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On induced modules of inertial-invariant support τ-tilting modules over blocks of finite groups

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Abstract. In this article, we prove that induced modules of support τ-tilting modules over blocks of finite groups satisfying inertial-invariant condition are also support τ-tilting modules.

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§1. Introduction and notation

Support τ-tilting modules introduced in [2] form an important class of modules. These correspond bijectively to various representation theorical objects such as two-term silting complexes, functorially finite torsion classes, left finite semibricks, two-term simple-minded collections and so on (see [2, 4, 5, 7]). Let k be an algebraically closed field of characteristic $p > 0$, $\tilde{G}$ a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B, that is, the block of $k\tilde{G}$ satisfying that $1_B 1_{\tilde{B}} \neq 0$, where 1_B and $1_{\tilde{B}}$ mean the respective identity elements of B and $\tilde{B}$. We denote the inertial group of the block B in $\tilde{G}$ by $I_{\tilde{G}}(B)$ and the second group cohomology of the factor group $I_{\tilde{G}}(B)/G$ with coefficients in the unit group $k^\times$ of the field k with trivial action by $H^2(I_{\tilde{G}}(B)/G, k^\times)$. In [8], a construction of support τ-tilting modules over $\tilde{B}$ from the ones over B using the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ were presented under the following conditions:

(1) Any left finite brick U in the category of B-modules is $I_{\tilde{G}}(B)$-invariant, that is, $xU \cong U$ as B-modules for any $x \in I_{\tilde{G}}(B)$.

(2) $H^2(I_{\tilde{G}}(B)/G, k^\times) = 1$.

(3) The group algebra $k[I_{\tilde{G}}(B)/G]$ is basic as a k-algebra.

This paper presents the following results, which relaxes the assumptions above. First, we state a construction of support τ-tilting modules over $k\tilde{G}$ from the ones over kG.

Main Theorem 1.1 (see Theorems 3.2 and 3.3)**.** *Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a support τ-tilting B-modules satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then the induced module $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module. In particular, the module $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $\tilde{B}$-module.*

We will demonstrate that there is a relation between $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules and support τ-tilting $k\tilde{G}$-modules. We denote by $\mathrm{s}\tau\text{-tilt}\, B$ the set of additive equivalence classes of support τ-tilting B-modules. Now we recall that the set $\mathrm{s}\tau\text{-tilt}\, B$ has a partially ordered set structure (see Definition-Proposition 2.2).

Main Theorem 1.2 (see Theorem 3.6)**.** *Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a B-modules satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then M is a support τ-tilting B-module if and only if $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module. Moreover, for any two $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules M and M', $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\, B$ if and only if $\mathrm{Ind}_G^{\tilde{G}}M \geq \mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\, k\tilde{G}$.*

Throughout this paper, we use the following notation and terminologies. The symbol k means an algebraically closed field of characteristic $p > 0$. Let Λ be a finite dimensional algebra over a field k. Modules mean finitely generated left modules. We denote by Λ-mod the module category of Λ. For a Λ-module U, we denote by $P(U)$ the projective cover of U, by $\Omega(U)$ the syzygy of U, by τU the Auslander–Reiten translate of U and by $\mathrm{add}\, U$ the full subcategory of Λ-mod whose objects are isomorphic to direct summands of finite direct sums of U.

This paper is organized as follows. In Section 2, we introduce basic terminologies and some known results for τ-tilting theory and modular representation theory of finite groups. In Section 3, we give some lemmas and the main result, and present applications and examples.

§2. Preliminaries

In this section, Λ is assumed to be a finite dimensional k-algebra.

2.1. Support τ-tilting modules

We recall the definitions and basic properties of support τ-tilting modules. For a Λ-module M, we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of M. In particular, $|\Lambda| := |{}_\Lambda\Lambda|$ means the number of isomorphism classes of simple Λ-modules.

Definition 2.1 ([2, Definition 0.1]). Let Λ be a finite dimensional k-algebra and M a Λ-module.

(1) We say that M is τ-rigid if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$.

(2) We say that M is τ-tilting if M is a τ-rigid module and $|M| = |\Lambda|$.

(3) We say that M is support τ-tilting if there exists an idempotent e of Λ such that M is a τ-tilting $\Lambda/\Lambda e\Lambda$-module.

For support τ-tilting Λ-modules M and M', we say that M and M' are additively equivalent if $\operatorname{add} M = \operatorname{add} M'$. Denote by s$\tau$-tilt Λ the set of additive equivalence classes of support τ-tilting Λ-modules.

Definition-Proposition 2.2 ([2, Theorem 2.7]). For $M, M' \in$ sτ-tilt Λ, we write $M \geq M'$ if there exist a positive integer r and an epimorphism

$$M^{\oplus r} \xrightarrow{\varphi} M'.$$

Then we get a partial order on sτ-tilt Λ.

We denote by $\mathcal{H}$(sτ-tilt Λ) the Hasse diagram for the partially ordered set sτ-tilt Λ.

Remark 2.3 ([1, Proposition 2.3 (a), (b)]). Since $e = 0$ is an idempotent of Λ and $\Lambda/\Lambda e\Lambda = \Lambda$, any τ-tilting module is a support τ-tilting module. Moreover, for any τ-rigid Λ-module M, the following conditions are equivalent:

(1) M is a support τ-tilting module.

(2) There exists a projective Λ-module P satisfying $\operatorname{Hom}_\Lambda(P, M) = 0$ and $|M| + |P| = |\Lambda|$.

Proposition 2.4 ([2, Corollary 2.13]). *Let M be a τ-rigid Λ-module and P a projective Λ-module satisfying* $\operatorname{Hom}_\Lambda(P, M) = 0$. *Then the following conditions are equivalent:*

(1) $|M| + |P| = |\Lambda|$, that is, M is a support τ-tilting Λ-module (see Remark 2.3).

(2) If $\mathrm{Hom}_\Lambda(M,\tau X)=0$, $\mathrm{Hom}_\Lambda(X,\tau M)=0$ *and* $\mathrm{Hom}_\Lambda(P,X)=0$, *then* $X\in\mathrm{add}\,M$ *for any* Λ*-module* X.

The following proposition plays an important role in the proof of our main result.

Proposition 2.5 ([6, Proposition 2.14]). *Let* Λ *be a finite dimensional* k*-algebra and* M *a* τ*-rigid* Λ*-module. Then* M *is a support* τ*-tilting* Λ*-module if and only if there exists an exact sequence*

$$\Lambda \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0$$

in Λ-mod *with* $M',M''\in\mathrm{add}\,M$ *and* f *a left* $\mathrm{add}\,M$*-approximation of* Λ*, that is, the map*

$$\mathrm{Hom}_\Lambda(M',X)\xrightarrow{\bullet\circ f}\mathrm{Hom}_\Lambda(\Lambda,X)$$

is surjective for any $X\in\mathrm{add}\,M$.

2.2. Modules over blocks of finite group

Let G be a finite group and H a subgroup of G. We denote by Res^G_H the restriction functor from kG-mod to kH-mod and $\mathrm{Ind}^G_H := {}_{kG}kG\otimes_{kH}\bullet$ the induction functor from kH-mod to kG-mod. The field k can always be regarded as a kG-module by defining $gx=x$ for any $g\in G$ and $x\in k$. This module is called the trivial module and is denoted by k_G.

Proposition 2.6 (see [3, Lemma 8.5, Lemma 8.6]). *Let* G *be a finite group,* K *a subgroup of* G*,* H *a subgroup of* K*. Then the following hold:*

(1) $\mathrm{Res}^G_H\cong\mathrm{Res}^K_H\mathrm{Res}^G_K$.

(2) $\mathrm{Ind}^G_H\cong\mathrm{Ind}^G_K\mathrm{Ind}^K_H$.

(3) The functors Res^G_H *and* Ind^G_H *are left and right adjoint to each other.*

(4) The functors Res^G_H *and* Ind^G_H *send projective modules to projective modules.*

Let G be a normal subgroup of a finite group $\tilde{G}$ and U a kG-module. For $\tilde{g}\in\tilde{G}$, we define a kG-module $\tilde{g}U$ consisting of symbols $\tilde{g}u$ as a set, where $u\in U$ and its kG-module structure is given by $\tilde{g}u+\tilde{g}u' := \tilde{g}(u+u')$, $\lambda(\tilde{g}u) := \tilde{g}(\lambda u)$ and $g(\tilde{g}u) := \tilde{g}(\tilde{g}^{-1}g\tilde{g}u)$ for any $u,u'\in U$, $\lambda\in k$ and $g\in G$. Let U be a kG-module. If U is projective or indecomposable, then xU is also projective or indecomposable, respectively.

Theorem 2.7 (Mackey's decomposition formula for normal subgroups)**.** *Let G be a normal subgroup of a finite group $\tilde{G}$ and U a kG-module. Then we have an isomorphism*

$$\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}U \cong \bigoplus_{x\in[\tilde{G}/G]} xU,$$

of kG-modules, where $[\tilde{G}/G]$ is a set of representatives of the factor group $\tilde{G}/G$.

We recall the definition of blocks of group algebras. Let G be a finite group. The group algebra kG has a unique decomposition

$$kG = B_0 \times \cdots \times B_l$$

into the direct product of indecomposable k-algebras B_i. We call each indecomposable direct product component B_i a block of kG and the decomposition above the block decomposition. We remark that any block B_i is a two-sided ideal of kG.

For any indecomposable kG-module U, there exists a unique block B_i of kG such that $U = B_iU$ and $B_jU = 0$ for all $j \neq i$. Then we say that U lies in the block B_i or simply U is a B_i-module. We denote by $B_0(kG)$ the principal block of kG, in which the trivial kG-module k_G lies.

Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$. We say that $\tilde{B}$ covers B (or that B is covered by $\tilde{B}$) if $1_B 1_{\tilde{B}} \neq 0$.

Remark 2.8 (see [3, Theorem 15.1, Lemma 15.3])**.** With the notation above, the following are equivalent:

(1) The block $\tilde{B}$ covers B.

(2) There exists a non-zero $\tilde{B}$-module U such that $\mathrm{Res}_G^{\tilde{G}}U$ has a non-zero direct summand lying in B.

(3) For any non-zero $\tilde{B}$-module U, there exists a non-zero direct summand of $\mathrm{Res}_G^{\tilde{G}}U$ lying in B.

We denote by $I_{\tilde{G}}(B)$ the inertial group of B in $\tilde{G}$, that is $I_{\tilde{G}}(B) := \left\{x \in \tilde{G} \;\middle|\; xBx^{-1} = B\right\}$.

Remark 2.9. The principal block $B_0(kG)$ of kG is covered by the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ and $I_{\tilde{G}}(B_0(kG)) = \tilde{G}$.

Remark 2.10. Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and M a B-module. Then xM is a B-module for $x \in \tilde{G}$ if and only if $x \in I_{\tilde{G}}(B)$.

Proposition 2.11 (see [9, Theorem 5.5.10, Theorem 5.5.12])**.** *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and β a block of $kI_{\tilde{G}}(B)$ covering B. Then the following hold:*

(1) For any B-module V, the induced module $\mathrm{Ind}_G^{I_{\tilde{G}}(B)} V$ is a direct sum of $kI_{\tilde{G}}(B)$-modules lying in blocks covering B.

(2) There exists a unique block $\tilde{B}$ of $k\tilde{G}$ covering B such that the induction functor

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon kI_{\tilde{G}}(B)\text{-mod} \to k\tilde{G}\text{-mod}$$

restricts to a Morita equivalence

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \colon \beta\text{-mod} \longrightarrow \tilde{B}\text{-mod}$$

and the mapping β to $\tilde{B}$ is a bijection between the set of blocks of $kI_{\tilde{G}(B)}$ covering B and the one of $k\tilde{G}$ covering B.

§3. The main results and their applications

In this section, we show some lemmas and our main theorem. After that, we give some applications and examples of the main results.

3.1. Main theorems and their proof

The next lemma plays a key role.

Lemma 3.1. *Let G be a normal subgroup of a finite group $\tilde{G}$ and M a kG-module satisfying $xM \cong M$ as kG-modules for any $x \in \tilde{G}$. Then the following hold:*

(1) $xP(M) \cong P(M)$ for any $x \in \tilde{G}$.

(2) $x\Omega(M) \cong \Omega(M)$ for any $x \in \tilde{G}$.

(3) $\mathrm{Ind}_G^{\tilde{G}}\Omega(M) \cong \Omega(\mathrm{Ind}_G^{\tilde{G}}M)$.

(4) $\tau(\mathrm{Ind}_G^{\tilde{G}}M) \cong \mathrm{Ind}_G^{\tilde{G}}\tau M$.

Proof. For any $x \in \tilde{G}$, we have an isomorphism $\phi \colon xM \to M$ by the assumption. We consider the following commutative diagram in kG-mod with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & x\Omega(M) & \longrightarrow & xP(M) & \xrightarrow{x\pi_M} & xM & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi''} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & \Omega(M) & \longrightarrow & P(M) & \xrightarrow[\pi_M]{} & M & \longrightarrow & 0. \end{array}$$

Since π_M is an essential epimorphism and ϕ is an isomorphism, the vertical morphisms ϕ' and ϕ'' are isomorphisms and so (1) and (2) hold.

By Proposition 2.6 (4), we have the following commutative diagram in $k\tilde{G}$-mod with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ind}_G^{\tilde{G}}\Omega(M) & \longrightarrow & \mathrm{Ind}_G^{\tilde{G}}P(M) & \xrightarrow{\mathrm{Ind}_G^{\tilde{G}}\pi_M} & \mathrm{Ind}_G^{\tilde{G}}M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}_{\mathrm{Ind}_G^{\tilde{G}}M}} & & \\ 0 & \longrightarrow & \Omega(\mathrm{Ind}_G^{\tilde{G}}M) & \longrightarrow & P(\mathrm{Ind}_G^{\tilde{G}}M) & \xrightarrow[\pi_{\mathrm{Ind}_G^{\tilde{G}}M}]{} & \mathrm{Ind}_G^{\tilde{G}}M & \longrightarrow & 0. \end{array}$$

Since $\pi_{\mathrm{Ind}_G^{\tilde{G}}M}$ is an essential epimorphism, we have that the vertical morphisms φ and φ' are split epimorphisms, that $\operatorname{Ker}\varphi \cong \operatorname{Ker}\varphi'$ are projective $k\tilde{G}$-modules and that $\Omega(\mathrm{Ind}_G^{\tilde{G}}M) \oplus \operatorname{Ker}\varphi' \cong \mathrm{Ind}_G^{\tilde{G}}\Omega(M)$. By Theorem 2.7 and (2), we have

$$\begin{aligned} \Omega(M)^{\oplus|\tilde{G}:G|} &\cong \bigoplus_{x\in[\tilde{G}/G]} x\Omega(M) \\ &\cong \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}\Omega(M) \\ &\cong \mathrm{Res}_G^{\tilde{G}}\Omega(\mathrm{Ind}_G^{\tilde{G}}M) \oplus \mathrm{Res}_G^{\tilde{G}}\operatorname{Ker}\varphi'. \end{aligned}$$

Since $\mathrm{Res}_G^{\tilde{G}}\operatorname{Ker}\varphi'$ is projective by Proposition 2.6 (4) and $\Omega(M)$ has no non-zero projective summands by the self-injectivity of the group algebra kG, we have that $\operatorname{Ker}\varphi \cong \operatorname{Ker}\varphi' = 0$. This finishes the proof of (3).

Finally, we prove the assertion (4). Since $k\tilde{G}$ and kG are symmetric k-algebras, it holds that $\tau M \cong \Omega\Omega(M)$ and $\tau(\mathrm{Ind}_G^{\tilde{G}}M) \cong \Omega\Omega(\mathrm{Ind}_G^{\tilde{G}}M)$ for any kG-module M. Therefore, (4) immediately follows from (3). □

Theorem 3.2. *Let G be a normal subgroup of a finite group $\tilde{G}$ and M a support τ-tilting kG-module satisfying $xM \cong M$ as kG-modules for any $x \in \tilde{G}$. Then the induced module $\mathrm{Ind}_G^{\tilde{G}}M$ of M is a support τ-tilting $k\tilde{G}$-module.*

Proof. A similar proof of [10, Theorem 4.2] works in this setting.

By Lemma 3.1 (3), Proposition 2.6 (3), Theorem 2.7, the $I_{\tilde{G}}(B)$-invariance of and the τ-rigidity of M, we have the following:

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}M) \cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \mathrm{Ind}_G^{\tilde{G}}\tau M)$$

$$
\begin{aligned}
&\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M, \tau M)\\
&\cong \mathrm{Hom}_{kG}(\bigoplus_{x\in[\tilde{G}/G]} xM, \tau M)\\
&\cong \bigoplus_{x\in[\tilde{G}/G]} \mathrm{Hom}_{kG}(M, \tau M)\\
&= 0.
\end{aligned}
$$

Therefore, we have that $\mathrm{Ind}_G^{\tilde{G}}M$ is τ-rigid. By Proposition 2.5, there exists an exact sequence

$$kG \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0 \tag{3.1.1}$$

with $M', M'' \in \operatorname{add} M$ and f a left $\operatorname{add} M$-approximation of kG. Applying the functor $\mathrm{Ind}_G^{\tilde{G}}$ to the exact sequence (3.1.1), we get the exact sequence

$$k\tilde{G} \cong \mathrm{Ind}_G^{\tilde{G}}kG \xrightarrow{\mathrm{Ind}_G^{\tilde{G}}f} \mathrm{Ind}_G^{\tilde{G}}M' \xrightarrow{\mathrm{Ind}_G^{\tilde{G}}f'} \mathrm{Ind}_G^{\tilde{G}}M'' \longrightarrow 0$$

satisfying that $\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M'' \in \operatorname{add}\mathrm{Ind}_G^{\tilde{G}}M$. Then by Proposition 2.5, we only have to prove that $\mathrm{Ind}_G^{\tilde{G}}f$ is a left $\operatorname{add}\mathrm{Ind}_G^{\tilde{G}}M$-approximation of $k\tilde{G}$, that is, the map

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', X) \xrightarrow{\bullet\circ\mathrm{Ind}_G^{\tilde{G}}f} \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, X) \tag{3.1.2}$$

is surjective for any $X \in \operatorname{add}\mathrm{Ind}_G^{\tilde{G}}M$. First we prove that the map

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M) \xrightarrow{\bullet\circ\mathrm{Ind}_G^{\tilde{G}}f} \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, \mathrm{Ind}_G^{\tilde{G}}M) \tag{3.1.3}$$

is surjective. By Proposition 2.6 (3), Theorem 2.7 and the assumption, we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M) & \xrightarrow{\bullet\circ\mathrm{Ind}_G^{\tilde{G}}f} & \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, \mathrm{Ind}_G^{\tilde{G}}M)\\
\wr\downarrow & & \downarrow\wr\\
\mathrm{Hom}_{kG}(M', \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M) & \xrightarrow{\bullet\circ f} & \mathrm{Hom}_{kG}(kG, \mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M)\\
\wr\downarrow & & \downarrow\wr\\
\mathrm{Hom}_{kG}(M', \bigoplus_{x\in[\tilde{G}/G]} xM) & \xrightarrow{\bullet\circ f} & \mathrm{Hom}_{kG}(kG, \bigoplus_{x\in[\tilde{G}/G]} xM)\\
\wr\downarrow & & \downarrow\wr\\
\mathrm{Hom}_{kG}(M', M^{\oplus|\tilde{G}:G|}) & \xrightarrow{\bullet\circ f} & \mathrm{Hom}_{kG}(kG, M^{\oplus|\tilde{G}:G|}).
\end{array}
$$

The map in the last row is surjective since f is a left add M-approximation of kG, which implies that the map in the first row, which is the map (3.1.3), is surjective. Hence, we get that

$$\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m}) \xrightarrow{\bullet\circ \mathrm{Ind}_G^{\tilde{G}}f} \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m}) \tag{3.1.4}$$

is surjective for any $m \in \mathbb{N}$. Now take $X \in \mathrm{add}\,\mathrm{Ind}_G^{\tilde{G}}M$ and $h \in \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, X)$ arbitrarily. Then there exist $m \in \mathbb{N}$ and a split exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m} \xrightarrow{\beta} Y \longrightarrow 0$$

in $k\tilde{G}$-mod. Let $\gamma\colon \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m} \to X$ be a retraction of α, that is, a $k\tilde{G}$-homomorphism satisfying $\gamma \circ \alpha = \mathrm{Id}_X$. Since the map (3.1.4) is surjective and $\alpha \circ h \in \mathrm{Hom}_{k\tilde{G}}(k\tilde{G}, \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m})$, there exists $h' \in \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M', \mathrm{Ind}_G^{\tilde{G}}M^{\oplus m})$ such that $h' \circ \mathrm{Ind}_G^{\tilde{G}}f = \alpha \circ h$. Hence, we have that

$$h = \mathrm{Id}_X \circ h = \gamma \circ \alpha \circ h = \gamma \circ h' \circ \mathrm{Ind}_G^{\tilde{G}}f.$$

Therefore, the map (3.1.2) is surjective. $\square$

The following result makes the assumption in Theorem 3.2 weaker not only in case where the module M is a kG-module but also in case where M is a B-module.

Theorem 3.3. *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG, $\tilde{B}$ a block of $k\tilde{G}$ covering B and M a support τ-tilting B-module satisfying $xM \cong M$ as B-modules for any $x \in I_{\tilde{G}}(B)$. Then $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module. In particular, $\tilde{B}\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $\tilde{B}$-module.*

Proof. Let $\tilde{B}_1 = \tilde{B}, \ldots, \tilde{B}_e$ be all the blocks of $k\tilde{G}$ covering B. By Proposition 2.11 (2), we can take $\beta_1, \ldots, \beta_e$ the blocks of $kI_{\tilde{G}}(B)$ satisfying the induction functor $\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}$ restricts to a Morita equivalence

$$\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\colon \beta_i\text{-mod} \longrightarrow \tilde{B}_i\text{-mod}$$

for any $i = 1, \ldots, e$. By Theorem 3.2, the induced module $\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$ is a support τ-tilting $kI_{\tilde{G}}(B)$-module and hence $\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$ is a support τ-tilting β_i-module for any $i = 1, \ldots, e$. Therefore, we have that $\mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}}\beta_i\mathrm{Ind}_G^{I_{\tilde{G}}(B)}M$

is a support τ-tilting $\tilde{B}_i$-module. By Proposition 2.11 (1) and Proposition 2.6 (2), we have

$$\begin{aligned}\bigoplus_{i=1}^{e} \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \beta_i \mathrm{Ind}_{G}^{I_{\tilde{G}}(B)} M &\cong \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \bigoplus_{i=1}^{e} \beta_i \mathrm{Ind}_{G}^{I_{\tilde{G}}(B)} M \\ &\cong \mathrm{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \mathrm{Ind}_{G}^{I_{\tilde{G}}(B)} M \\ &\cong \mathrm{Ind}_{G}^{\tilde{G}} M.\end{aligned}$$

Hence, $\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $k\tilde{G}$-module. Therefore, we get that $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $\tilde{B}$-module. □

Corollary 3.4. *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. If $M \geq M'$ in* sτ-tilt B *for $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules M and M', then $\tilde{B}\mathrm{Ind}_G^{\tilde{G}} M \geq \tilde{B}\mathrm{Ind}_G^{\tilde{G}} M'$ in* sτ-tilt $\tilde{B}$.

Proof. The statement follows from the exactness of the induction functor $\mathrm{Ind}_G^{\tilde{G}}$ and from Theorem 3.3. □

We present a relation between the orders of $I_{\tilde{G}}(B)$-invariant support τ-tilting B-modules and support τ-tilting $k\tilde{G}$-modules.

Proposition 3.5. *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. For an $I_{\tilde{G}}(B)$-invariant B-module M, if the induced module $\mathrm{Ind}_G^{\tilde{G}} M$ is a support τ-tilting $k\tilde{G}$-module, then M is a support τ-tilting B-module.*

Proof. By Remark 2.3, we can take a projective $k\tilde{G}$-module $\tilde{P}$ satisfying that $\mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}} M) = 0$ and $|\tilde{P}| + |\mathrm{Ind}_G^{\tilde{G}} M| = |\tilde{B}|$. By Proposition 2.4 and Proposition 2.6 (4), it is enough to show the following:

(1) M is a τ-rigid B-module.

(2) $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}} \tilde{P}, M) = 0$.

(3) If $\mathrm{Hom}_B(M, \tau X) = 0$, $\mathrm{Hom}_B(X, \tau M) = 0$ and $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}} \tilde{P}, X) = 0$, then $X \in \mathrm{add}\, M$ for any B-module X.

First, we have the following:

$$\begin{aligned}&\mathrm{Hom}_B(M, \tau M)^{\oplus |I_{\tilde{G}}(B):G|} \\ &\cong \mathrm{Hom}_{kG}(\bigoplus_{x \in [I_{\tilde{G}}(B)/G]} xM, \tau M) \qquad \text{(the } I_{\tilde{G}}(B)\text{-invariance of } M)\end{aligned}$$

$$\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{x\in[I_{\tilde{G}}(B)/G]} xM \oplus \bigoplus_{\substack{x\in[\tilde{G}/G]\\ x\notin I_{\tilde{G}}(B)}} xM, \tau M\Big) \qquad \text{(Remark 2.10)}$$

$$\cong \mathrm{Hom}_{kG}\Big(\bigoplus_{x\in[\tilde{G}/G]} xM, \tau M\Big)$$

$$\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M, \tau M) \qquad \text{(Theorem 2.7)}$$

$$\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \mathrm{Ind}_G^{\tilde{G}}\tau M) \qquad \text{(Proposition 2.6 (3))}$$

$$\cong \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}M) \qquad \text{(Lemma 3.1)}$$

$$= 0. \qquad \text{(the } \tau\text{-rigidity of } \mathrm{Ind}_G^{\tilde{G}}M)$$

Hence, we have that the B-module M is a τ-rigid B-module. Also, we have that

$$\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, M) \cong \mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}M) = 0.$$

For a B-module X, we assume that $\mathrm{Hom}_B(M, \tau X) = 0$, $\mathrm{Hom}_B(X, \tau M) = 0$ and $\mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X) = 0$. By the assumptions and similar arguments as above, we have that $\mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}M, \tau\mathrm{Ind}_G^{\tilde{G}}X) = \mathrm{Hom}_{k\tilde{G}}(\mathrm{Ind}_G^{\tilde{G}}X, \tau\mathrm{Ind}_G^{\tilde{G}}M) = 0$. Also, we have that

$$\begin{aligned}\mathrm{Hom}_{k\tilde{G}}(\tilde{P}, \mathrm{Ind}_G^{\tilde{G}}X) &\cong \mathrm{Hom}_{kG}(\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X)\\ &\cong \mathrm{Hom}_B(B\mathrm{Res}_G^{\tilde{G}}\tilde{P}, X)\\ &= 0.\end{aligned}$$

Hence, we have that $\mathrm{Ind}_G^{\tilde{G}}X \in \mathrm{add}\,\mathrm{Ind}_G^{\tilde{G}}M$ by Proposition 2.4. Therefore, we have that $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}X \in \mathrm{add}\,\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M$. In particular, we have that $X \in \mathrm{add}\,M$ since $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}X \cong \bigoplus_{x\in[\tilde{G}/G]} xX$ and $\mathrm{Res}_G^{\tilde{G}}\mathrm{Ind}_G^{\tilde{G}}M \cong \bigoplus_{x\in[\tilde{G}/G]} M$, which implies that M is a support τ-tilting B-module by Proposition 2.4. □

Theorem 3.6. *Let G be a normal subgroup of a finite group $\tilde{G}$, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. For $I_{\tilde{G}}(B)$-invariant B-modules M and M', the following hold:*

(1) M is a support τ-tilting B-module if and only if $\mathrm{Ind}_G^{\tilde{G}}M$ is a support τ-tilting $k\tilde{G}$-module.

(2) Assume that M and M' are support τ-tilting B-modules. Then $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\,B$ if and only if $\mathrm{Ind}_G^{\tilde{G}}M \geq \mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$.

Proof. (1) follows from Theorem 3.3 and Proposition 3.5. To prove (2), by Corollary 3.4, we only need to show that if $\mathrm{Ind}_G^{\tilde{G}}M \geq \mathrm{Ind}_G^{\tilde{G}}M'$ in $\mathrm{s}\tau\text{-tilt}\,k\tilde{G}$ then $M \geq M'$ in $\mathrm{s}\tau\text{-tilt}\,B$, but this follows from the fact the restriction functor $\mathrm{Res}_G^{\tilde{G}}$ is an exact functor, the $I_{\tilde{G}}(B)$-invariance of M and Theorem 2.7. □

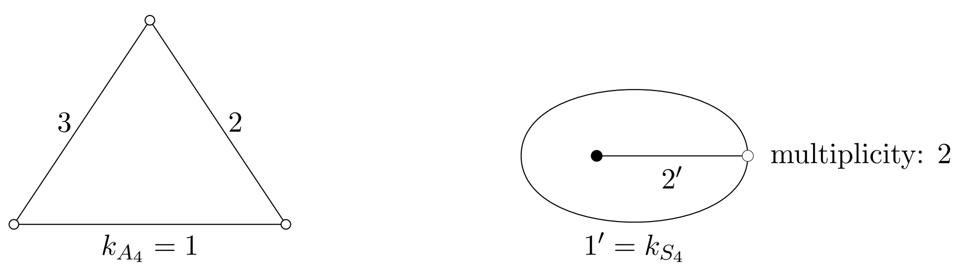
3.2. Examples

Finally, we illustrate our main results with the following examples.

Example 3.7. Let $\tilde{G}$ be a finite group, G a normal subgroup of $\tilde{G}$ with cyclic Sylow p-subgroup such that the quotient group $\tilde{G}/G$ is a p-group, B a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B. Then any support τ-tilting B-module is $I_{\tilde{G}}(B)$-invariant (see [8, Lemma 4.2 (2)]). Therefore, we can always apply Theorem 3.3.

Example 3.8. Let G_1 and G_2 be arbitrary finite groups and M a support τ-tilting kG_1-module. Then the group G_1 is a normal subgroup of the direct product group $G_1 \times G_2$, and it is clear that $M \cong xM$ for any $x \in G_1 \times G_2$. Therefore, the induced module $\mathrm{Ind}_{G_1}^{G_1 \times G_2} M \cong kG_2 \otimes_k M$ is support τ-tilting $k[G_1 \times G_2]$-module by Theorem 3.2.

Example 3.9. Let k be an algebraically closed field of characteristic $p = 2$, G the alternating group A_4 of degree 4 and $\tilde{G}$ the symmetric group S_4 of degree 4. The principal blocks of kA_4 and kS_4 are themselves, respectively. Moreover, the block kA_4 is covered by kS_4. The algebras kA_4 and kS_4 are Brauer graph algebras associated to the Brauer graphs in Figure 1(a) and Figure 1(b), respectively:



(a) The Brauer graph of kA_4 (b) The Brauer graph of kS_4

Figure 1: Brauer graphs

Now we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, kA_4)$ of the partially ordered set $\mathrm{s}\tau\text{-tilt}\, kA_4$ as follows:

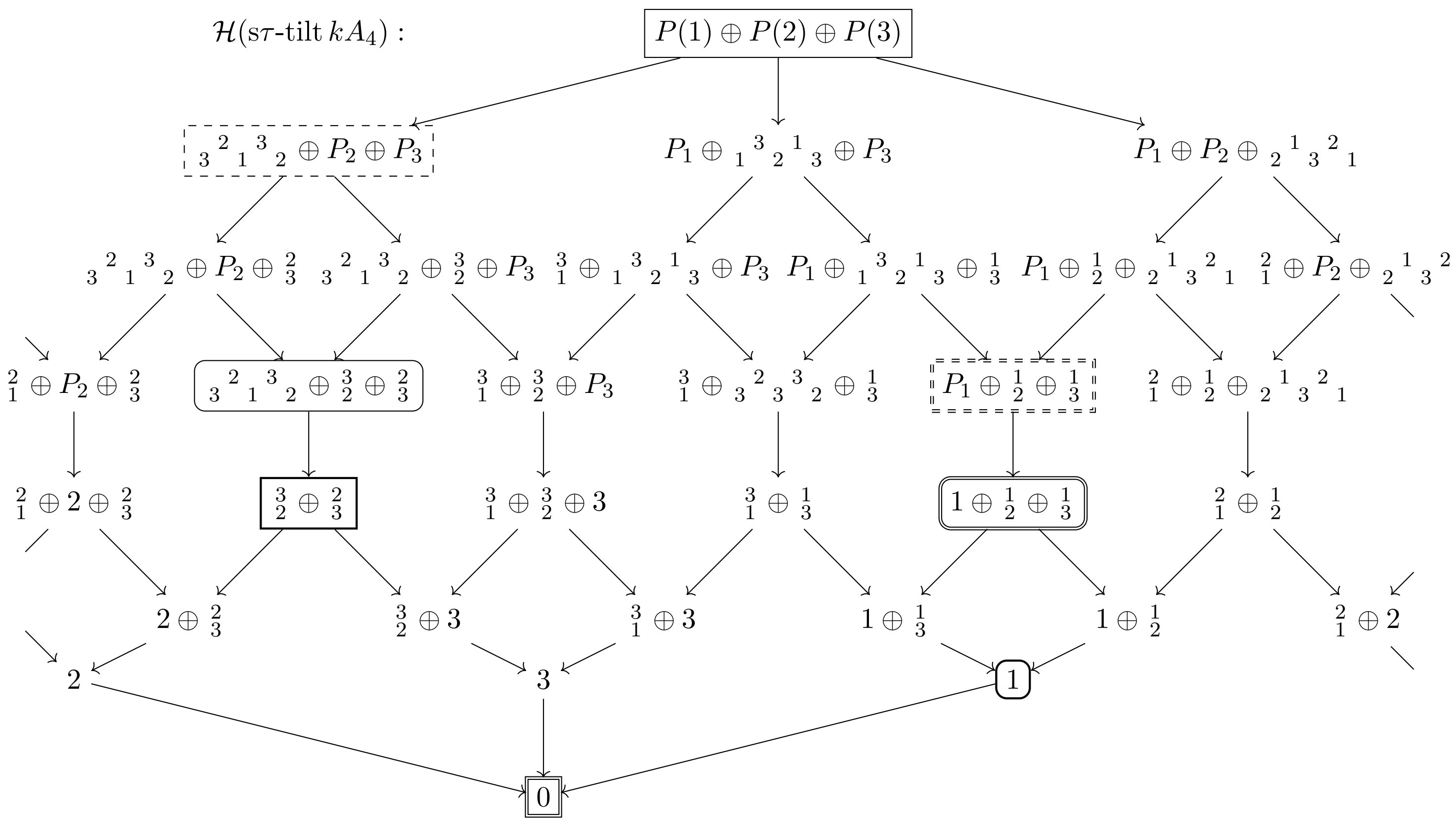


Figure 2: The Hasse diagram of sτ-tilt kA_4

The enclosed support τ-tilting modules in Figure 2 are all the invariant support τ-tilting modules under the action of S_4. Next we draw the Hasse diagram $\mathcal{H}(\mathrm{s}\tau\text{-tilt}\, kS_4)$ of partially ordered set $\mathrm{s}\tau\text{-tilt}\, kS_4$ as follows:

$\mathcal{H}(\mathrm{s}\tau\text{-tilt}(kS_4))$:

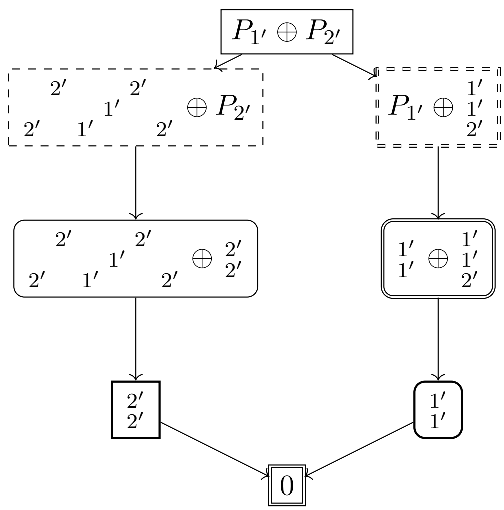


Figure 3: The Hasse diagram of $\mathrm{s}\tau\text{-tilt}\, kS_4$

The induction functor $\mathrm{Ind}_{A_4}^{S_4}$ takes each enclosed S_4-invariant support τ-tilting kA_4-module in Figure 2 to that in Figure 3 with the same square. We remark that even if support τ-tilting kA_4-module M is basic, its induction $\mathrm{Ind}_{A_4}^{S_4} M$ is not necessarily basic. For example, the induced module $\mathrm{Ind}_{A_4}^{S_4}(1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}) \cong \begin{smallmatrix}1'\\1'\end{smallmatrix} \oplus \begin{smallmatrix}1'\\1'\\2'\end{smallmatrix} \oplus \begin{smallmatrix}1'\\1'\\2'\end{smallmatrix}$ is not basic.

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