Multiplicative Quiver Varieties and Riemann-Hilbert Correspondence

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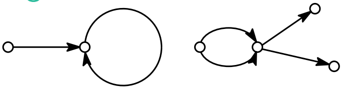
Additive / Multiplicative Quiver Varieties

Fix a finite quiver (directed graph)

e.g.



I: vertex set, Ω: arrow set

Main objects in this talk

- Additive quiver variety (Nakajima 1994)

$$\mathfrak{M}_{\zeta,\theta}(v), \quad v \in \mathbb{Z}_{\geq 0}^{I},\ \zeta \in \mathbb{C}^{I},\ \theta \in \mathbb{Q}^{I}$$

- Multiplicative quiver variety (Crawley-Boevey and Shaw 2004, Y 2007)

$$\mathcal{M}_{q,\theta}(v), \quad v \in \mathbb{Z}_{\geq 0}^{I},\ q \in (\mathbb{C}^{*})^{I},\ \theta \in \mathbb{Q}^{I}$$

Additive Quiver Variety

Take $V = (V_i)_{i \in I}$: family of $\mathbb{C}$-vec sps with $\dim V_i = v_i$

Set

$$\mathbf{M}_V := \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{\mathrm{s}(h)}, V_{\mathrm{t}(h)}) \oplus \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{\mathrm{t}(h)}, V_{\mathrm{s}(h)})$$

It has a natural action of $G_V := \prod_{i \in I} \mathrm{GL}(V_i)$

Define $\mu_i \colon \mathbf{M}_{\mathbf{V}} \to \mathfrak{gl}(V_i)$ $(i \in I)$ by

$$\mu_i(x, y) = \sum_{\substack{h \in \Omega \\ \mathrm{t}(h) = i}} x_h y_h - \sum_{\substack{h \in \Omega \\ \mathrm{s}(h) = i}} y_h x_h$$

$$\mathfrak{M}^{\mathrm{reg}}_{\zeta, \theta}(v) := \left\{ (x, y) \in \mathbf{M}_V \;\middle|\; \begin{array}{l} \mu_i(x, y) = \zeta_i 1_{V_i} \\ (x, y) : \theta\text{-stable} \end{array} \right\} / G_V$$

Symplectic Structure

$\mathfrak{M}^{\mathbf{reg}}_{\zeta,\theta}(v)$ has a (holomorphic) symplectic structure

Why?

- $\mathbf{M}_V$ is a (complex) symplectic vector space
- $G_V \curvearrowright \mathbf{M}_V$ is Hamiltonian with moment map μ
- $\mathfrak{M}^{\mathbf{reg}}_{\zeta,\theta}(v)$ is a level set of μ modulo G_V-action

$\Longrightarrow$ $\mathfrak{M}^{\mathbf{reg}}_{\zeta,\theta}(v)$ is a symplectic quotient

Remark

- $\mathfrak{M}^{\mathbf{reg}}_{\zeta,\theta}(v)$ has in fact a hyper-Kähler structure
- $\mathfrak{M}^{\mathbf{reg}}_{\zeta,\theta}(v)$ does not depend (up to isom) on the orientation of (I, Ω)

Set

$$\mathbf{M}_V^\circ := \{\, (x, y) \in \mathbf{M}_V \mid \det(1 + y_h x_h) \neq 0 \,\}$$

Fix a total ordering on Ω and define $\Phi_i \colon \mathbf{M}_V^\circ \to \mathrm{GL}(V_i)$ by

$$\Phi_i(x_h, y_h) := \prod_{\substack{h \in \Omega \\ \mathrm{t}(h)=i}} (1_{V_i} + x_h y_h) \prod_{\substack{h \in \Omega \\ \mathrm{s}(h)=i}} (1_{V_i} + y_h x_h)^{-1}$$

$$\mathcal{M}_{q,\theta}^{\mathrm{reg}}(v) := \left\{ (x, y) \in \mathbf{M}_V^\circ \;\middle|\; \begin{array}{l} \Phi_i(x, y) = q_i 1_{V_i} \\ (x, y) : \theta\text{-stable} \end{array} \right\} / G_V$$

Symplectic Structure

$\mathcal{M}^{\mathrm{reg}}_{q,\theta}(v)$ has a holomorphic symplectic structure

Why?

- [Alekseev-Malkin-Meinrenken 1998] Theory of quasi-Hamiltonian action and group-valued moment map
- [Van den Bergh 2007] $G_V \curvearrowright \mathbf{M}^\circ_V$ is quasi-Hamiltonian with group-valued moment map Φ
- $\mathcal{M}^{\mathrm{reg}}_{q,\theta}(v)$ is a level set of Φ modulo by G_V-action

$\Longrightarrow \mathcal{M}^{\mathrm{reg}}_{q,\theta}(v)$ is a group-valued symplectic quotient

Remark

$\mathcal{M}^{\mathrm{reg}}_{q,\theta}(v)$ does not depend (up to isom) both on

- the orientation of (I, Ω)
- the ordering on Ω

Surprising Similarity: Additive Reflection Functor

Let $i \in I$ be a loopfree vertex

Let $(\ ,\ )$ be a symmetric bilinear form on $\mathbb{Z}^I$ associated to (I, Ω);

$$(\alpha, \beta) := 2 \sum_{i \in I} \alpha_i \beta_i - \sum_{h \in \Omega} (\alpha_{\mathrm{t}(h)} \beta_{\mathrm{s}(h)} + \alpha_{\mathrm{s}(h)} \beta_{\mathrm{t}(h)})$$

Define $s_i \colon \mathbb{Z}^I \to \mathbb{Z}^I$ by $s_i(\alpha) = \alpha - (\alpha, e_i) e_i$

Set $r_i := {}^t s_i$

Theorem (Lusztig, Maffei, Nakajima)

If $v \neq e_i$, there is an isomorphism of algebraic varieties

$$\mathfrak{M}^{\mathrm{reg}}_{\zeta,\theta}(v) \simeq \mathfrak{M}^{\mathrm{reg}}_{r_i(\zeta), r_i(\theta)}(s_i(v))$$

Surprising Similarity: Multiplicative Reflection Functor

There is $u_i\colon (\mathbb{C}^*)^I \to (\mathbb{C}^*)^I$ s.t. $u_i(\exp\zeta) = \exp r_i(\zeta)$

Theorem (Crawley-Boevey–Shaw, Y)

If $v \neq e_i$, there is an isomorphism of algebraic varieties

$$\mathcal{M}^{\mathrm{reg}}_{q,\theta}(v) \simeq \mathcal{M}^{\mathrm{reg}}_{u_i(q),r_i(\theta)}(s_i(v))$$

λ-parabolic Connections

Take distinct pts $p_1, p_2, \ldots, p_n \in \mathbb{P}^1$ ($D := \sum p_i$)
Let $\lambda = (\lambda_i^j \mid i = 1, \ldots, n,\ j = 1, \ldots, r)$, $\lambda_i^j \in \mathbb{C}$
A **λ-parabolic connection** on $(\mathbb{P}^1, D)$ consists of —

- E: holomorphic vector bundle on $\mathbb{P}^1$
- ∇: holomorphic connection on $E|_{\mathbb{P}^1 \setminus D}$ s.t. near each p_i,

$$\nabla = d + \frac{A(z)}{z - p_i} dz, \quad A(z)\text{: holomorphic}$$

- full filtration $E_{p_i} = E_i^0 \supset E_i^1 \supset \cdots \supset E_i^r = 0$ s.t.

$$(A(0) - \lambda_i^j 1)(E_i^{j-1}) \subset E_i^j$$

Moduli of λ-parabolic Connections

$\alpha = (\alpha_i^j)_{1 \le i \le n, 1 \le j \le r}$ with $0 \le \alpha_i^j < \alpha_i^{j+1} < 1$
defines the α-stability condition for λ-parabolic connections

Theorem (Inaba-Iwasaki-Saito)

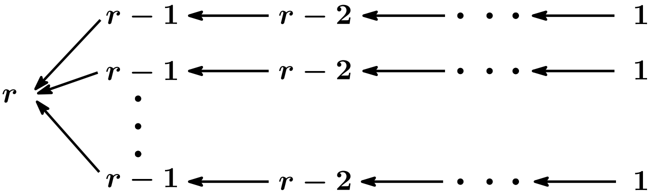
The moduli space $\mathcal{M}_{\lambda,\alpha}(r)$ of α-stable λ-parabolic connections on $(\mathbb{P}^1, D)$ of rank r has naturally a holomorphic symplectic structure

Theorem

Assume α is generic and $\alpha_i^j - \mathrm{Re}\,\lambda_i^j \in \mathbb{Q}$

Then there is a symplectic biholomorphic map between

- $\mathcal{M}_{\lambda,\alpha}(r)$*; and*
- $\mathcal{M}_{q,\theta}(v)$ *associated to the full star-shaped quiver with n legs*



and some q, θ

Monodromy Map

$$\begin{array}{ccc} \mathcal{M}_{q,\theta}(v) & \xrightarrow[\text{not algebraic!}]{\text{symp biholo}} & \mathcal{M}_{\lambda,\alpha}(r) \\ \Big\downarrow \pi & \circlearrowright & \Big\downarrow \text{monodromy map} \\ \mathcal{M}_{q,0}(v) & \overset{\iota}{\hookrightarrow} & \mathbf{Hom}(\pi_1(\mathbb{P}^1 \setminus D), \mathbf{GL}_r(\mathbb{C})) /\!/ \mathbf{GL}_r(\mathbb{C}) \end{array}$$

- ι is a closed immersion of algebraic varieties
- π is a natural projective map

Direct Relation between Additive and Multiplicative

Theorem

There exist

- U*: open neighborhood of* $[0] \in \mathcal{M}_{1,0}(v)$
- U'*: open neighborhood of* $[0] \in \mathfrak{M}_{0,0}(v)$

and

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\tilde{f}} & \pi^{-1}(U') \\ \downarrow \pi & \circlearrowright & \downarrow \pi \\ (U, [0]) & \xrightarrow{f} & (U', [0]) \end{array}$$

such that

- $f,\ \tilde{f}$*: complex analytic isomorphism*
- $\tilde{f} \colon \pi^{-1}(U)^{\mathrm{reg}} \to \pi^{-1}(U')^{\mathrm{reg}}$ *symplectomorphic*