Near-wall expansions of individual terms in transport equation for dissipation rate

Nomenclature

$a_p, b_p, c_p, \cdots$	coefficients in near-wall expansions of pressure fluctuation.
$b_i, c_i, d_i, \cdots (i = 1, 2, \cdots)$	coefficients in near-wall expansions of velocity fluctuations.
$b_\theta, d_\theta, e_\theta, \cdots$	coefficients in near-wall expansions of temperature fluctuation.
k	turbulent kinetic energy, $\overline{u_i' u_i'} \big/ 2$
k_θ	temperature variance, $\overline{\theta'\theta'}/2$
p	pressure
q_w	heat flux at wall
u_i, u, v, w	velocity component
x_1, x	streamwise direction
x_2, y	wall-normal direction
x_3, z	spanwise direction
Greek	
ε	dissipation rate of turbulent kinetic energy, $\nu \left(\partial u_i'/\partial x_j\right)^2$
ε_θ	dissipation rate of temperature variance, $\kappa \left(\partial \theta'/\partial x_j\right)^2$
κ	thermal diffusivity
θ	temperature
ν	kinematic viscosity
ρ	density
Superscripts	
$(\quad)'$	fluctuation component
$\overline{(\quad)}$	statistically averaged component

Near-wall expansions of individual terms in transport equations for ε and ε_θ are examined analytically.

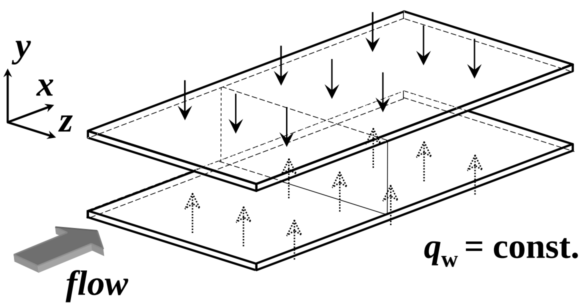

Fig. 1: Configuration of the channel.

The configuration applied here is a fully developed turbulent channel flow as shown in Fig. 1. The flow is driven by a uniform pressure gradient. The temperature field is imposed by uniform heating over both walls with a constant time-averaged heat-flux. The temperature is treated as passive scalar. The periodic boundary conditions are imposed in the streamwise (x) and spanwise (z) directions. The walls are non-slip. Further details of the numerical procedures can be found in Kozuka *et al.* (2008), Abe *et al.* (2004) and Kawamura *et al.* (1999).

1. Near-wall expansions of individual terms in ε-budget

- Viscous diffusion

$$
\begin{aligned}
V_\varepsilon &= \nu^2 \frac{\partial^2}{\partial x_j^2} \overline{\left(\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right)} \\
&= \nu^2 \frac{\partial^2}{\partial y^2} \left\{ \overline{\left(\frac{\partial u'}{\partial x_k} \frac{\partial u'}{\partial x_k} \right)} + \overline{\left(\frac{\partial v'}{\partial x_k} \frac{\partial v'}{\partial x_k} \right)} + \overline{\left(\frac{\partial w'}{\partial x_k} \frac{\partial w'}{\partial x_k} \right)} \right\} \\
&= \nu^2 \frac{\partial^2}{\partial y^2} \left\{ \overline{\left(\frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x} \right)} + \overline{\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right)} + \overline{\left(\frac{\partial u'}{\partial z} \frac{\partial u'}{\partial z} \right)} \right\} \\
&+ \nu^2 \frac{\partial^2}{\partial y^2} \left\{ \overline{\left(\frac{\partial v'}{\partial x} \frac{\partial v'}{\partial x} \right)} + \overline{\left(\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right)} + \overline{\left(\frac{\partial v'}{\partial z} \frac{\partial v'}{\partial z} \right)} \right\} \\
&+ \nu^2 \frac{\partial^2}{\partial y^2} \left\{ \overline{\left(\frac{\partial w'}{\partial x} \frac{\partial w'}{\partial x} \right)} + \overline{\left(\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y} \right)} + \overline{\left(\frac{\partial w'}{\partial z} \frac{\partial w'}{\partial z} \right)} \right\} \qquad (1)
\end{aligned}
$$

Near-wall expansions of the velocity fluctuations in terms of y can be given in Eq. (2) $\sim$ (4). By substituting Eq. (2) $\sim$ (4) into Eq. (1), the each component in the viscous diffusion term can be expanded as follows:

$$
\begin{aligned}
u' &= b_1 y + c_1 y^2 + d_1 y^3 + \cdots \qquad (2) \\
v' &= c_2 y^2 + d_2 y^3 + \cdots \qquad (3) \\
w' &= b_3 y + c_3 y^2 + d_3 y^3 + \cdots \qquad (4)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial u'}{\partial x} \frac{\partial u'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial x} y + \frac{\partial c_1}{\partial x} y^2 + \frac{\partial d_1}{\partial x} y^3 + \cdots \right)^2} = \overline{\left(\frac{\partial b_1}{\partial x} \right)^2} y^2 + 2 \overline{\left(\frac{\partial b_1}{\partial x} \right) \left(\frac{\partial c_1}{\partial x} \right)} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}} &= \overline{(b_1 + 2c_1 y + 3d_1 y^2 + \cdots)^2} = \overline{b_1^2} + 4\overline{b_1 c_1} y + \left(4\overline{c_1^2} + 6\overline{b_1 d_1} \right) y^2 \\
&+ \left(8\overline{b_1 e_1} + 12\overline{c_1 d_1} \right) y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial z} \frac{\partial u'}{\partial z}} &= \overline{\left(\frac{\partial b_1}{\partial z} y + \frac{\partial c_1}{\partial z} y^2 + \frac{\partial d_1}{\partial z} y^3 + \cdots \right)^2} = \overline{\left(\frac{\partial b_1}{\partial z} \right)^2} y^2 + 2 \overline{\left(\frac{\partial b_1}{\partial z} \right) \left(\frac{\partial c_1}{\partial z} \right)} y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial x} \frac{\partial v'}{\partial x}} &= \overline{\left(\frac{\partial c_2}{\partial x} y^2 + \frac{\partial d_2}{\partial x} y^3 + \frac{\partial e_2}{\partial x} y^4 + \cdots \right)^2} = \overline{\left(\frac{\partial c_2}{\partial x} \right)^2} y^4 + \cdots \\
\overline{\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y}} &= \overline{(2c_2 y + 3d_2 y^2 + 4e_2 y^3 + \cdots)^2} = 4\overline{c_2^2} y^2 + 12\overline{c_2 d_2} y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial z} \frac{\partial v'}{\partial z}} &= \overline{\left(\frac{\partial c_2}{\partial z} y^2 + \frac{\partial d_2}{\partial z} y^3 + \frac{\partial e_2}{\partial z} y^4 + \cdots \right)^2} = \overline{\left(\frac{\partial c_2}{\partial z} \right)^2} y^4 + \cdots \\
\overline{\frac{\partial w'}{\partial x} \frac{\partial w'}{\partial x}} &= \overline{\left(\frac{\partial b_3}{\partial x} y + \frac{\partial c_3}{\partial x} y^2 + \frac{\partial d_3}{\partial x} y^3 + \cdots \right)^2} = \overline{\left(\frac{\partial b_3}{\partial x} \right)^2} y^2 + 2 \overline{\left(\frac{\partial b_3}{\partial x} \right) \left(\frac{\partial c_3}{\partial x} \right)} y^3 + \cdots \\
\overline{\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y}} &= \overline{(b_3 + 2c_3 y + 3d_3 y^2 + \cdots)^2} = \overline{b_3^2} + 4\overline{b_3 c_3} y + \left(4\overline{c_3^2} + 6\overline{b_3 d_3} \right) y^2 + \cdots \\
&+ \left(8\overline{b_3 e_3} + 12\overline{c_3 d_3} \right) y^3 + \cdots \\
\overline{\frac{\partial w'}{\partial z} \frac{\partial w'}{\partial z}} &= \overline{\left(\frac{\partial b_3}{\partial z} y + \frac{\partial c_3}{\partial z} y^2 + \frac{\partial d_3}{\partial z} y^3 + \cdots \right)^2} = \overline{\left(\frac{\partial b_3}{\partial z} \right)^2} y^2 + 2 \overline{\left(\frac{\partial b_3}{\partial z} \right) \left(\frac{\partial c_3}{\partial z} \right)} y^3 + \cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the viscous diffusion term can be expressed as below.

$$
\begin{aligned}
V_\varepsilon &= \nu^2\left[2\left\{\overline{\left(\frac{\partial b_1}{\partial x}\right)^2}+\overline{\left(\frac{\partial b_1}{\partial z}\right)^2}+\overline{\left(\frac{\partial b_3}{\partial x}\right)^2}+\overline{\left(\frac{\partial b_3}{\partial z}\right)^2}\right\}+8\left(\overline{c_1^2}+\overline{c_2^2}+\overline{c_3^2}\right)\right.\\
&+ \left. 12\left(\overline{b_1d_1}+\overline{b_3d_3}\right)\right]\\
&+ \nu^2\left[12\left\{\overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_1}{\partial x}\right)}+\overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_1}{\partial z}\right)}+\overline{\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial c_3}{\partial x}\right)}+\overline{\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial c_3}{\partial z}\right)}\right\}\right.\\
&+ \left. 72\left(\overline{c_1d_1}+\overline{c_2d_2}+\overline{c_3d_3}\right)+48\left(\overline{b_1e_1}+\overline{b_3e_3}\right)\right]y+O\left(y^2\right) \qquad (5)
\end{aligned}
$$

- Dissipation

$$
\begin{aligned}
\gamma_\varepsilon &= 2\nu^2\overline{\left(\frac{\partial^2 u_i'}{\partial x_k\partial x_j}\right)^2}=2\nu^2\left\{\overline{\left(\frac{\partial^2 u'}{\partial x_k\partial x_j}\right)^2}+\overline{\left(\frac{\partial^2 v'}{\partial x_k\partial x_j}\right)^2}+\overline{\left(\frac{\partial^2 w'}{\partial x_k\partial x_j}\right)^2}\right\}\\
&= 2\nu^2\left\{\overline{\left(\frac{\partial^2 u'}{\partial x^2}\right)^2}+2\overline{\left(\frac{\partial^2 u'}{\partial x\partial y}\right)^2}+2\overline{\left(\frac{\partial^2 u'}{\partial x\partial z}\right)^2}+\overline{\left(\frac{\partial^2 u'}{\partial y^2}\right)^2}+2\overline{\left(\frac{\partial^2 u'}{\partial y\partial z}\right)^2}+\overline{\left(\frac{\partial^2 u'}{\partial z^2}\right)^2}\right\}\\
&+ 2\nu^2\left\{\overline{\left(\frac{\partial^2 v'}{\partial x^2}\right)^2}+2\overline{\left(\frac{\partial^2 v'}{\partial x\partial y}\right)^2}+2\overline{\left(\frac{\partial^2 v'}{\partial x\partial z}\right)^2}+\overline{\left(\frac{\partial^2 v'}{\partial y^2}\right)^2}+2\overline{\left(\frac{\partial^2 v'}{\partial y\partial z}\right)^2}+\overline{\left(\frac{\partial^2 v'}{\partial z^2}\right)^2}\right\}\\
&+ 2\nu^2\left\{\overline{\left(\frac{\partial^2 w'}{\partial x^2}\right)^2}+2\overline{\left(\frac{\partial^2 w'}{\partial x\partial y}\right)^2}+2\overline{\left(\frac{\partial^2 w'}{\partial x\partial z}\right)^2}+\overline{\left(\frac{\partial^2 w'}{\partial y^2}\right)^2}+2\overline{\left(\frac{\partial^2 w'}{\partial y\partial z}\right)^2}+\overline{\left(\frac{\partial^2 w'}{\partial z^2}\right)^2}\right\}
\end{aligned}
$$

(6)

By substituting Eq. (2) $\sim$ (4) into Eq. (6), the each component in dissipation term can be expanded as follows:

$$
\begin{aligned}
\overline{\left(\frac{\partial^2 u'}{\partial x^2}\right)^2} &= \overline{\left(\frac{\partial^2 b_1}{\partial x^2}y+\frac{\partial^2 c_1}{\partial x^2}y^2+\frac{\partial^2 d_1}{\partial x^2}y^3+\cdots\right)^2}\\
&= \overline{\left(\frac{\partial^2 b_1}{\partial x^2}\right)^2}y^2+2\overline{\left(\frac{\partial^2 b_1}{\partial x^2}\right)\left(\frac{\partial^2 c_1}{\partial x^2}\right)}y^3+\cdots\\
2\overline{\left(\frac{\partial^2 u'}{\partial x\partial y}\right)^2} &= 2\overline{\left(\frac{\partial b_1}{\partial x}+2\frac{\partial c_1}{\partial x}y+3\frac{\partial d_1}{\partial x}y^2+\cdots\right)^2}\\
&= 2\left[\overline{\left(\frac{\partial b_1}{\partial x}\right)^2}+4\overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_1}{\partial x}\right)}y+\left\{4\overline{\left(\frac{\partial c_1}{\partial x}\right)^2}+6\overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial d_1}{\partial x}\right)}\right\}y^2+\cdots\right]\\
2\overline{\left(\frac{\partial^2 u'}{\partial x\partial z}\right)^2} &= 2\overline{\left(\frac{\partial^2 b_1}{\partial x\partial z}y+\frac{\partial^2 c_1}{\partial x\partial z}y^2+\frac{\partial^2 d_1}{\partial x\partial z}y^3+\cdots\right)^2}\\
&= 2\overline{\left(\frac{\partial^2 b_1}{\partial x\partial z}\right)^2}y^2+4\overline{\left(\frac{\partial^2 b_1}{\partial x\partial z}\right)\left(\frac{\partial^2 c_1}{\partial x\partial z}\right)}y^3+\cdots\\
\overline{\left(\frac{\partial^2 u'}{\partial y^2}\right)^2} &= \overline{\left(2c_1+6d_1y+\cdots\right)^2}=4\overline{c_1^2}+24\overline{c_1d_1}y+\cdots\\
2\overline{\left(\frac{\partial^2 u'}{\partial y\partial z}\right)^2} &= 2\overline{\left(\frac{\partial b_1}{\partial z}+2\frac{\partial c_1}{\partial z}y+3\frac{\partial d_1}{\partial z}y^2+\cdots\right)^2}\\
&= 2\left[\overline{\left(\frac{\partial b_1}{\partial z}\right)^2}+4\overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_1}{\partial z}\right)}y+\left\{4\overline{\left(\frac{\partial c_1}{\partial z}\right)^2}+6\overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial d_1}{\partial z}\right)}\right\}y^2+\cdots\right]
\end{aligned}
$$

$$\begin{aligned}
\overline{\left(\frac{\partial^2 u'}{\partial z^2}\right)^2} &= \overline{\left(\frac{\partial^2 b_1}{\partial z^2}y + \frac{\partial^2 c_1}{\partial z^2}y^2 + \frac{\partial^2 d_1}{\partial z^2}y^3 + \cdots\right)^2} \\
&= \overline{\left(\frac{\partial^2 b_1}{\partial z^2}\right)^2}y^2 + 2\overline{\left(\frac{\partial^2 b_1}{\partial z^2}\right)\left(\frac{\partial^2 c_1}{\partial z^2}\right)}y^3 + \cdots \\
\overline{\left(\frac{\partial^2 v'}{\partial x^2}\right)^2} &= \overline{\left(\frac{\partial^2 c_2}{\partial x^2}y^2 + \cdots\right)^2} = \overline{\left(\frac{\partial^2 c_2}{\partial x^2}\right)^2}y^4 + \cdots \\
2\overline{\left(\frac{\partial^2 v'}{\partial x \partial y}\right)^2} &= 2\overline{\left(2\frac{\partial c_2}{\partial x}y + 3\frac{\partial d_2}{\partial x}y^2 + \cdots\right)^2} \\
&= 2\left\{4\overline{\left(\frac{\partial c_2}{\partial x}\right)^2}y^2 + 12\overline{\left(\frac{\partial c_2}{\partial x}\right)\left(\frac{\partial d_2}{\partial x}\right)}y^3 \cdots\right\} \\
2\overline{\left(\frac{\partial^2 v'}{\partial x \partial z}\right)^2} &= 2\overline{\left(\frac{\partial^2 c_2}{\partial x \partial z}y^2 + \cdots\right)^2} = 2\overline{\left(\frac{\partial^2 c_2}{\partial x \partial z}\right)^2}y^4 + \cdots \\
\overline{\left(\frac{\partial^2 v'}{\partial y^2}\right)^2} &= \overline{(2c_2 + 6d_2 y + \cdots)^2} = 4\overline{c_2^2} + 24\overline{c_2 d_2}y + \cdots \\
2\overline{\left(\frac{\partial^2 v'}{\partial y \partial z}\right)^2} &= 2\overline{\left(2\frac{\partial c_2}{\partial z}y + 3\frac{\partial d_2}{\partial z}y^2 + \cdots\right)^2} \\
&= 2\left\{4\overline{\left(\frac{\partial c_2}{\partial z}\right)^2}y^2 + 12\overline{\left(\frac{\partial c_2}{\partial z}\right)\left(\frac{\partial d_2}{\partial z}\right)}y^3 \cdots\right\} \\
\overline{\left(\frac{\partial^2 v'}{\partial z^2}\right)^2} &= \overline{\left(\frac{\partial^2 c_2}{\partial z^2}y^2 + \cdots\right)^2} = \overline{\left(\frac{\partial^2 c_2}{\partial z^2}\right)^2}y^4 + \cdots \\
\overline{\left(\frac{\partial^2 w'}{\partial x^2}\right)^2} &= \overline{\left(\frac{\partial^2 b_3}{\partial x^2}y + \frac{\partial^2 c_3}{\partial x^2}y^2 + \frac{\partial^2 d_3}{\partial x^2}y^3 + \cdots\right)^2} \\
&= \overline{\left(\frac{\partial^2 b_3}{\partial x^2}\right)^2}y^2 + 2\overline{\left(\frac{\partial^2 b_3}{\partial x^2}\right)\left(\frac{\partial^2 c_3}{\partial x^2}\right)}y^3 + \cdots \\
2\overline{\left(\frac{\partial^2 w'}{\partial x \partial y}\right)^2} &= 2\overline{\left(\frac{\partial b_3}{\partial x} + 2\frac{\partial c_3}{\partial x}y + 3\frac{\partial d_3}{\partial x}y^2 + \cdots\right)^2} \\
&= 2\left[\overline{\left(\frac{\partial b_3}{\partial x}\right)^2} + 4\overline{\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial c_3}{\partial x}\right)}y + \left\{4\overline{\left(\frac{\partial c_3}{\partial x}\right)^2} + 6\overline{\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial d_3}{\partial x}\right)}\right\}y^2 + \cdots\right] \\
2\overline{\left(\frac{\partial^2 w'}{\partial x \partial z}\right)^2} &= 2\overline{\left(\frac{\partial^2 b_3}{\partial x \partial z}y + \frac{\partial^2 c_3}{\partial x \partial z}y^2 + \frac{\partial^2 d_3}{\partial x \partial z}y^3 + \cdots\right)^2} \\
&= 2\overline{\left(\frac{\partial^2 b_3}{\partial x \partial z}\right)^2}y^2 + 4\overline{\left(\frac{\partial^2 b_3}{\partial x \partial z}\right)\left(\frac{\partial^2 c_3}{\partial x \partial z}\right)}y^3 + \cdots \\
\overline{\left(\frac{\partial^2 w'}{\partial y^2}\right)^2} &= \overline{(2c_3 + 6d_3 y + \cdots)^2} = 4\overline{c_3^2} + 24\overline{c_3 d_3}y + \cdots \\
2\overline{\left(\frac{\partial^2 w'}{\partial y \partial z}\right)^2} &= 2\overline{\left(\frac{\partial b_3}{\partial z} + 2\frac{\partial c_3}{\partial z}y + 3\frac{\partial d_3}{\partial z}y^2 + \cdots\right)^2} \\
&= 2\left[\overline{\left(\frac{\partial b_3}{\partial z}\right)^2} + 4\overline{\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial c_3}{\partial z}\right)}y + \left\{4\overline{\left(\frac{\partial c_3}{\partial z}\right)^2} + 6\overline{\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial d_3}{\partial z}\right)}\right\}y^2 + \cdots\right] \\
\overline{\left(\frac{\partial^2 w'}{\partial z^2}\right)^2} &= \overline{\left(\frac{\partial^2 b_3}{\partial z^2}y + \frac{\partial^2 c_3}{\partial z^2}y^2 + \frac{\partial^2 d_3}{\partial z^2}y^3 + \cdots\right)^2}
\end{aligned}$$

$$= \overline{\left(\frac{\partial^2 b_3}{\partial z^2}\right)^2} y^2 + 2\overline{\left(\frac{\partial^2 b_3}{\partial z^2}\right)\left(\frac{\partial^2 c_3}{\partial z^2}\right)} y^3 + \cdots$$

Thus, the dissipation term can be expanded as follows:

$$\begin{aligned}
\gamma_\varepsilon &= 2\nu^2 \left[2\left\{ \overline{\left(\frac{\partial b_1}{\partial x}\right)^2} + \overline{\left(\frac{\partial b_1}{\partial z}\right)^2} + \overline{\left(\frac{\partial b_3}{\partial x}\right)^2} + \overline{\left(\frac{\partial b_3}{\partial z}\right)^2} \right\} + 4\left(\overline{c_1^2} + \overline{c_2^2} + \overline{c_3^2}\right) \right] \\
&+ 2\nu^2 \left[8\left\{ \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_1}{\partial x}\right)} + \overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_1}{\partial z}\right)} + \overline{\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial c_3}{\partial x}\right)} + \overline{\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial c_3}{\partial z}\right)} \right\} \right. \\
&+ \left. 24\left(\overline{c_1 d_1} + \overline{c_2 d_2} + \overline{c_3 d_3}\right) \right] y + O\left(y^2\right)
\end{aligned} \tag{7}$$

- Turbulent production

$$\begin{aligned}
P_\varepsilon^{(4)} &= -2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u_i'}{\partial x_j}} \\
&= -2\nu \left\{ \overline{\frac{\partial u'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u'}{\partial x_j}} + \overline{\frac{\partial v'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial v'}{\partial x_j}} + \overline{\frac{\partial w'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial w'}{\partial x_j}} \right\}
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\overline{\frac{\partial u'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial u'}{\partial x_j}} = \\
&+ \left(\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}} + \overline{\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}} + \overline{\frac{\partial u'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial x}} + \overline{\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial u'}{\partial y}} + \overline{\frac{\partial u'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial u'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial x}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial u'}{\partial z}} \right)
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\overline{\frac{\partial v'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial v'}{\partial x_j}} = \\
&+ \left(\overline{\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}} + \overline{\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y}} + \overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial v'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial v'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x}} + \overline{\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}} + \overline{\frac{\partial v'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial v'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial x}} + \overline{\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial y}} + \overline{\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial v'}{\partial z}} \right)
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\overline{\frac{\partial w'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\frac{\partial w'}{\partial x_j}} = \\
&+ \left(\overline{\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial w'}{\partial x}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial w'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial w'}{\partial x}} + \overline{\frac{\partial w'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial w'}{\partial y}} + \overline{\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial z}} \right) \\
&+ \left(\overline{\frac{\partial w'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial x}} + \overline{\frac{\partial w'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}} + \overline{\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}} \right)
\end{aligned} \tag{11}$$

With use of Eq. (2) $\sim$ (4), individual components in Eq. (9) can be expanded as below.

$$
\begin{aligned}
\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial x}\right)^3} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial y}} &= \overline{b_1\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_1}{\partial x}\right)} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z}} &= \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial b_1}{\partial z}\right)} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial x}} &= \overline{b_1^2\left(\frac{\partial b_1}{\partial x}\right)} y + \left\{\overline{b_1^2\left(\frac{\partial c_1}{\partial x}\right)} + 4\overline{b_1 c_1\left(\frac{\partial b_1}{\partial x}\right)}\right\} y^2 + \cdots \\
\overline{\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial u'}{\partial y}} &= 2\overline{b_1^2 c_2} y + \left(3\overline{b_1^2 d_2} + 8\overline{b_1 c_2^2}\right) y^2 + \cdots \\
\overline{\frac{\partial u'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial u'}{\partial z}} &= \overline{b_1 b_3\left(\frac{\partial b_1}{\partial z}\right)} y + \left\{\overline{b_1 b_3\left(\frac{\partial c_1}{\partial z}\right)} + 2\overline{(b_1 c_3 + b_3 c_1)\left(\frac{\partial b_1}{\partial z}\right)}\right\} y^2 + \cdots \\
\overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial z}\right)^2\left(\frac{\partial b_1}{\partial x}\right)} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}} &= \overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right) b_1} y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial u'}{\partial z}} &= \overline{\left(\frac{\partial b_1}{\partial z}\right)^2\left(\frac{\partial b_3}{\partial z}\right)} y^3 + \cdots
\end{aligned}
$$

Individual components in Eq. (10) and (11) can be also expanded as below.

$$
\begin{aligned}
\overline{\frac{\partial v'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)^2} y^5 + \cdots \\
\overline{\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y}} &= 2\overline{c_2\left(\frac{\partial c_2}{\partial x}\right)^2} y^5 + \cdots \\
\overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial v'}{\partial z}} &= \overline{\left(\frac{\partial c_2}{\partial x}\right)\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial c_2}{\partial z}\right)} y^5 + \cdots \\
\overline{\frac{\partial v'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial x}} &= 2\overline{b_1 c_2\left(\frac{\partial c_2}{\partial x}\right)} y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}} &= 8\overline{c_2^3} y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial v'}{\partial z}} &= 2\overline{c_2 b_3\left(\frac{\partial c_2}{\partial z}\right)} y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial x}} &= \overline{\left(\frac{\partial c_2}{\partial z}\right)\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_2}{\partial x}\right)} y^5 + \cdots \\
\overline{\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial y}} &= 2\overline{c_2\left(\frac{\partial c_2}{\partial z}\right)^2} y^5 + \cdots \\
\overline{\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial v'}{\partial z}} &= \overline{\left(\frac{\partial c_2}{\partial z}\right)^2\left(\frac{\partial b_3}{\partial z}\right)} y^5 + \cdots \\
\overline{\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial x}\frac{\partial w'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial b_3}{\partial x}\right)^2} y^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial w'}{\partial x}\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}} &= \overline{b_3\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)}y^3+\cdots \\
\overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial z}} &= \overline{\left(\frac{\partial b_3}{\partial x}\right)^2\left(\frac{\partial b_3}{\partial z}\right)}y^3+\cdots \\
\overline{\frac{\partial w'}{\partial y}\frac{\partial u'}{\partial y}\frac{\partial w'}{\partial x}} &= \overline{b_1b_3\left(\frac{\partial b_3}{\partial x}\right)}y+\left\{\overline{b_1b_3\left(\frac{\partial c_3}{\partial x}\right)}+2\overline{(b_1c_3+b_3c_1)\left(\frac{\partial b_3}{\partial x}\right)}\right\}y^2+\cdots \\
\overline{\frac{\partial w'}{\partial y}\frac{\partial v'}{\partial y}\frac{\partial w'}{\partial y}} &= 2\overline{b_3^2c_2}y+\left(3\overline{b_3^2d_2}+8\overline{b_3c_2c_3}\right)y^2+\cdots \\
\overline{\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial z}} &= \overline{b_3^2\left(\frac{\partial b_3}{\partial z}\right)}y+\left\{\overline{b_3^2\left(\frac{\partial c_3}{\partial z}\right)}+4\overline{b_3c_3\left(\frac{\partial b_3}{\partial z}\right)}\right\}y^2+\cdots \\
\overline{\frac{\partial w'}{\partial z}\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial b_3}{\partial z}\right)^2}y^3+\cdots \\
\overline{\frac{\partial w'}{\partial z}\frac{\partial v'}{\partial z}\frac{\partial w'}{\partial y}} &= \overline{b_3\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right)}y^3+\cdots \\
\overline{\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}} &= \overline{\left(\frac{\partial b_3}{\partial z}\right)^3}y^3+\cdots
\end{aligned}
$$

By substituting the expanded components in (9), (10) and (11) into Eq. (8), the near-wall expansion of the turbulent production can be derived as follows:

$$
\begin{aligned}
P_\varepsilon^{(4)} &= -2\nu\left[\overline{b_1^2\left(\frac{\partial b_1}{\partial x}\right)}+2\overline{b_1^2c_2}+\overline{b_1b_3\left(\frac{\partial b_1}{\partial z}\right)}+\overline{b_1b_3\left(\frac{\partial b_3}{\partial x}\right)}+2\overline{b_3^2c_2}+\overline{b_3^2\left(\frac{\partial b_3}{\partial z}\right)}\right]y \\
&+ O\left(y^2\right) \qquad (12)
\end{aligned}
$$

- Turbulent diffusion

$$
\begin{aligned}
T_\varepsilon = &-\frac{\partial}{\partial x_j}\overline{\left(u_j'\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)}=-\frac{\partial}{\partial y}\overline{\left(v'\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\right)} \\
= &-\frac{\partial}{\partial y}\left\{\overline{\left(v'\frac{\partial u'}{\partial x_k}\frac{\partial u'}{\partial x_k}\right)+\left(v'\frac{\partial v'}{\partial x_k}\frac{\partial v'}{\partial x_k}\right)+\left(v'\frac{\partial w'}{\partial x_k}\frac{\partial w'}{\partial x_k}\right)}\right\} \\
= &-\frac{\partial}{\partial y}\left\{\overline{\left(v'\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}\right)+\left(v'\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right)+\left(v'\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}\right)}\right\} \\
&-\frac{\partial}{\partial y}\left\{\overline{\left(v'\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial x}\right)+\left(v'\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}\right)+\left(v'\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial z}\right)}\right\} \\
&-\frac{\partial}{\partial y}\left\{\overline{\left(v'\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial x}\right)+\left(v'\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial y}\right)+\left(v'\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}\right)}\right\} \qquad (13)
\end{aligned}
$$

By substituting Eq. (2) $\sim$ (4) into Eq. (13), the each component can be expanded as follows:

$$
\begin{aligned}
\overline{\left(v'\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x}\right)} &= \overline{c_2\left(\frac{\partial b_1}{\partial x}\right)^2}y^4+\cdots \\
\overline{\left(v'\frac{\partial u'}{\partial y}\frac{\partial u'}{\partial y}\right)} &= \overline{b_1^2c_2}y^2+\cdots
\end{aligned}
$$

$$
\begin{aligned}
\overline{\left(v'\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial z}\right)} &= \overline{c_2\left(\frac{\partial b_1}{\partial z}\right)^2} y^4 + \cdots \\
\overline{\left(v'\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial x}\right)} &= \overline{c_2\left(\frac{\partial c_2}{\partial x}\right)^2} y^6 + \cdots \\
\overline{\left(v'\frac{\partial v'}{\partial y}\frac{\partial v'}{\partial y}\right)} &= 4\overline{c_2^3} y^4 + \cdots \\
\overline{\left(v'\frac{\partial v'}{\partial z}\frac{\partial v'}{\partial z}\right)} &= \overline{c_2\left(\frac{\partial c_2}{\partial z}\right)^2} y^6 + \cdots \\
\overline{\left(v'\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial x}\right)} &= \overline{c_2\left(\frac{\partial b_3}{\partial x}\right)^2} y^4 + \cdots \\
\overline{\left(v'\frac{\partial w'}{\partial y}\frac{\partial w'}{\partial y}\right)} &= \overline{c_2 b_3^2} y^2 + \cdots \\
\overline{\left(v'\frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z}\right)} &= \overline{c_2\left(\frac{\partial b_3}{\partial z}\right)^2} y^4 + \cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the turbulent diffusion term can be given as follows:

$$T_\varepsilon = -2\left(\overline{b_1^2 c_2} + \overline{c_2 b_3^2}\right) y + O\left(y^2\right) \cdots \tag{14}$$

- Pressure diffusion

$$
\begin{aligned}
\pi_\varepsilon &= -2\frac{\nu}{\rho}\frac{\partial}{\partial x_k}\overline{\left(\frac{\partial p'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}\right)} = -2\frac{\nu}{\rho}\frac{\partial}{\partial y}\overline{\left(\frac{\partial p'}{\partial x_j}\frac{\partial v'}{\partial x_j}\right)} \\
&= -2\frac{\nu}{\rho}\frac{\partial}{\partial y}\left\{\overline{\left(\frac{\partial p'}{\partial x}\frac{\partial v'}{\partial x}\right)} + \overline{\left(\frac{\partial p'}{\partial y}\frac{\partial v'}{\partial y}\right)} + \overline{\left(\frac{\partial p'}{\partial z}\frac{\partial v'}{\partial z}\right)}\right\}
\end{aligned} \tag{15}
$$

Near-wall expansion of pressure fluctuation in terms of y can be given as follows:

$$p' = a_p + b_p y + c_p y^2 + d_p y^3 + \cdots \tag{16}$$

By substituting Eq. (2) $\sim$ (4) and (16) into Eq. (15), the each component can be expanded as follows:

$$
\begin{aligned}
\overline{\left(\frac{\partial p'}{\partial x}\frac{\partial v'}{\partial x}\right)} &= \overline{\left(\frac{\partial a_p}{\partial x} + \frac{\partial b_p}{\partial x}y + \cdots\right)\left(\frac{\partial c_2}{\partial x}y^2 + \frac{\partial d_2}{\partial x}y^3 + \cdots\right)} \\
&= \overline{\left(\frac{\partial a_p}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)} y^2 + \left\{\overline{\left(\frac{\partial b_p}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)} + \overline{\left(\frac{\partial a_p}{\partial x}\right)\left(\frac{\partial d_2}{\partial x}\right)}\right\} y^3 + \cdots \\
\overline{\left(\frac{\partial p'}{\partial y}\frac{\partial v'}{\partial y}\right)} &= \overline{\left(b_p + 2c_p y + \cdots\right)\left(2c_2 y + 3d_2 y^2 + \cdots\right)} \\
&= 2\overline{b_p c_2} y + \left(3\overline{b_p d_2} + 4\overline{c_p c_2}\right) y^2 + \cdots \\
\overline{\left(\frac{\partial p'}{\partial z}\frac{\partial v'}{\partial z}\right)} &= \overline{\left(\frac{\partial a_p}{\partial z} + \frac{\partial b_p}{\partial z}y + \cdots\right)\left(\frac{\partial c_2}{\partial z}y^2 + \frac{\partial d_2}{\partial z}y^3 + \cdots\right)} \\
&= \overline{\left(\frac{\partial a_p}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right)} y^2 + \left\{\overline{\left(\frac{\partial b_p}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right)} + \overline{\left(\frac{\partial a_p}{\partial z}\right)\left(\frac{\partial d_2}{\partial z}\right)}\right\} y^3 + \cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the pressure diffusion term can be written as follows:

$$\begin{aligned}\pi_\varepsilon &= -2\frac{\nu}{\rho}\left[2\overline{b_p c_2} + 2\left\{\overline{\left(\frac{\partial a_p}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)} + \left(3\overline{b_p d_2} + 4\overline{c_p c_2}\right) + \overline{\left(\frac{\partial a_p}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right)}\right\} y\right.\\ &+ \left. O\left(y^2\right)\right]\end{aligned} \tag{17}$$

- Gradient production

$$P_\varepsilon^{(3)} = -2\nu\overline{u_j'\frac{\partial u_i'}{\partial x_k}}\frac{\partial^2\overline{u_i}}{\partial x_j \partial x_k} \tag{18}$$

$$\begin{aligned} &= -2\nu\overline{v'\frac{\partial u'}{\partial y}}\frac{\partial^2\overline{u}}{\partial y^2}\\ &= -2\nu\overline{\left(c_2 y^2 + d_2 y^3 + \cdots\right)\left(b_1 + 2c_1 y + \cdots\right)}\frac{\partial^2\overline{u}}{\partial y^2}\\ &= -2\nu\left\{\overline{b_1 c_2}y^2 + \left(2\overline{c_1 c_2} + \overline{b_1 d_2}\right)y^3 + O\left(y^4\right)\right\}\frac{\partial^2\overline{u}}{\partial y^2}\end{aligned} \tag{19}$$

- Mixed production

$$\begin{aligned}P_\varepsilon^{(1)} &= -2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\frac{\partial\overline{u_i}}{\partial x_j} = -2\nu\overline{\frac{\partial u'}{\partial x_k}\frac{\partial v'}{\partial x_k}}\frac{\partial\overline{u}}{\partial y}\\ &= -2\nu\left(\overline{\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}} + \overline{\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial y}} + \overline{\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial z}}\right)\frac{\partial\overline{u}}{\partial y}\end{aligned} \tag{20}$$

$$\begin{aligned}\overline{\frac{\partial u'}{\partial x}\frac{\partial v'}{\partial x}} &= \overline{\left(\frac{\partial b_1}{\partial x}y + \frac{\partial c_1}{\partial x}y^2\cdots\right)\left(\frac{\partial c_2}{\partial x}y^2 + \frac{\partial d_2}{\partial x}y^3 + \cdots\right)} = \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial c_2}{\partial x}\right)}y^3 + \cdots\\ \overline{\frac{\partial u'}{\partial y}\frac{\partial v'}{\partial y}} &= \overline{\left(b_1 + 2c_1 y + \cdots\right)\left(2c_2 y + 3d_2 y^2 + \cdots\right)} = 2\overline{b_1 c_2}y + \left(3\overline{b_1 d_2} + 4\overline{c_1 c_2}\right)y^2 + \cdots\\ \overline{\frac{\partial u'}{\partial z}\frac{\partial v'}{\partial z}} &= \overline{\left(\frac{\partial b_1}{\partial z}y + \frac{\partial c_1}{\partial z}y^2\cdots\right)\left(\frac{\partial c_2}{\partial z}y^2 + \frac{\partial d_2}{\partial z}y^3 + \cdots\right)} = \overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial c_2}{\partial z}\right)}y^3 + \cdots\end{aligned}$$

Therefore, the near-wall expansion of the mixed production term can be given as follows:

$$P_\varepsilon^{(1)} = -2\nu\left\{2\overline{b_1 c_2}y + \left(3\overline{b_1 d_2} + 4\overline{c_1 c_2}\right)y^2 + O\left(y^3\right)\right\}\frac{\partial\overline{u}}{\partial y} \tag{21}$$

- Production by mean velocity gradient

$$\begin{aligned}P_\varepsilon^{(2)} &= -2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}}\frac{\partial\overline{u_j}}{\partial x_k} = -2\nu\overline{\frac{\partial u_i'}{\partial x}\frac{\partial u_i'}{\partial y}}\frac{\partial\overline{u}}{\partial y}\\ &= -2\nu\left(\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial y}} + \overline{\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial y}}\right)\frac{\partial\overline{u}}{\partial y}\end{aligned} \tag{22}$$

$$\begin{aligned}\overline{\frac{\partial u'}{\partial x}\frac{\partial u'}{\partial y}} &= \overline{\left(\frac{\partial b_1}{\partial x}y + \frac{\partial c_1}{\partial x}y^2 + \cdots\right)\left(b_1 + 2c_1 y + \cdots\right)}\\ &= \overline{b_1\left(\frac{\partial b_1}{\partial x}\right)}y + \left\{\overline{b_1\left(\frac{\partial c_1}{\partial x}\right)} + 2\overline{c_1\left(\frac{\partial b_1}{\partial x}\right)}\right\}y^2 + \cdots\end{aligned}$$

$$\begin{aligned}
\overline{\frac{\partial v'}{\partial x}\frac{\partial v'}{\partial y}} &= \overline{\left(\frac{\partial c_2}{\partial x}y^2 + \frac{\partial d_2}{\partial x}y^3 + \cdots\right)(2c_2y + 3d_2y^2 + \cdots)} = \overline{2c_2\left(\frac{\partial c_2}{\partial x}\right)}y^3 + \cdots \\
\overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial y}} &= \overline{\left(\frac{\partial b_3}{\partial x}y + \frac{\partial c_3}{\partial x}y^2 + \cdots\right)(b_3 + 2c_3y + \cdots)} \\
&= \overline{b_3\left(\frac{\partial b_3}{\partial x}\right)}y + \left\{\overline{b_3\left(\frac{\partial c_3}{\partial x}\right)} + \overline{2c_3\left(\frac{\partial b_3}{\partial x}\right)}\right\}y^2 + \cdots
\end{aligned}$$

Therefore, the near-wall expansion of the production by mean velocity gradient term can be given as follows:

$$\begin{aligned}
P_\varepsilon^{(2)} = -2\nu \Bigg[&\left\{\overline{b_1\left(\frac{\partial b_1}{\partial x}\right)} + \overline{b_3\left(\frac{\partial b_3}{\partial x}\right)}\right\} y \\
&+ \left\{\overline{b_1\left(\frac{\partial c_1}{\partial x}\right)} + \overline{2c_1\left(\frac{\partial b_1}{\partial x}\right)} + \overline{b_3\left(\frac{\partial c_3}{\partial x}\right)} + \overline{2c_3\left(\frac{\partial b_3}{\partial x}\right)}\right\} y^2 + O\left(y^3\right)\Bigg] \frac{\partial \overline{u}}{\partial y} \qquad (23)
\end{aligned}$$

2. Near-wall expansions of individual terms in ε_θ-budget

- Viscous diffusion

$$V_{\varepsilon_\theta} = \frac{\partial}{\partial x_k}\left\{\frac{\partial}{\partial x_k}\kappa^2\overline{\left(\frac{\partial \theta'}{\partial x_m}\right)^2}\right\} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\kappa^2\left\{\overline{\left(\frac{\partial \theta'}{\partial x}\right)^2}+\overline{\left(\frac{\partial \theta'}{\partial y}\right)^2}+\overline{\left(\frac{\partial \theta'}{\partial z}\right)^2}\right\}\right] \tag{24}$$

In case of the current thermal boundary condition, near-wall expansion of the temperature fluctuation can be given as follows:

$$\theta' = b_\theta y + d_\theta y^3 + e_\theta y^4 + \cdots \tag{25}$$

By substituting Eq. (25) into Eq. (24), the each component in the viscous diffusion term can be written as follows:

$$\begin{aligned}
\overline{\left(\frac{\partial \theta'}{\partial x}\right)^2} &= \overline{\left(\frac{\partial b_\theta}{\partial x}y + \frac{\partial d_\theta}{\partial x}y^3 + \frac{\partial e_\theta}{\partial x}y^4 + \cdots\right)^2} \\
&= \overline{\left(\frac{\partial b_\theta}{\partial x}\right)^2}y^2 + 2\overline{\left(\frac{\partial b_\theta}{\partial x}\right)\left(\frac{\partial d_\theta}{\partial x}\right)}y^4 + \cdots \\
\overline{\left(\frac{\partial \theta'}{\partial y}\right)^2} &= \overline{(b_\theta + 3d_\theta y^2 + 4e_\theta y^3 + \cdots)^2} \\
&= \overline{b_\theta^2} + 6\overline{b_\theta d_\theta}y^2 + 8\overline{b_\theta e_\theta}y^3 + \cdots \\
\overline{\left(\frac{\partial \theta'}{\partial z}\right)^2} &= \overline{\left(\frac{\partial b_\theta}{\partial z}y + \frac{\partial d_\theta}{\partial z}y^3 + \frac{\partial e_\theta}{\partial z}y^4 + \cdots\right)^2} \\
&= \overline{\left(\frac{\partial b_\theta}{\partial z}\right)^2}y^2 + 2\overline{\left(\frac{\partial b_\theta}{\partial z}\right)\left(\frac{\partial d_\theta}{\partial z}\right)}y^4 + \cdots
\end{aligned}$$

Thus, the viscous diffusion term can be expanded as follows:

$$V_{\varepsilon_\theta} = \kappa^2\left[2\left\{\overline{\left(\frac{\partial b_\theta}{\partial x}\right)^2} + 6\overline{b_\theta d_\theta} + \overline{\left(\frac{\partial b_\theta}{\partial z}\right)^2}\right\} + 48\overline{b_\theta e_\theta}y + \mathrm{O}\left(y^2\right)\right] \tag{26}$$

- Dissipation

$$\begin{aligned}
\gamma_{\varepsilon_\theta} &= 2\kappa^2\overline{\left(\frac{\partial^2 \theta'}{\partial x_k \partial x_j}\right)^2} \\
&= 2\kappa^2\left[\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial x}\right)\right\}^2} + 2\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial y}\right)\right\}^2} + 2\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2}\right] \\
&+ 2\kappa^2\left[\overline{\left\{\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial y}\right)\right\}^2} + 2\overline{\left\{\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2} + \overline{\left\{\frac{\partial}{\partial z}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2}\right]
\end{aligned} \tag{27}$$

Using Eq. (25), following equations can be obtained.

$$\begin{aligned}
\frac{\partial \theta'}{\partial x} &= \frac{\partial b_\theta}{\partial x}y + \frac{\partial d_\theta}{\partial x}y^3 + \frac{\partial e_\theta}{\partial x}y^4 + \cdots \\
\frac{\partial \theta'}{\partial y} &= b_\theta + 3d_\theta y^2 + 4e_\theta y^3 + \cdots \\
\frac{\partial \theta'}{\partial z} &= \frac{\partial b_\theta}{\partial z}y + \frac{\partial d_\theta}{\partial z}y^3 + \frac{\partial e_\theta}{\partial z}y^4 + \cdots \\
\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial x}\right) &= \frac{\partial^2 b_\theta}{\partial x^2}y + \frac{\partial^2 d_\theta}{\partial x^2}y^3 + \frac{\partial^2 e_\theta}{\partial x^2}y^4 + \cdots
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial y}\right) &= \frac{\partial b_\theta}{\partial x} + 3\frac{\partial d_\theta}{\partial x}y^2 + 4\frac{\partial e_\theta}{\partial x}y^3 + \cdots \\
\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial z}\right) &= \frac{\partial^2 b_\theta}{\partial x \partial z}y + \frac{\partial^2 d_\theta}{\partial x \partial z}y^3 + \frac{\partial^2 e_\theta}{\partial x \partial z}y^4 + \cdots \\
\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial y}\right) &= 6d_\theta y + 12e_\theta y^2 + \cdots \\
\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial z}\right) &= \frac{\partial b_\theta}{\partial z} + 3\frac{\partial d_\theta}{\partial z}y^2 + 4\frac{\partial e_\theta}{\partial z}y^3 + \cdots \\
\frac{\partial}{\partial z}\left(\frac{\partial \theta'}{\partial z}\right) &= \frac{\partial^2 b_\theta}{\partial z^2}y + \frac{\partial^2 d_\theta}{\partial z^2}y^3 + \frac{\partial^2 e_\theta}{\partial z^2}y^4 + \cdots
\end{aligned}
$$

By substituting the near-wall expansions above into individual components in Eq. (27), the each term can be expanded as follows:

$$
\begin{aligned}
\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial x}\right)\right\}^2} &= \overline{\left\{\frac{\partial^2 b_\theta}{\partial x^2}y + \frac{\partial^2 d_\theta}{\partial x^2}y^3 + \frac{\partial^2 e_\theta}{\partial x^2}y^4 + \cdots\right\}^2} = \overline{\left(\frac{\partial^2 b_\theta}{\partial x^2}\right)^2}y^2 + \cdots \\
2\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial y}\right)\right\}^2} &= 2\overline{\left\{\frac{\partial b_\theta}{\partial x} + 3\frac{\partial d_\theta}{\partial x}y^2 + 4\frac{\partial e_\theta}{\partial x}y^3 + \cdots\right\}^2} = 2\overline{\left(\frac{\partial b_\theta}{\partial x}\right)^2} + 12\overline{\frac{\partial b_\theta}{\partial x}\frac{\partial d_\theta}{\partial x}}y^2 + \cdots \\
2\overline{\left\{\frac{\partial}{\partial x}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2} &= 2\overline{\left\{\frac{\partial^2 b_\theta}{\partial x \partial z}y + \frac{\partial^2 d_\theta}{\partial x \partial z}y^3 + \frac{\partial^2 e_\theta}{\partial x \partial z}y^4 + \cdots\right\}^2} = 2\overline{\left(\frac{\partial^2 b_\theta}{\partial x \partial z}\right)^2}y^2 + \cdots \\
\overline{\left\{\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial y}\right)\right\}^2} &= \overline{(6d_\theta y + 12e_\theta y^2 + \cdots)^2} = 36\overline{d_\theta^2}y^2 + \cdots \\
2\overline{\left\{\frac{\partial}{\partial y}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2} &= 2\overline{\left\{\frac{\partial b_\theta}{\partial z} + 3\frac{\partial d_\theta}{\partial z}y^2 + 4\frac{\partial e_\theta}{\partial z}y^3 + \cdots\right\}^2} = 2\overline{\left(\frac{\partial b_\theta}{\partial z}\right)^2} + 12\overline{\frac{\partial b_\theta}{\partial z}\frac{\partial d_\theta}{\partial z}}y^2 + \cdots \\
\overline{\left\{\frac{\partial}{\partial z}\left(\frac{\partial \theta'}{\partial z}\right)\right\}^2} &= \overline{\left\{\frac{\partial^2 b_\theta}{\partial z^2}y + \frac{\partial^2 d_\theta}{\partial z^2}y^3 + \frac{\partial^2 e_\theta}{\partial z^2}y^4 + \cdots\right\}^2} = \overline{\left(\frac{\partial^2 b_\theta}{\partial z^2}\right)^2}y^2 + \cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the dissipation term can be given as below.

$$
\begin{aligned}
\gamma_{\varepsilon_\theta} &= 2\kappa^2\left[2\left\{\overline{\left(\frac{\partial b_\theta}{\partial x}\right)^2} + \overline{\left(\frac{\partial b_\theta}{\partial z}\right)^2}\right\}\right] \\
&+ 2\kappa^2\left[\overline{\left(\frac{\partial^2 b_\theta}{\partial x^2}\right)^2} + 12\overline{\left(\frac{\partial b_\theta}{\partial x}\right)\left(\frac{\partial d_\theta}{\partial x}\right)} + 2\overline{\left(\frac{\partial^2 b_\theta}{\partial x \partial z}\right)^2} + 36\overline{d_\theta^2}\right. \\
&+ \left.12\overline{\left(\frac{\partial b_\theta}{\partial z}\right)\left(\frac{\partial d_\theta}{\partial z}\right)} + \overline{\left(\frac{\partial^2 b_\theta}{\partial z^2}\right)^2}\right]y^2 + \mathrm{O}\left(y^3\right) \qquad (28)
\end{aligned}
$$

- Turbulent production

$$
\begin{aligned}
P_{\varepsilon_\theta}^{(4)} &= -\ 2\kappa\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}\frac{\partial \theta'}{\partial x_j}} \\
&= -\ 2\kappa\left(\overline{\frac{\partial u_j'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial x_j}} + \overline{\frac{\partial u_j'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial x_j}}\right) \\
&= -\ 2\kappa\left(\overline{\frac{\partial u'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial x}} + \overline{\frac{\partial v'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial y}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial z}}\right) \\
&\quad -\ 2\kappa\left(\overline{\frac{\partial u'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial x}} + \overline{\frac{\partial v'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial y}} + \overline{\frac{\partial w'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial z}}\right) \\
&\quad -\ 2\kappa\left(\overline{\frac{\partial u'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial x}} + \overline{\frac{\partial v'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial y}} + \overline{\frac{\partial w'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial z}}\right)
\end{aligned} \tag{29}
$$

Using Eq. (2) $\sim$ (4), following equations can be obtained.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial b_1}{\partial x}y + \frac{\partial c_1}{\partial x}y^2 + \frac{\partial d_1}{\partial x}y^3 + \cdots \\
\frac{\partial v}{\partial x} &= \frac{\partial c_2}{\partial x}y^2 + \frac{\partial d_2}{\partial x}y^3 + \cdots \\
\frac{\partial w}{\partial x} &= \frac{\partial b_3}{\partial x}y + \frac{\partial c_3}{\partial x}y^2 + \frac{\partial d_3}{\partial x}y^3 + \cdots \\
\frac{\partial u}{\partial y} &= b_1 + 2c_1y + 3d_1y^2 + \cdots \\
\frac{\partial v}{\partial y} &= 2c_2y + 3d_2y^2 + \cdots \\
\frac{\partial w}{\partial y} &= b_3 + 2c_3y + 3d_3y^2 + \cdots \\
\frac{\partial u}{\partial z} &= \frac{\partial b_1}{\partial z}y + \frac{\partial c_1}{\partial z}y^2 + \frac{\partial d_1}{\partial z}y^3 + \cdots \\
\frac{\partial v}{\partial z} &= \frac{\partial c_2}{\partial z}y^2 + \frac{\partial d_2}{\partial z}y^3 + \cdots \\
\frac{\partial w}{\partial z} &= \frac{\partial b_3}{\partial z}y + \frac{\partial c_3}{\partial z}y^2 + \frac{\partial d_3}{\partial z}y^3 + \cdots
\end{aligned}
$$

By substituting the near-wall expansions above into individual components in Eq. (29), the each term can be expanded as follows:

$$
\begin{aligned}
\overline{\frac{\partial u'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial x}} &= \overline{\frac{\partial b_1}{\partial x}y \cdot \frac{\partial b_\theta}{\partial x}y \cdot \frac{\partial b_\theta}{\partial x}y + \cdots} = \overline{\left(\frac{\partial b_1}{\partial x}\right)\left(\frac{\partial b_\theta}{\partial x}\right)^2}y^3 + \cdots \\
\overline{\frac{\partial v'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial y}} &= \overline{\frac{\partial c_2}{\partial x}y^2 \cdot \frac{\partial b_\theta}{\partial x}y \cdot b_\theta + \cdots} = \overline{b_\theta\left(\frac{\partial c_2}{\partial x}\right)\left(\frac{\partial b_\theta}{\partial x}\right)}y^3 + \cdots \\
\overline{\frac{\partial w'}{\partial x}\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial z}} &= \overline{\frac{\partial b_3}{\partial x}y \cdot \frac{\partial b_\theta}{\partial x}y \cdot \frac{\partial b_\theta}{\partial z}y + \cdots} = \overline{\left(\frac{\partial b_3}{\partial x}\right)\left(\frac{\partial b_\theta}{\partial x}\right)\left(\frac{\partial b_\theta}{\partial z}\right)}y^3 + \cdots \\
\overline{\frac{\partial u'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial x}} &= \overline{b_1 \cdot b_\theta \cdot \frac{\partial b_\theta}{\partial x}y + \cdots} = \overline{b_1b_\theta\left(\frac{\partial b_\theta}{\partial x}\right)}y + \cdots \\
\overline{\frac{\partial v'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial y}} &= \overline{2c_2y \cdot b_\theta \cdot b_\theta + \cdots} = 2\overline{c_2b_\theta^2}y + \cdots \\
\overline{\frac{\partial w'}{\partial y}\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial z}} &= \overline{b_3 \cdot b_\theta \cdot \frac{\partial b_\theta}{\partial z}y + \cdots} = \overline{b_3b_\theta\left(\frac{\partial b_\theta}{\partial z}\right)}y + \cdots
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial u'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial x}} &= \overline{\frac{\partial b_1}{\partial z}y\cdot\frac{\partial b_\theta}{\partial z}y\cdot\frac{\partial b_\theta}{\partial x}y+\cdots} = \overline{\left(\frac{\partial b_1}{\partial z}\right)\left(\frac{\partial b_\theta}{\partial z}\right)\left(\frac{\partial b_\theta}{\partial x}\right)}y^3+\cdots \\
\overline{\frac{\partial v'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial y}} &= \overline{\frac{\partial c_2}{\partial z}y^2\cdot\frac{\partial b_\theta}{\partial z}y\cdot b_\theta+\cdots} = \overline{b_\theta\left(\frac{\partial c_2}{\partial z}\right)\left(\frac{\partial b_\theta}{\partial z}\right)}y^3+\cdots \\
\overline{\frac{\partial w'}{\partial z}\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial z}} &= \overline{\frac{\partial b_3}{\partial z}y\cdot\frac{\partial b_\theta}{\partial z}y\cdot\frac{\partial b_\theta}{\partial z}y+\cdots} = \overline{\left(\frac{\partial b_3}{\partial z}\right)\left(\frac{\partial b_\theta}{\partial z}\right)\left(\frac{\partial b_\theta}{\partial z}\right)}y^3+\cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the turbulent production term can be written as below.

$$
P_{\varepsilon_\theta}^{(4)} = -2\kappa\left[\overline{b_1 b_\theta\left(\frac{\partial b_\theta}{\partial x}\right)} + 2\overline{c_2 b_\theta^2} + \overline{b_3 b_\theta\left(\frac{\partial b_\theta}{\partial z}\right)}\right]y + \mathrm{O}\left(y^2\right) \tag{30}
$$

- Turbulent diffusion

$$
\begin{aligned}
T_{\varepsilon_\theta} &= -\kappa\frac{\partial}{\partial x_j}\overline{\left(u_j'\frac{\partial \theta'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}\right)} \\
&= -\kappa\left\{\frac{\partial}{\partial y}\overline{\left(v'\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial x}\right)} + \frac{\partial}{\partial y}\overline{\left(v'\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial y}\right)} + \frac{\partial}{\partial y}\overline{\left(v'\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial z}\right)}\right\}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\overline{\left(v'\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial x}\right)} &= \overline{c_2 y^2\cdot\frac{\partial b_\theta}{\partial x}y\cdot\frac{\partial b_\theta}{\partial x}y+\cdots} = \overline{c_2\left(\frac{\partial b_\theta}{\partial x}\right)^2}y^4+\cdots \\
\overline{\left(v'\frac{\partial \theta'}{\partial y}\frac{\partial \theta'}{\partial y}\right)} &= \overline{c_2 y^2\cdot b_\theta\cdot b_\theta+\cdots} = \overline{c_2 b_\theta^2}y^2+\cdots \\
\overline{\left(v'\frac{\partial \theta'}{\partial z}\frac{\partial \theta'}{\partial z}\right)} &= \overline{c_2 y^2\cdot\frac{\partial b_\theta}{\partial z}y\cdot\frac{\partial b_\theta}{\partial z}y+\cdots} = \overline{c_2\left(\frac{\partial b_\theta}{\partial z}\right)^2}y^4+\cdots
\end{aligned}
$$

Therefore, the near-wall expansion of the turbulent diffusion term can be written as below.

$$
T_{\varepsilon_\theta} = -\kappa\left\{2\overline{c_2 b_\theta^2}y + O\left(y^2\right)\right\} \tag{32}
$$

- Gradient production

$$
P_{\varepsilon_\theta}^{(3)} = -2\kappa\overline{u_j'\frac{\partial \theta'}{\partial x_k}}\frac{\partial^2\overline{\theta}}{\partial x_j\partial x_k} = -2\kappa\overline{v'\frac{\partial \theta'}{\partial y}}\frac{\partial^2\overline{\theta}}{\partial y^2} \tag{33}
$$

$$
= -2\kappa\left(\overline{c_2 b_\theta}y^2+\cdots\right)\frac{\partial^2\overline{\theta}}{\partial y^2} \tag{34}
$$

- Production by mean temperature gradient

$$
\begin{aligned}
P_{\varepsilon_\theta}^{(1)} &= -2\kappa\overline{\frac{\partial u_j'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}\frac{\partial\overline{\theta}}{\partial x_j} = -2\kappa\overline{\frac{\partial v'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}\frac{\partial\overline{\theta}}{\partial y} \\
&= -2\kappa\left(\overline{\frac{\partial v'}{\partial x}\frac{\partial \theta'}{\partial x}}\frac{\partial\overline{\theta}}{\partial y} + \overline{\frac{\partial v'}{\partial y}\frac{\partial \theta'}{\partial y}}\frac{\partial\overline{\theta}}{\partial y} + \overline{\frac{\partial v'}{\partial z}\frac{\partial \theta'}{\partial z}}\frac{\partial\overline{\theta}}{\partial y}\right)
\end{aligned} \tag{35}
$$

$$\begin{aligned}
\overline{\frac{\partial v'}{\partial x}\frac{\partial \theta'}{\partial x}} &= \overline{\frac{\partial c_2}{\partial x}y^2 \cdot \frac{\partial b_\theta}{\partial x}y + \cdots} \\
\overline{\frac{\partial v'}{\partial y}\frac{\partial \theta'}{\partial y}} &= \overline{2c_2 y \cdot b_\theta + \cdots} \\
\overline{\frac{\partial v'}{\partial z}\frac{\partial \theta'}{\partial z}} &= \overline{\frac{\partial c_2}{\partial z}y^2 \cdot \frac{\partial b_\theta}{\partial z}y + \cdots}
\end{aligned}$$

$$P^{(1)}_{\varepsilon_\theta} = -2\kappa \left(2\overline{c_2 b_\theta}y + \cdots\right)\frac{\partial \overline{\theta}}{\partial y} \tag{36}$$

- Production by mean velocity gradient

$$P^{(2)}_{\varepsilon_\theta} = -2\kappa \overline{\frac{\partial \theta'}{\partial x_j}\frac{\partial \theta'}{\partial x_k}}\frac{\partial \overline{u_j}}{\partial x_k} = -2\kappa \overline{\frac{\partial \theta'}{\partial x}\frac{\partial \theta'}{\partial y}}\frac{\partial \overline{u}}{\partial y} \tag{37}$$

$$= -2\kappa \overline{\left(\frac{\partial b_\theta}{\partial x}y \cdot b_\theta + \cdots\right)}\frac{\partial \overline{u}}{\partial y} = -2\kappa \left\{\overline{b_\theta\left(\frac{\partial b_\theta}{\partial x}\right)}y + \cdots\right\}\frac{\partial \overline{u}}{\partial y} \tag{38}$$

Reference

[1] Kozuka, M., Seki, Y. and Kawamura, H., 2008, "DNS of turbulent heat transfer in a channel flow with a high spatial resolution", *Proc. of 7th Int. Symp. on Engineering Turbulence Modelling and Measurements(ETMM7)*, Vol. 1, pp. 163-168.

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[3] Kawamura, H., Abe, H. and Matsuo, Y., 1999, "DNS of turbulent heat transfer in a channel flow with respect to Reynolds and Prandtl number effects", *Int. J. Heat and Fluid Flow*, **20**, pp. 196-207.