_2]_{w_2-i_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}}. \end{aligned}$$

From the negative $\mathcal{R}(p,q)$-Vandermonde formula [5, 6], we have

$$\begin{aligned} E\big([W_2]_{i_2,\mathcal{R}(p^m,q^m)}\big) =&[n+i_2-1]_{i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}\\ &\times \frac{[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-\beta_2+i_2]_{n+i_2,\mathcal{R}(p^{-m},q^{-m})}} \end{aligned}$$

and the relation (3.10) follows by using

$$\begin{aligned} [\beta-\beta_1-\beta_2+i_2]_{n+i_2,\mathcal{R}(p^{-m},q^{-m})} =&[\beta-\beta_1-\beta_2+i_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}\\ &\times [\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}. \end{aligned}$$

Furthermore, the conditional distribution W_1, given that $W_2 = w_2$, is an inverse $\mathcal{R}(p,q)$-Pólya probability distribution, with mass function:

$$\begin{aligned} P(W_1=w_1|W_1=w_1) =&F_2(p,q)\begin{bmatrix} n+w_2+w_1-1\\ w_1\end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}\\ &\times \frac{[\beta_1]_{w_1,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta]_{n+w_2+w_1,\mathcal{R}(p^{-m},q^{-m})}}. \end{aligned}$$

From the same procedure using to obtain (3.10), the relation (3.11) holds. Moreover, the expected value of the $\mathcal{R}(p,q)$-function of $\underline{W}=\big(W_1,W_2\big)$

$$\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}$$

can be computed by using

$$E\bigg(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg) = E\bigg(E\bigg(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}|W_2\bigg)\bigg).$$

Clearly we have

$$E(\hat{\alpha}) :=E\bigg(\frac{[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg)$$

$$
\begin{aligned}
&= \sum_{w_2=0}^{\infty} \frac{[n+i_1+w_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{m},q^{m})}} F_2(p,q) \\
&\times \begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \\
&\times \frac{[\beta_2]_{w_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}}
\end{aligned}
$$

and from the relations

$$
\frac{\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} n+i_1+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} = \frac{[n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[n+i_1+w_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}},
$$

$$
[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{m},q^{m})} = \frac{[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{mi_1(\beta-\beta_1-n-w_2)-m\binom{i_2+1}{2}}},
$$

$$
\begin{aligned}
[\beta-\beta_1]_{n+i_1+w_2,\mathcal{R}(p^{-m},q^{-m})} =& [\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})} \\
& \times [\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{-m},q^{-m})},
\end{aligned}
$$

with the negative $\mathcal{R}(p,q)$-Vandermonde formula [5, 6], we obtain:

$$
\begin{aligned}
E\big(\bar{\alpha}\big) :=& E\bigg(\frac{[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^{m},q^{m})}}\bigg) \\
=& \frac{[n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{-mi_1(\beta-\beta_1-n-\beta_2)+m\binom{i_2+1}{2}}} \\
& \times \sum_{w_2=0}^{\infty} F_2(p,q) \begin{bmatrix} n+i_2+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \\
& \times \frac{[\beta_2]_{w_2,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+i_1+w_2,\mathcal{R}(p^{-m},q^{-m})}} \\
=& \frac{[n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{-mi_1(\beta-\beta_1-n-\beta_2)+m\binom{i_2+1}{2}}} \\
& \times \frac{[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-\beta_2]_{n+i_1,\mathcal{R}(p^{-m},q^{-m})}}.
\end{aligned}
$$

Also, according to

$$
\begin{aligned}
[\beta-\beta_1-\beta_2]_{n+i_1,\mathcal{R}(p^{-m},q^{-m})} =& [\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})} \\
& \times [\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{-m},q^{-m})}
\end{aligned}
$$

and

$$[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^m,q^m)} = \frac{[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{mi_1(\beta-\beta_1-\beta_2-n)-m\binom{i_2+1}{2}}},$$

we get

$$E\bigg(\frac{[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg) = \frac{[n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^m,q^m)}}.$$

Then,

$$\begin{aligned} E\bigg(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg) =& \frac{[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1+i_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}} \\ & \times E\bigg(\frac{[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg) \\ =& \frac{[n+i_1-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^m,q^m)}} \\ & \times \frac{[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1+i_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}. \end{aligned}$$

In addition, we compute the mean of the $\mathcal{R}(p,q)$-function

$$\widetilde{W} = \frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}[W_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}.$$

For instance, we use the relation

$$\begin{aligned} E\big(\widetilde{W}\big) =& E\bigg(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}[W_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg) \\ =& E\bigg(E\bigg(\frac{[W_1]_{i_1,\mathcal{R}(p^m,q^m)}[W_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}|W_2\bigg)\bigg) \\ =& \frac{[\beta_1]_{i_1,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1+i_1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}} \\ & \times E\bigg(\frac{[W_2]_{i_2,\mathcal{R}(p^m,q^m)}[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg). \end{aligned}$$

Clearly we have

$$E\big(\overline{W}\big) := E\bigg(\frac{[W_2]_{i_2,\mathcal{R}(p^m,q^m)}[n+i_1+W_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-W_2]_{i_1,\mathcal{R}(p^m,q^m)}}\bigg)$$

$$
\begin{aligned}
&= \sum_{w_2=0}^{\infty} \frac{[w_2]_{i_2,\mathcal{R}(p^m,q^m)}[n+i_1+w_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^m,q^m)}} \\
&\quad \times F_2(p,q)\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{w_2,\mathcal{R}(p^{-m},q^{-m})} \\
&\quad \times \frac{[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{[\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})}},
\end{aligned}
$$

and using the relations

$$
\frac{\begin{bmatrix} n+w_2-1 \\ w_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}}{\begin{bmatrix} n+i_1+w_2-1 \\ w_2-i_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})}} = \frac{[n+i_1+i_2-1]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}}{[w_2]_{i_2,\mathcal{R}(p^m,q^m)}[n+i_1+w_2-1]_{i_1,\mathcal{R}(p^{-m},q^{-m})}},
$$

$$
\begin{aligned}
[w_2]_{i_2,\mathcal{R}(p^m,q^m)} &= \big(\tau_1\tau_2\big)^{mi_2\,w_2-m\binom{i_2+1}{2}}\,[w_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}, \\
[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)} &= \big(\tau_1\tau_2\big)^{mi_2\,\beta_2-m\binom{i_2+1}{2}}\,[\beta_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}, \\
[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^m,q^m)} &= \frac{[\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{mi_1(\beta-\beta_1-n-\beta_2)-m\binom{i_2+1}{2}}},
\end{aligned}
$$

$$
\begin{aligned}
[\beta-\beta_1]_{n+i_1+w_2,\mathcal{R}(p^{-m},q^{-m})} =& [\beta-\beta_1]_{n+w_2,\mathcal{R}(p^{-m},q^{-m})} \\
& \times [\beta-\beta_1-n-w_2]_{i_1,\mathcal{R}(p^{-m},q^{-m})},
\end{aligned}
$$

with the negative $\mathcal{R}(p,q)$-Vandermonde formula [5, 6], we obtain:

$$
\begin{aligned}
E\big(\overline{W}\big) =& \frac{[n+i_1+i_2-1]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{-mi_1(\beta-\beta_1-n-\beta_2)+m\binom{i_2+1}{2}}} \\
& \times \sum_{w_2=0}^{\infty} F_2(p,q)\begin{bmatrix} n+i_1+w_2-1 \\ w_2-i_2 \end{bmatrix}_{\mathcal{R}(p^{-m},q^{-m})} \\
& \times \frac{[\beta_2-i_2]_{w_2-i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1]_{n+i_1+w_2,\mathcal{R}(p^{-m},q^{-m})}} \\
=& \frac{[n+i_1+i_2-1]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{-mi_1(\beta-\beta_1-n-\beta_2)+m\binom{i_2+1}{2}}} \\
& \times \frac{[\beta_2]_{i_2,\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-\beta_2+i_2]_{n+i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}}.
\end{aligned}
$$

Furthermore, by using

$$
[\beta-\beta_1-\beta_2+i_2]_{n+i_1+i_2,\mathcal{R}(p^{-m},q^{-m})} = [\beta-\beta_1-\beta_2+i_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}
$$

$$\times [\beta-\beta_1-\beta_2]_{n+i_1,\mathcal{R}(p^{-m},q^{-m})},$$

$$\begin{aligned}[\beta-\beta_1-\beta_2]_{n+i_1,\mathcal{R}(p^{-m},q^{-m})} =&[\beta-\beta_1-\beta_2]_{n,\mathcal{R}(p^{-m},q^{-m})}\\ &\times [\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{-m},q^{-m})}.\end{aligned}$$

and

$$[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{m},q^{m})} = \frac{[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{-m},q^{-m})}}{\big(\tau_1\tau_2\big)^{mi_1(\beta-\beta_1-\beta_2-n)-m\binom{i_2+1}{2}}},$$

we obtain

$$E\big(\overline{W}\big) = \frac{[n+i_1+i_2-1]_{i_1+i_2,\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{i_2,\mathcal{R}(p^{m},q^{m})}}{[\beta-\beta_1-\beta_2-n]_{i_1,\mathcal{R}(p^{m},q^{m})}[\beta-\beta_1-\beta_2+i_2]_{i_2,\mathcal{R}(p^{-m},q^{-m})}}.$$

Then, the relation (3.13) follows and the proof is achieved. □

Corollary 3.9. *The $\mathcal{R}(p,q)$-covariance of the two random variables $\ddot{W} = [W_2]_{\mathcal{R}(p^m,q^m)}$ and $\widehat{W} = \frac{[W_1]_{\mathcal{R}(p^m,q^m)}}{[\beta-\beta_1-n-W_2]_{\mathcal{R}(p^m,q^m)}}$ is given by:*

$$\begin{aligned}Cov\big(\widehat{W},\ddot{W}\big) =&\frac{[n]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{\mathcal{R}(p^{m},q^{m})}[\beta_1]_{\mathcal{R}(p^{m},q^{m})}}{\nabla(n,\beta)}\\ &\times\bigg([n+1]_{\mathcal{R}(p^{-m},q^{-m})}-[n]_{\mathcal{R}(p^{-m},q^{-m})}\bigg),\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}\nabla(n,\beta) =&[\beta-\beta_1+1]_{\mathcal{R}(p^{-m},q^{-m})}[\beta-\beta_1-\beta_2-n]_{\mathcal{R}(p^{m},q^{m})}\\ &\times[\beta-\beta_1-\beta_2+1]_{\mathcal{R}(p^{-m},q^{-m})}\end{aligned}$$

and the random vector $\big(W_1,W_2\big)$ satisfy the bivariate $\mathcal{R}(p,q)$-Pólya distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q.

Proof. The $\mathcal{R}(p,q)$-covariance of $\widehat{W}$ and $\ddot{W}$, is defined by

$$Cov\big(\widehat{W},\ddot{W}\big) = E\big(\widehat{W}\ddot{W}\big)-E\big(\widehat{W}\big)E\big(\ddot{W}\big).$$

From the relations (3.10), (3.12), and (3.13), with $i_1=i_2=1$, we have:

$$E\big(\widehat{W}\ddot{W}\big) = \frac{[n+1]_{\mathcal{R}(p^{-m},q^{-m})}[n]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_1]_{\mathcal{R}(p^{m},q^{m})}[\beta_2]_{\mathcal{R}(p^{m},q^{m})}}{\nabla(n,\beta)}$$

and

$$E\big(\widehat{W}\big)E\big(\ddot{W}\big) = \frac{[n]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_2]_{\mathcal{R}(p^{m},q^{m})}[n]_{\mathcal{R}(p^{-m},q^{-m})}[\beta_1]_{\mathcal{R}(p^{m},q^{m})}}{\nabla(n,\beta)}.$$

After computation, the relation (3.15) holds and the proof is achieved. □

Remark 3.10. Particular cases of multivariate inverse Pólya distribution are deduced as: the probability function of the multivariate inverse (p,q)-Pólya distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2,\cdots,\beta_k)$, β, p, and q, is given by:

$$\begin{aligned} P(W_1=w_1,\ldots,W_k=w_k)=&F_k(p,q)\begin{bmatrix} n+w_k-1\\ w_1,w_2,\cdots,w_k\end{bmatrix}_{p^{-m},q^{-m}}\\ &\times\frac{\prod_{j=1}^{k}[\beta_j]_{w_j,p^{-m},q^{-m}}[\beta_{k+1}]_{n,p^{-m},q^{-m}}}{[\beta]_{n+w_k,p^{-m},q^{-m}}}, \end{aligned}$$

where

$$F_k(p,q)=p^{-m\sum_{j=1}^{k}w_j(\beta_{j+1}-n+1)}q^{-m\sum_{j=1}^{k}(n+w_k-w_j)(\beta_j-w_j)}$$

for $w_j\in\mathbb{N}\cup\{0\}$, $j\in\{1,2,\cdots,k\}$, $\beta_{k+1}=\beta-\sum_{j=1}^{k}\beta_j$, and $w_j=\sum_{i=1}^{j}w_i$, for $j\in\{1,2,\cdots,k\}$.

The probability function of the bivariate inverse (p,q)-Pólya distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q, is:

$$\begin{aligned} P(W_1=w_1,W_2=w_2)=&F_2(p,q)\begin{bmatrix} n+w_2-1\\ w_1,w_2\end{bmatrix}_{p^{-m},q^{-m}}\\ &\times\frac{\prod_{j=1}^{2}[\beta_j]_{w_j,p^{-m},q^{-m}}[\beta_3]_{n,p^{-m},q^{-m}}}{[\beta]_{n+w_2,p^{-m},q^{-m}}}, \end{aligned}$$

where

$$F_2(p,q)=p^{-m\sum_{j=1}^{2}w_j(\beta_{j+1}-n+1)}q^{-m\sum_{j=1}^{2}(n+w_2-w_j)(\beta_j-w_j)}$$

for $w_j\in\mathbb{N}\cup\{0\}$, $j\in\{1,2\}$, $\beta_3=\beta-\sum_{j=1}^{2}\beta_j$, and $w_j=\sum_{i=1}^{j}w_i$, for $j\in\{1,2\}$. Moreover, for $i_1\in\mathbb{N}\cup\{0\}$ and $i_2\in\mathbb{N}\cup\{0\}$, it's factorial moments are given by:

$$E\big([W_2]_{i_2,p^m,q^m}\big)=\frac{[n+i_2-1]_{i_2,p^{-m},q^{-m}}[\beta_2]_{i_2,p^m,q^m}}{[\beta-\beta_1-\beta_2+i_2]_{i_2,p^{-m},q^{-m}}},$$

$$E\big([W_1]_{i_1,p^m,q^m}|W_2=w_2\big)=\frac{[n+w_2+i_1-1]_{i_1,p^{-m},q^{-m}}[\beta_1]_{i_1,p^m,q^m}}{[\beta-\beta_1+i_1]_{i_1,p^{-m},q^{-m}}},$$

$$\begin{aligned} E\bigg(\frac{[W_1]_{i_1,p^m,q^m}}{[\beta-\beta_1-n-W_2]_{i_1,p^m,q^m}}\bigg)=&\frac{[n+i_1-1]_{i_1,p^{-m},q^{-m}}}{[\beta-\beta_1+i_1]_{i_1,p^{-m},q^{-m}}}\\ &\times\frac{[\beta_1]_{i_1,p^m,q^m}}{[\beta-\beta_1-\beta_2-n]_{i_1,p^m,q^m}}, \end{aligned}$$

and

$$E\bigg(\frac{[W_1]_{i_1,p^m,q^m}[W_2]_{i_2,p^m,q^m}}{[\beta-\beta_1-n-W_2]_{i_1,p^m,q^m}}\bigg) = \frac{[n+i_1+i_2-1]_{i_1+i_2,p^{-m},q^{-m}}}{[\beta-\beta_1+i_1]_{i_1,p^{-m},q^{-m}}} \times \frac{[\beta_1]_{i_1,p^m,q^m}}{[\beta-\beta_1-\beta_2+i_2]_{i_2,p^{-m},q^{-m}}} \times \frac{[\beta_2]_{i_2,p^m,q^m}}{[\beta-\beta_1-\beta_2-n]_{i_1,p^m,q^m}}.$$

Furthermore, the (p,q)-covariance of the random $\ddot{W}=[W_2]_{p^m,q^m}$ and $\widehat{W}=\frac{[W_1]_{p^m,q^m}}{[\beta-\beta_1-n-W_2]_{p^m,q^m}}$ is determined by:

$$Cov\big(\widehat{W},\ddot{W}\big)=\frac{[n]_{p^{-m},q^{-m}}[\beta_2]_{p^m,q^m}[\beta_1]_{p^m,q^m}}{\nabla(n,\beta)}\bigg([n+1]_{p^{-m},q^{-m}}-[n]_{p^{-m},q^{-m}}\bigg),$$

where

$$\nabla(n,\beta)=[\beta-\beta_1+1]_{p^{-m},q^{-m}}[\beta-\beta_1-\beta_2-n]_{p^m,q^m} \times[\beta-\beta_1-\beta_2+1]_{p^{-m},q^{-m}}$$

the random vector $\big(W_1,W_2\big)$ satisfy the bivariate inverse (p,q)-Pólya distribution, with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q.

3.3. Multivariate $\mathcal{R}(p,q)$-hypergeometric distribution

In this section, the multivariate $\mathcal{R}(p,q)$-hypergeometric and the negative multivariate $\mathcal{R}(p,q)$-hypergeometric distribution are determined.

We consider a sequence of independent Bernoulli trials and suppose that the probability of success at the ith trial is given as follows:

$$p_i=\frac{\theta\tau_2^{i-1}}{\tau_1^{i-1}+\theta\tau_2^{i-1}},\quad i\in\mathbb{N},\quad 0<\theta<1.$$

We denote by H_j the number of successes after the (s_{j-1})th trial and until the (s_j)th trial, with $j\in\{1,2,\dots,k+1\}$, $s_0=0$, $s_j=\sum_{i=1}^{j}r_i$, $j\in\{1,2,\dots,k+1\}$, and $s_{k+1}=r$. Thus, the random variables H_j are independent and the probability distribution of the $\mathcal{R}(p,q)$-binomial probability distribution of the first kind is given by:

$$(3.16)\quad P(H_j=h_j)=\begin{bmatrix}r_j\\h_j\end{bmatrix}_{\mathcal{R}(p,q)}\frac{(\theta\tau_2^{s_{j-1}})^{h_j}\tau_1^{\binom{r_j-h_j}{2}}\tau_2^{\binom{h_j}{2}}}{\big(\tau_1^{s_{j-1}}\oplus\theta\tau_2^{s_{j-1}}\big)^{r_j}_{\mathcal{R}(p,q)}},\quad h_j\in\{0,1,\dots,r_j\},$$

where

$$\big(a\oplus b\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n}\big(a\tau_1^{i-1}+b\tau_2^{i-1}\big).$$

Corollary 3.11. *The conditional probability function of the random vector* $\underline{H}=(H_1,H_2,\ldots,H_k)$*, given that* $H_1+H_2+\ldots+H_{k+1}=n$*, is the multivariate* $\mathcal{R}(p,q)$*-hypergeometric distribution. Its probability distribution is given by:*

$$P(\underline{H}=\underline{h}) = \mathcal{H}_k(p,q)\begin{bmatrix} n \\ h_1,\cdots,h_k\end{bmatrix}_{\mathcal{R}(p,q)}\frac{\prod_{j=1}^{k+1}\begin{bmatrix}\beta_j\\ h_j\end{bmatrix}_{\mathcal{R}(p,q)}}{\begin{bmatrix}\beta\\ n\end{bmatrix}_{\mathcal{R}(p,q)}}, \tag{3.17}$$

where

$$\mathcal{H}_k(p,q) = \tau_1^{\sum_{j=1}^k h_j(\beta_{j+1}-h_{j+1})}\tau_2^{\sum_{j=1}^k (n-y_j)(\beta_j-h_j)}$$

for $h_j\in\{0,1,\cdots,n\}$*,* $j\in\{1,2,\cdots,k\}$ *with*

$$\sum_{j=1}^k h_j\le n,\;\; h_{k+1}=n-\sum_{j=1}^k h_j,\;\; \beta_{k+1}=\beta-\sum_{j=1}^k\beta_j,\;\; \text{and } y_j=\sum_{i=1}^j h_i$$

for $j\in\{1,2,\cdots,k\}$.

Remark 3.12. The multivariate $\mathcal{R}(p,q)$-hypergeometric can also be obtained by taking $m=-1$ in the relation (3.2).

3.3.1. Bivariate $\mathcal{R}(p,q)$-hypergeometric distribution

The probability distribution of the random vector $\underline{H}=\big(H_1,H_2\big)$ is called the bivariate $\mathcal{R}(p,q)$-hypergeometric distribution with parameters n, $\underline{\beta}=(\beta_1,\beta_2)$, p and q. The probability function is given by the following relation:

$$P(H_1=h_1,H_2=h_2) = \Psi_2(p,q)\begin{bmatrix} n \\ h_1,h_2\end{bmatrix}_{\mathcal{R}(p,q)}\frac{\prod_{j=1}^{3}[\beta_j]_{h_j,\mathcal{R}(p,q)}}{[\beta]_{n,\mathcal{R}(p,q)}}, \tag{3.18}$$

where

$$\Psi_2(p,q) = \tau_1^{\sum_{j=1}^2 h_j(\beta_{j+1}-h_{j+1})}\tau_2^{\sum_{j=1}^2 (n-x_j)(\beta_j-h_j)},$$

$h_j\in\{0,1,\ldots,n\}$, $j\in\{1,2\}$, $h_1+h_2\le n$, $h_3=n-h_1+h_2$, $\beta_3=\beta-\beta_1+\beta_2$, $x_1=h_1$, and $x_2=h_1+h_2$.

Proposition 3.13. *The* $\mathcal{R}(p,q)$*-factorial moments of the bivariate* $\mathcal{R}(p,q)$*-hypergeometric distribution, with parameters* n*,* $\underline{\beta}=(\beta_1,\beta_2),\beta$*,* p *and* q*, are presented as follows:*

$$E\big([H_1]_{i_1,\mathcal{R}(p^{-1},q^{-1})}\big) = \frac{[n]_{i_1,\mathcal{R}(p^{-1},q^{-1})}[\beta_1]_{i_1,\mathcal{R}(p,q)}}{[\beta]_{i_1,\mathcal{R}(p,q)}},\; i_1\in\{0,1,\cdots,n\},$$

$$E\big([H_2]_{i_1,\mathcal{R}(p^{-1},q^{-1})}|H_1=h_1\big)=\frac{[n-h_1]_{i_2,\mathcal{R}(p^{-1},q^{-1})}[\beta_2]_{i_2,\mathcal{R}(p,q)}}{[\beta-\beta_1]_{i_2,\mathcal{R}(p,q)}},$$

with $i_2\in\{0,1,\cdots,n-h_1\}$,

$$E\big([H_2]_{i_2,\mathcal{R}(p^{-1},q^{-1})}\big)=\frac{\tau_2^{i_2\beta_1}\,[n]_{i_2,\mathcal{R}(p^{-1},q^{-1})}[\beta_2]_{i_2,\mathcal{R}(p,q)}}{[\beta]_{i_2,\mathcal{R}(p,q)}},$$

where $i_2\in\{0,1,\cdots,n\}$, *and*

$$E\big([H_1]_{i_1,\mathcal{R}(p^{-1},q^{-1})}[H_2]_{i_2,\mathcal{R}(p^{-1},q^{-1})}\big)=\frac{[n]_{i_1+i_2,\mathcal{R}(p^{-1},q^{-1})}}{\tau_2^{-i_2\beta_1}}\times\frac{[\beta_1]_{i_1,\mathcal{R}(p,q)}[\beta_2]_{i_2,\mathcal{R}(p,q)}}{[\beta]_{i_1+i_2,\mathcal{R}(p,q)}},$$

where $i_1\in\{0,1,\cdots,n-i_2\}$ *and* $i_2\in\{0,1,\cdots,n\}$.

Corollary 3.14. *The* $\mathcal{R}(p,q)$*-covariance of* $[H_1]_{\mathcal{R}(p^{-1},q^{-1})}$ *and* $[H_2]_{\mathcal{R}(p^{-1},q^{-1})}$ *is derived by:*

$$Cov\big([H_1]_{\mathcal{R}(p^{-1},q^{-1})},[H_2]_{\mathcal{R}(p^{-1},q^{-1})}\big)=\frac{[n]_{\mathcal{R}(p^{-1},q^{-1})}[\beta_1]_{\mathcal{R}(p,q)}[\beta_2]_{\mathcal{R}(p,q)}}{\tau_2^{-\beta_1}}\frac{\Delta(n,\beta)}{[\beta]_{\mathcal{R}(p,q)}},$$

where

$$\Delta(n,\beta)=\frac{[n-1]_{\mathcal{R}(p^{-1},q^{-1})}}{[\beta-1]_{\mathcal{R}(p,q)}}-\frac{[n]_{\mathcal{R}(p^{-1},q^{-1})}}{[\beta_1]_{\mathcal{R}(p,q)}}$$

and $\underline{H}=\big(H_1,H_2\big)$ *a random vector satisfying the bivariate* $\mathcal{R}(p,q)$*-hypergeometric probability distribution, with parameters* n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p *and* q.

Remark 3.15. Multivariate hypergeometric distributions are deduced as: the probability distribution of the multivariate (p,q)-hypergeometric distribution is given by:

$$P(H_1=h_1,\ldots,H_k=h_k)=\mathcal{H}_k(p,q)\begin{bmatrix}n\\ h_1,\cdots,h_k\end{bmatrix}_{p,q}\frac{\prod_{j=1}^{k+1}\begin{bmatrix}\alpha_j\\ h_j\end{bmatrix}_{p,q}}{\begin{bmatrix}\alpha\\ n\end{bmatrix}_{p,q}},$$

where

$$\mathcal{H}_k(p,q)=p^{\sum_{j=1}^k h_j(\alpha_{j+1}-h_{j+1})}\,q^{\sum_{j=1}^k(n-y_j)(\alpha_j-h_j)}$$

for $h_j\in\{0,1,\cdots,n\}$, $j\in\{1,2,\ldots,k\}$, with $\sum_{j=1}^k h_j\leq n$, $h_{k+1}=n-\sum_{j=1}^k h_j$, $\alpha_{k+1}=\alpha-\sum_{j=1}^k\alpha_j$, and $y_j=\sum_{i=1}^j h_i$, for $j\in\{1,2,\cdots,k\}$.

The probability function of the bivariate (p,q)-hypergeometric distribution with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q. is given by the following relation:

$$P(H_1=h_1,H_2=h_2)=\Psi_2(p,q)\begin{bmatrix}n\\ h_1,h_2\end{bmatrix}_{p,q}\frac{\prod_{j=1}^3[\beta_j]_{h_j,p,q}}{[\beta]_{n,p,q}},$$

where $\Psi_2(p,q) = p^{\sum_{j=1}^{2} h_j(\beta_{j+1}-h_{j+1})}\, q^{\sum_{j=1}^{2}(n-x_j)(\beta_j-h_j)}$, $h_j \in \{0,1,\ldots,n\}$, $j \in \{1,2\}$, $h_1+h_2 \leq n$, $h_3 = n-h_1+h_2$, $\beta_3 = \beta-\beta_1+\beta_2$, $x_1 = h_1$, and $x_2 = h_1+h_2$.

Besides, its (p,q)-factorial moments are presented as follows:

$$E\big([H_1]_{i_1,p^{-1},q^{-1}}\big) = \frac{[n]_{i_1,p^{-1},q^{-1}}[\beta_1]_{i_1,p,q}}{[\beta]_{i_1,p,q}},\ i_1 \in \{0,1,\cdots,n\},$$

$$E\big([H_2]_{i_1,p^{-1},q^{-1}}|H_1 = h_1\big) = \frac{[n-h_1]_{i_2,p^{-1},q^{-1}}[\beta_2]_{i_2,p,q}}{[\beta-\beta_1]_{i_2,p,q}},$$

where $i_2 \in \{0,1,\cdots,n-h_1\}$,

$$E\big([H_2]_{i_2,p^{-1},q^{-1}}\big) = \frac{[n]_{i_2,p^{-1},q^{-1}}[\beta_2]_{i_2,p,q}q^{i_2\beta_1}}{[\beta]_{i_2,p,q}},\ i_2 \in \{0,1,\cdots,n\},$$

and

$$E\big([H_1]_{i_1,p^{-1},q^{-1}}[H_2]_{i_2,p^{-1},q^{-1}}\big) = \frac{[n]_{i_1+i_2,p^{-1},q^{-1}}[\beta_1]_{i_1,p,q}[\beta_2]_{i_2,p,q}}{q^{-i_2\beta_1}\,[\beta]_{i_1+i_2,p,q}},$$

where $i_1 \in \{0,1,\cdots,n-i_2\}$ and $i_2 \in \{0,1,\cdots,n\}$. Furthermore, the (p,q)-covariance of $[H_1]_{p^{-1},q^{-1}}$ and $[H_2]_{p^{-1},q^{-1}}$ is derived by:

$$Cov\big([H_1]_{p^{-1},q^{-1}},[H_2]_{p^{-1},q^{-1}}\big) = \frac{[n]_{p^{-1},q^{-1}}[\beta_1]_{p,q}[\beta_2]_{p,q}}{q^{-\beta_1}[\beta]_{p,q}}\Delta(n,\beta),$$

where

$$\Delta(n,\beta) = \frac{[n-1]_{p^{-1},q^{-1}}}{[\beta-1]_{p,q}} - \frac{[n]_{p^{-1},q^{-1}}}{[\beta_1]_{p,q}}.$$

3.4. Multivariate negative $\mathcal{R}(p,q)$-hypergeometric distribution

We consider a sequence of independent Bernoulli trials and suppose that the conditional probability of success at a trial, given that $j-1$ successes occur in the previous trials, is determined by:

$$p_j = 1-\theta\tau_1^{1-j}\tau_2^{j-1},\quad j \in \mathbb{N},\quad 0<\theta<1.$$

We denotes by V_j the number of failures after the (s_{j-1})th success and until the occurrence of the (s_j)th success, for $j \in \{1,2,\cdots,k+1\}$, with $s_0 = 0$, $s_j = \sum_{i=1}^{j} r_i$, $j \in \{1,2,\cdots,k+1\}$. Thus, the random variables V_j are independent and the $\mathcal{R}(p,q)$-binomial probability distribution of the second kind is presented by:

$$(3.19)\qquad P(V_j = v_j) = \begin{bmatrix} r_j+v_j-1 \\ v_j \end{bmatrix}_{\mathcal{R}(p,q)} (\theta\tau_2^{s_{j-1}})^{v_j}\big(\tau_1^{s_{j-1}} \ominus \theta\tau_2^{s_{j-1}}\big)^{r_j}_{\mathcal{R}(p,q)},$$

with $v_j \in \mathbb{N} \cup \{0\}$, and

$$\big(a \ominus b\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n} \big(a\tau_1^{i-1} - b\tau_2^{i-1}\big).$$

Corollary 3.16. *The conditional probability function of the random vector $\underline{V} = (V_1, V_2, \cdots, V_k)$, given that $V_1 + V_2 + \cdots + V_{k+1} = n$, is the multivariate negative $\mathcal{R}(p,q)$-hypergeometric distribution with probability function:*

$$P(\underline{V} = \underline{v}) = \big(\tau_1\tau_2\big)^{\sum_{j=1}^{k} r_j(n-y_j)} \frac{\prod_{j=1}^{k+1} \begin{bmatrix} r_j+v_j-1 \\ v_j \end{bmatrix}_{\mathcal{R}(p,q)}}{\begin{bmatrix} r+n-1 \\ n \end{bmatrix}_{\mathcal{R}(p,q)}}. \tag{3.20}$$

Equivalently,

$$P(\underline{V} = \underline{v}) = \begin{bmatrix} n \\ v_1, v_2, \cdots, v_k \end{bmatrix}_{\mathcal{R}(p,q)} (\tau_1\tau_2)^{\sum_{j=1}^{k} r_j(n-y_j)} \times \frac{\prod_{j=1}^{k+1} [r_j + v_j - 1]_{v_j,\mathcal{R}(p,q)}}{[r+n-1]_{n,\mathcal{R}(p,q)}}, \tag{3.21}$$

for $x_j \in \{0, 1, \cdots, n\}$, $j \in \{1, 2, \cdots, k\}$, with $\sum_{j=1}^{k} x_j \leq n$, and, where $x_{k+1} = n - \sum_{j=1}^{k} x_j$, $r_{k+1} = r - \sum_{j=1}^{k} r_j$, and $y_j = \sum_{i=1}^{j} x_i$, $j \in \{1, 2, \cdots, k\}$.

3.4.1. Bivariate negative $\mathcal{R}(p,q)$-hypergeometric distribution

Let $\underline{V} = \big(V_1, V_2\big)$ be the random vector. Then, the probability function of the bivariate negative $\mathcal{R}(p,q)$-hypergeometric probability distribution with parameters n, $\underline{\beta} = (\beta_1, \beta_2), \beta$, p and q, is given by the following relation:

$$P(V_1 = v_1, V_2 = v_2) = \Psi_2(p,q) \begin{bmatrix} n \\ v_1, v_2 \end{bmatrix}_{\mathcal{R}(p^{-1},q^{-1})} \frac{\prod_{j=1}^{3} [\beta_j]_{x_j,\mathcal{R}(p^{-1},q^{-1})}}{[\beta]_{n,\mathcal{R}(p^{-1},q^{-1})}},$$

where

$$\Psi_2(p,q) = \tau_1^{-\sum_{j=1}^{2} v_j(\beta_{j+1}-v_{j+1})} \tau_2^{-\sum_{j=1}^{2}(n-x_j)(\beta_j - v_j)},$$

$v_j \in \{0, 1, \ldots, n\}$, $j \in \{1, 2\}$, $v_1 + v_2 \leq n$, $v_3 = n - v_1 + v_2$, $\beta_3 = \beta - \beta_1 + \beta_2$, $x_1 = v_1$, and $x_2 = v_1 + v_2$.

Proposition 3.17. *The $\mathcal{R}(p,q)$-factorial moments of the bivariate negative $\mathcal{R}(p,q)$-hypergeometric probability distribution, with parameters n, p and $\underline{\beta} = (\beta_1, \beta_2), \beta$, q, are derived as follows:*

$$E\big([V_1]_{i_1,\mathcal{R}(p,q)}\big) = \frac{[n]_{i_1,\mathcal{R}(p,q)}[\beta_1]_{i_1,\mathcal{R}(p^{-1},q^{-1})}}{[\beta]_{i_1,\mathcal{R}(p^{-1},q^{-1})}}, \; i_1 \in \{0, 1, \cdots, n\},$$

$$E\big([V_2]_{i_1,\mathcal{R}(p,q)}|V_1=v_1\big)=\frac{[n-v_1]_{i_2,\mathcal{R}(p,q)}[\beta_2]_{i_2,\mathcal{R}(p^{-1},q^{-1})}}{[\beta-\beta_1]_{i_2,\mathcal{R}(p^{-1},q^{-1})}},$$

with $i_2\in\{0,1,\cdots,n-v_1\}$,

$$E\big([V_2]_{i_2,\mathcal{R}(p,q)}\big)=\frac{[n]_{i_2,\mathcal{R}(p,q)}[\beta_2]_{i_2,\mathcal{R}(p^{-1},q^{-1})}\tau_2^{-i_2\beta_1}}{[\beta]_{i_2,\mathcal{R}(p^{-1},q^{-1})}},$$

with $i_2\in\{0,1,\cdots,n\}$, *and*

$$E\big([V_1]_{i_1,\mathcal{R}(p,q)}[V_2]_{i_2,\mathcal{R}(p,q)}\big)=\frac{[n]_{i_1+i_2,\mathcal{R}(p,q)}[\beta_1]_{i_1,\mathcal{R}(p^{-1},q^{-1})}[\beta_2]_{i_2,\mathcal{R}(p^{-1},q^{-1})}}{\tau_2^{i_2\beta_1}\,[\beta]_{i_1+i_2,\mathcal{R}(p^{-1},q^{-1})}},$$

where $i_1\in\{0,1,\cdots,n-i_2\}$ *and* $i_2\in\{0,1,\cdots,n\}$.

Corollary 3.18. *The* $\mathcal{R}(p,q)$*-covariance of* $[V_1]_{\mathcal{R}(p,q)}$ *and* $[V_2]_{\mathcal{R}(p,q)}$ *is determined by:*

$$Cov\big([V_1]_{\mathcal{R}(p,q)},[V_2]_{\mathcal{R}(p,q)}\big)=\frac{[n]_{\mathcal{R}(p,q)}[\beta_1]_{\mathcal{R}(p^{-1},q^{-1})}[\beta_2]_{\mathcal{R}(p^{-1},q^{-1})}}{\tau_2^{\beta_1}[\beta]_{\mathcal{R}(p^{-1},q^{-1})}}\Delta(n,\beta),$$

where

$$\Delta(n,\beta)=\frac{[n-1]_{\mathcal{R}(p,q)}}{[\beta-1]_{\mathcal{R}(p^{-1},q^{-1})}}-\frac{[n]_{\mathcal{R}(p,q)}}{[\beta_1]_{\mathcal{R}(p^{-1},q^{-1})}},$$

and $\underline{V}=\big(V_1,V_2\big)$ *is a random vector obeying the bivariate negative* $\mathcal{R}(p,q)$*-hypergeometric probability distribution, with parameters* n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p *and* q.

Remark 3.19. The bivariate negative hypergeometric distribution related to the quantum algebras is also interesting for the lecture. Then, the mass function of the bivariate negative (p,q)-hypergeometric probability distribution with parameters n, $\underline{\beta}=(\beta_1,\beta_2),\beta$, p and q, is given by the following relation:

$$P(V_1=v_1,V_2=v_2)=\Psi_2(p,q)\begin{bmatrix}n\\v_1,v_2\end{bmatrix}_{p^{-1},q^{-1}}\frac{\prod_{j=1}^{3}[\beta_j]_{y_j,p^{-1},q^{-1}}}{[\beta]_{n,p^{-1},q^{-1}}},$$

where $\Psi_2(p,q)=p^{-\sum_{j=1}^{2}v_j(\beta_{j+1}-v_{j+1})}q^{-\sum_{j=1}^{2}(n-x_j)(\beta_j-v_j)}$, $v_j\in\{0,1,\ldots,n\}$, $j\in\{1,2\}$, $v_1+v_2\le n$, $v_3=n-v_1+v_2$, $\beta_3=\beta-\beta_1+\beta_2$, $x_1=v_1$, and $x_2=v_1+v_2$. Furthermore, its (p,q)-factorial moments are derived as follows:

$$E\big([V_1]_{i_1,p,q}\big)=\frac{[n]_{i_1,p,q}[\beta_1]_{i_1,p^{-1},q^{-1}}}{[\beta]_{i_1,p^{-1},q^{-1}}},\ i_1\in\{0,1,\cdots,n\},$$

$$E\big([V_2]_{i_1,p,q}|V_1=v_1\big)=\frac{[n-v_1]_{i_2,p,q}[\beta_2]_{i_2,p^{-1},q^{-1}}}{[\beta-\beta_1]_{i_2,p^{-1},q^{-1}}},$$

with $i_2 \in \{0, 1, \cdots, n - v_1\}$,

$$E\big([V_2]_{i_2,p,q}\big) = \frac{[n]_{i_2,p,q}[\beta_2]_{i_2,p^{-1},q^{-1}}q^{-i_2\beta_1}}{[\beta]_{i_2,p^{-1},q^{-1}}},\ i_2 \in \{0, 1, \cdots, n\},$$

and

$$E\big([V_1]_{i_1,p,q}[V_2]_{i_2,p,q}\big) = \frac{[n]_{i_1+i_2,p,q}[\beta_1]_{i_1,p^{-1},q^{-1}}[\beta_2]_{i_2,p^{-1},q^{-1}}}{q^{i_2\beta_1}\,[\beta]_{i_1+i_2,p^{-1},q^{-1}}},$$

where $i_1 \in \{0, 1, \cdots, n - i_2\}$ and $i_2 \in \{0, 1, \cdots, n\}$. Moreover, the (p, q)-covariance of $[V_1]_{p,q}$ and $[V_2]_{p,q}$ is determined by:

$$Cov\big([V_1]_{p,q}, [V_2]_{p,q}\big) = \frac{[n]_{p,q}[\beta_1]_{p^{-1},q^{-1}}[\beta_2]_{p^{-1},q^{-1}}}{q^{\beta_1}[\beta]_{p^{-1},q^{-1}}}\Delta(n, \beta),$$

where

$$\Delta(n, \beta) = \frac{[n-1]_{p,q}}{[\beta - 1]_{p^{-1},q^{-1}}} - \frac{[n]_{p,q}}{[\beta_1]_{p^{-1},q^{-1}}},$$

and $\underline{V} = \big(V_1, V_2\big)$ is a random vector obeying the bivariate negative (p, q)-hypergeometric probability distribution, with parameters n, $\underline{\beta} = (\beta_1, \beta_2), \beta$, p and q.

§4. Concluding remarks

The multivariate Pólya, inverse Pólya, hypergeometric, and negative hypergeometric probability distributions in the framework of generalized quantum deformed algebras have been constructed and discussed. Moreover, the $\mathcal{R}(p, q)$-factorial moments and covariance the corresponding bivariate probability distributions have been investigated. The (p, q)-distributions and properties have also been deduced from the formalism as particular case.

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