Hamiltonians for isomonodromic deformations

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Non-autonomous Hamiltonian system

Consider a non-autonomous Hamiltonian system

$$\begin{cases} \dfrac{\partial q_j}{\partial t_i} = \dfrac{\partial H_i}{\partial p_j}, \\ \dfrac{\partial p_j}{\partial t_i} = -\dfrac{\partial H_i}{\partial q_j} \end{cases} \quad (i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$$

with Hamiltonians $H_i(p_1, \ldots, p_n, q_1, \ldots, q_n, t_1, \ldots, t_m)$ $(i = 1, 2, \ldots, m)$.
The system corresponds to the distribution $\mathcal{H} \subset TM$ on $M := \mathbb{C}^{2n} \times \mathbb{C}^m$ spanned by the vector fields

$$X_i = \frac{\partial}{\partial t_i} + \sum_{j=1}^{n} \left(\frac{\partial H_i}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H_i}{\partial q_j} \frac{\partial}{\partial p_j} \right) \quad (i = 1, 2, \ldots, m).$$

Fundamental two-form

The phase space M is a fiber bundle over the time space $B = \mathbb{C}^m$ with each fiber $\mathbb{C}^{2n}$ equipped with the symplectic form $\sum dp_j \wedge dq_j$.

The fundamental two-form for the system is

$$\Omega := \sum_{j=1}^{n} dp_j \wedge dq_j - \sum_{i=1}^{m} dH_i \wedge dt_i.$$

Properties of the fundamental two-form

(1) The restriction of Ω to each fiber is the symplectic form.

(2) $\mathcal{H}$ is Ω-orthogonal to any vector tangent to a fiber.

(3) $\dfrac{\partial H_i}{\partial t_j} - \dfrac{\partial H_j}{\partial t_i} = \{H_i, H_j\}$ $(1 \leq i, j \leq m)$ if and only if $\mathcal{H} = \operatorname{Ker}\Omega$, i.e., $\iota(X_i)\Omega = 0$ $(i = 1, 2, \ldots, m)$. In this case the distribution is integrable.

Fundamental two-form for IMD

Question

What are the fundamental two-forms for the isomonodromic deformations?

We know the fundamental two-forms for

- Various "Painlevé type" equations
  (Okamoto, Noumi–Yamada, Sasano, Fuji–Suzuki–Tsuda,
  Kawakami–Nakamura–Sakai, e.t.c.)
- Schlesinger equation and its higher genus version
- Jimbo–Miwa–Môri–Sato's equation
- Simply-laced isomonodromy system

Assumption

Consider the isomonodromic deformations of a system of linear ODEs

$$\frac{dy}{dx} = A(x)y, \quad A(x) \in M_n(\mathbb{C}(x))$$

with poles $a_0 = \infty$, $a_1, \ldots, a_m$. Put $x_0 = 1/x$, $x_i = x - a_i$ $(i = 1, 2, \ldots, m)$.

Assumption

For each i the system has a formal fundamental solution $\widehat{Y}_i = \widehat{F}_i e^{\Lambda_i} x_i^{L_i}$, where

- $\widehat{F}_i = F_0^{(i)} + F_1^{(i)} x_i + F_2^{(i)} x_i^2 + \cdots \in \mathrm{GL}_n(\mathbb{C}[\![x_i]\!])$.
- $\Lambda_i = \dfrac{\Lambda_1^{(i)}}{x_i} + \cdots + \dfrac{\Lambda_{r_i}^{(i)}}{x_i^{r_i}}$ with each $\Lambda_j^{(i)}$ diagonal (irregular type).
- $L_i \in \mathfrak{h}_i := \{\, X \in M_n(\mathbb{C}) \mid [X, \Lambda_i] = 0 \,\}$ with $\mathrm{ad}_{L_i} \in \mathrm{End}(\mathfrak{h}_i)$ having no nonzero integral eigenvalues (matrix of exponents of formal monodromy).

We may further assume $F_0^{(0)} = I_n$ by normalizing A.

Local existence of IMD

Deform $a_1, \ldots, a_m$ and $\Lambda_0, \ldots, \Lambda_m$ (with small parameter $t \in \Delta \subset \mathbb{C}^N$) so that

- $a_i(t) \neq a_j(t)$ $(0 \leq i < j \leq m)$.
- the pole order of the difference of any two diagonal elements of each $\Lambda_i(t)$ is constant.

Along such a deformation we may deform the system $dy/dx = A(x)y$ so that

$$\frac{\partial A}{\partial t_i} = \frac{\partial B_i}{\partial x} + [B_i, A] \quad (i = 1, 2, \ldots, N)$$

for some matrix-valued functions $B_i(x, t)$ (isomonodromic deformation).

Jimbo–Miwa–Ueno's approach

For $t \in \Delta$ let $\mathcal{M}_t$ be the set of tuples $(A(x), g_0 = I_n, g_1, \ldots, g_m)$, where

- the system $dy/dx = A(x)y$ has poles at $a_0(t) = \infty, a_1(t), \ldots, a_m(t)$;
- the system has the formal fundamental solution $\widehat{F}_i e^{\Lambda_i(t)} x_i^{L_i}$ for some L_i and $\widehat{F}_i \in \mathrm{GL}_n(\mathbb{C}[\![x_i]\!])$ with $\widehat{F}_i(0) = g_i$.

Following Jimbo–Miwa–Ueno, we can describe the isomonodromic deformations as an Ehresmann connection (isomonodromy connection) on the fiber bundle

$$\mathcal{M} := \bigcup_{t \in \Delta} \mathcal{M}_t \to \Delta.$$

Each fiber $\mathcal{M}_t$ has a canonical symplectic structure [Boalch].

Fundamental two-form for IMD

It is known that $\widehat{F}_i, L_i$ are unique for given $(A, g_0, \ldots, g_m) \in \mathcal{M}$ and so define functions on $\mathcal{M}$.

Theorem [Y]

For each i define a one-form θ_i on $\mathcal{M}$ by

$$\theta_i = \operatorname*{res}_{x_i=0} \operatorname{tr}\left[\left(\Lambda_i' + \frac{L_i}{x_i}\right)\widehat{F}_i^{-1} d\widehat{F}_i\right].$$

Then the two-form $\Omega := \sum_{i=0}^m d\theta_i$ satisfies the following conditions:

- the restriction of Ω to each fiber is the symplectic form.
- $\iota(X)\Omega = 0$ for any horizontal vector X.

Corollary

The isomonodromy connection on $\mathcal{M}$ is flat and symplectic.

How to find Hamiltonians

Question

What are the Hamiltonians for the IMD?

Approach

Hiroe–Y's quiver description gives a trivialization $\mathcal{M} \simeq \mathcal{M}_* \times \Delta$ of the symplectic fiber bundle $\mathcal{M} \to \Delta$.

Let ω_* be the symplectic form on $\mathcal{M}_*$ and $\mathrm{pr}\colon \mathcal{M} \to \mathcal{M}_*$ be the first projection.

If Ω is expressed as

$$\Omega = \mathrm{pr}^* \omega_* - d\varpi, \quad \varpi = \sum_{i=1}^{N} H_i dt_i$$

then H_i are Hamiltonians (ϖ is called the Hamiltonian one-form).

Triangular decomposition

The irregular type $\Lambda_i(t) = \sum_{j=1}^{r_i} \Lambda_j^{(i)} x_i^{-j}$ defines

$$\mathfrak{h}_j^{(i)} = \mathfrak{z}(\Lambda_{j+1}^{(i)}, \Lambda_{j+2}^{(i)}, \ldots, \Lambda_{r_i}^{(i)}) \subset \mathfrak{gl}_n(\mathbb{C}) \quad (j = 0, 1, \ldots, r_i - 1).$$

Since Λ_i is diagonal we may assume that each $\mathfrak{h}_j^{(i)}$ consists of block diagonal matrices

$$\mathrm{diag}(C_1, C_2, \ldots, C_l), \quad C_k \in M_{n_k}(\mathbb{C})$$

for some partition $n = \sum_{k=1}^{l} n_k$ (determined from i, j).
Let

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{n}_{j,-}^{(i)} \oplus \mathfrak{h}_j^{(i)} \oplus \mathfrak{n}_{j,+}^{(i)}$$

be the block triangular decomposition. It does not depend on $t \in \Delta$.

Hamiltonians

We can uniquely decompose $\widehat{F}_i = g_i F_i^- F_i^+ e^{h_i}$, where

- $F_i^{\pm} - I_n \in \mathfrak{n}_{1,\pm}^{(i)} x_i + \mathfrak{n}_{2,\pm}^{(i)} x_i^2 + \cdots + \mathfrak{n}_{r_i-1,\pm}^{(i)} x_i^{r_i-1}$,
- $h_i = h_1^{(i)} x_i + h_2^{(i)} x_i^2 + \cdots$ with $h_j^{(i)} \in \mathfrak{h}_j^{(i)}$ for $j < r_i$.

Theorem [Y]

The one-form

$$\varpi = \sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}\left(h_i d_{\mathrm{irr}} \Lambda_i'\right) + \frac{1}{2} \sum_{i=1}^{m} \operatorname*{res}_{x_i=0} \mathrm{tr}(A^2) da_i$$

is the Hamiltonian one-form for the isomonodromy connection w.r.t. Hiroe–Y's trivialization.

Some remarks

- In general ϖ is different to the one-form of Jimbo–Miwa–Ueno

$$-\sum_{i=0}^{m} \operatorname*{res}_{x_i=0} \operatorname{tr}\left[(\Lambda_i + L_i d\log x_i)\widehat{F}_i^{-1}\widehat{F}_i'\right].$$

- If we write $\varpi = \sum_{i=1}^{N} H_i dt_i$ then

$$\frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} = \{H_i, H_j\} \quad (1 \leq i, j \leq N).$$

- Our construction looks closely related to Fedorov's description of isomonodromic deformations as autonomous Hamiltonian systems.