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Permanence for multi-species nonautonomous Lotka-Volterra cooperative systems

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Abstract. In this paper, we establish sufficient conditions under which Lotka-Volterra cooperative systems are permanent for the n-dimensional case. We improve the result of [G. Lu and Z. Lu, Permanence for two species Lotka-Volterra systems with delays, Math. Biosci. Engi. 5 (2008), 477-484] for the 2-dimensional case in that no restrictions of the size of time delays are needed. When the interval of time delays is constant, we further show that the restriction of the size of time delays is not required for the case $n = 2$, but it is required for the case $n \geq 3$ to obtain lower bounds of solutions. An example is offered to illustrate the feasibility of our results.

Keywords: Lotka-Volterra systems, delay differential equations, permanence.
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1 Introduction

In this paper, we consider the following n-dimensional Lotka-Volterra system with multiple delays:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)\left[r_1(t) - \displaystyle\sum_{\substack{j=1\\ j\neq 2}}^{n}\sum_{l=0}^{m} a_{1j}^l(t)x_j(t-l\tau) + a_{12}^1(t)x_2(t-\tau)\right], \\ \dfrac{dx_i(t)}{dt} = x_i(t)\left[r_i(t) - \displaystyle\sum_{\substack{j=1\\ j\neq i+1}}^{n}\sum_{l=0}^{m} a_{ij}^l(t)x_j(t-l\tau) + \sum_{l=0}^{m} a_{i\,i+1}^l(t)x_{i+1}(t-l\tau)\right],\ i=2,\dots,n-1, \\ \dfrac{dx_n(t)}{dt} = x_n(t)\left[r_n(t) - \displaystyle\sum_{j=2}^{n}\sum_{l=0}^{m} a_{nj}^l(t)x_j(t-l\tau) + \sum_{l=0}^{m} a_{n1}^l(t)x_1(t-l\tau)\right],\ t\geq 0, \end{cases} \tag{1.1}$$

where $\tau \geq 0$, each $r_i(t)$ and $a_{ij}^l(t)$ are bounded continuous functions on $[0, +\infty)$ and

$$\inf_{0\leq t<+\infty} r_i(t) > 0,\ \sum_{l=0}^{m}\left(\inf_{0\leq t<+\infty} a_{ii}^l(t)\right) > 0,\ a_{ij}^l(t) \geq 0,\ 1\leq i,j\leq n \ and \ 0\leq l\leq m,$$

with initial conditions $x_i(\theta) = \phi_i(\theta) \geq 0,\ \ \theta \in [-m\tau, 0),\ \ \phi_i(0) > 0\ (1 \leq i \leq n)$. Our method of analysis is motivated by the idea developed by Nakata and Muroya [8] and they consider the specific case that the interval of time delays is constant. We will discuss the harmlessness of the length of time delays.

Many authors have since studied the permanence of population systems governed by Lotka-Volterra differential systems (see [1-15] and references therein). Ahmad and Lazer [1] have established the average conditions for persistence on the nonautonomous Lotka-Volterra competitive systems with no delays and Xu and Chen [10] have studied the delayed nonautonomous 3-species Lotka-Volterra predator-prey systems without dominating instantaneous negative feedback. On the other hand, there are few papers concerning permanence for multi-species Lotka-Volterra cooperative systems with delays compared to competitive and prey-predator systems, while it is well known that size of time delays may harm permanence for the cooperative systems (see [4, 6, 7]). For example, Lin and Lu [4] consider the following 2-dimensional Lotka-Volterra cooperative systems with delays and obtain sufficient conditions which ensure the system to be permanent.

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)(r_1 - a_1x_1(t) - a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})), \\ \dfrac{dx_2(t)}{dt} = x_2(t)(r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t-\tau_{22})), \end{cases} \tag{1.2}$$

where r_i, a_i, a_{ij} and τ_{ij} are constants with $r_i > 0$, $a_i > 0$ and $\tau_{ij} \geq 0$ $(i, j = 1, 2)$. We assume that system (1.2) is cooperative, that is, $a_{12} > 0$ and $a_{21} > 0$.

Theorem A (See [4, Theorem 1.1].) *System* (1.2) *is permanent if*

$$a_1 - a_{11} > 0, \ a_2 - a_{22} > 0, \ \textit{and} \ (a_1 - a_{11})(a_2 - a_{22}) > a_{12}a_{21}. \tag{1.3}$$

Muroya [7] proved that system (1.2) is persistent if the conditions (1.3) hold. Later, Lin and Lu [4] show that system (1.2) may have unbounded solutions if the conditions (1.3) fail and is permanent if the condition holds. However, in real ecosystems, it is rarely seen that the effects of both intraspecific and interspecific competition occur instantaneously. (See [2, 10].)

Recently, Lu and Lu [6] study the permanence with the following 2-species cooperative system with delays.

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)(r_1 - a_{11}^1 x_1(t-\tau) - a_{11}^2 x_1(t-2\tau) + a_{12}^1 x_2(t-\tau)), \\ \dfrac{dx_2(t)}{dt} = x_2(t)(r_2 + a_{21}^0 x_1(t) + a_{21}^1 x_1(t-\tau) - a_{22}^0 x_2(t) - a_{22}^1 x_2(t-\tau)), \end{cases} \tag{1.4}$$

where $r_i > 0$, $a_{ij}^l > 0$ and $\tau \geq 0$ are constants for $i, j = 1, 2$ and $l = 0, 1, 2$. Note that there is no undelayed intraspecific competition on the first equation of system (1.4). They obtain the following result:

Theorem B (See [6, Theorem 1.3].) *Assume that $a_{21}^0 = 0$ and that there exist constants $C_i > 0$, $D_i \geq 0$ such that $\frac{dx_i(t)}{dt} \leq C_i x_i(t) + D_i$ $(i = 1, 2)$. Then system* (1.4) *is permanent if*

$$\left\{a_{11}^2 (1 - 2r_1\tau) + a_{11}^1 (1 - r_1\tau)\right\} a_{22}^0 - a_{12}^1 a_{21}^1 > 0. \tag{1.5}$$

It is obvious that both the size of time delays $\tau \geq 0$ and growth rate $r_1 > 0$ must be sufficiently small to satisfy the condition (1.5).

Nakata and Muroya [8] establish new sufficient conditions for system (1.4) to be permanent. Remarkably, their conditions no longer depend on the size of time delays. They obtain the following result:

Theorem C (See [8, Corollary 1.2].) *System* (1.4) *is permanent if*

$$a_{11}^1 > a_{21}^0, \ a_{11}^2 > a_{21}^1, \ a_{22}^0 > a_{12}^1. \tag{1.6}$$

2 Main result

In this paper, motivated by the result of Nakata and Muroya [8] for the case $n = 2$, we generalize to a class of n-dimensional Lotka-Volterra systems.

For a continuous bounded function $f(t)$ defined on $[0, +\infty)$, we first set

$$f^L = \inf_{0 \leq t < +\infty} f(t), \ f^M = \sup_{0 \leq t < +\infty} f(t),$$

and

$$P = \begin{cases} \left(\dfrac{r_1^M + r_2^M}{2\sum_{l=0}^{m} \sqrt{b_1^{l+L} b_2^{lL}}} \right)^2 \exp\big((r_1^M + r_2^M)m\tau\big), & if \ n \geq 3, \\ \left(\dfrac{r_1^M + r_2^M}{2\sum_{l=0}^{m} \sqrt{b_1^{l-L} b_2^{lL}}} \right)^2 \exp\big((r_1^M + r_2^M)m\tau\big), & if \ n = 2, \end{cases}$$

where

$$\begin{cases} b_1^0(t) = a_{11}^0(t), \ b_1^{l\pm}(t) = a_{11}^l(t) \pm a_{21}^{l-1}(t-\tau), \ l = 1, \dots, m, \\ b_2^l(t) = a_{22}^l(t-\tau) - a_{12}^{l+1}(t), \ l = 0, \dots, m-1, \ b_2^m(t) = a_{22}^m(t-\tau), \\ b_3^0(t) = a_{13}^0(t), \ b_3^l(t) = a_{13}^l(t) - a_{23}^{l-1}(t-\tau), \ l = 1, \dots, m. \end{cases} \tag{2.1}$$

Next, we put m_i and M_i $(i=1,\dots,n)$ as follows.

$$\begin{cases} A_i^L=\sum_{l=0}^m a_{ii}^{lL},\ \ A_i^M=\sum_{l=0}^m a_{ii}^{lM},\ \ 1\le i\le n,\\ M_1=-\dfrac{a_{12}^{1M}P}{r_1^M}+\left\{\dfrac{a_{12}^{1M}P}{r_1^M}+\left(1/A_1^L\right)\left(r_1^M+\dfrac{a_{12}^{1M}P}{x_1^*}\right)\right\}\exp\left(r_1^M m\tau\right),\\ M_n=\left\{\left(r_n^M+\sum_{l=0}^m a_{n1}^{lM}M_1\right)/A_n^L\right\}\exp\left(\left(r_n^M+\sum_{l=0}^m a_{n1}^{lM}M_1\right)m\tau\right),\\ M_{n-i}=\left\{\left(r_{n-i}^M+\sum_{l=0}^m a_{n-i\ n-(i-1)}^{lM}M_{n-(i-1)}\right)/A_{n-i}^L\right\}\\ \qquad\qquad\times\exp\left(\left(r_{n-i}^M+\sum_{l=0}^m a_{n-i\ n-(i-1)}^{lM}M_{n-(i-1)}\right)m\tau\right),\ \ i=1,\dots,n-2,\\ m_i=\left\{\left(r_i^L-\sum_{\substack{j=1\\ j\ne i,i+1}}^n\sum_{l=0}^m a_{ij}^{lM}M_j\right)/A_i^M\right\}\exp\left(\left(r_i^L-\sum_{\substack{j=1\\ j\ne i+1}}^n\sum_{l=0}^m a_{ij}^{lM}M_j\right)m\tau\right),\ \ i=1,\dots,n, \end{cases}$$

where $x=x_1^*$ is a unique positive solution of the equation:

$$x(r_1^M-A_1^Lx)+a_{12}^{1M}P=0.$$

Our main theorem is as follows.

Theorem 2.1. *For system* (1.1)*, consider the case $n\ge 3$. Assume that*

$$a_{23}^m(t)\equiv 0,\ \ b_2^{lL}\ge 0,\ \ b_3^{lL}\ge 0,\ \ l=0,1,\dots,m,\ \ \sum_{l=0}^m\sqrt{b_1^{l+L}b_2^{lL}}>0, \tag{2.2}$$

and

$$r_i^L-\sum_{\substack{j=1\\ j\ne i,i+1}}^n\sum_{l=0}^m a_{ij}^{lM}M_j>0,\ \ i=1,\dots,n-1,\ \ r_n^L-\sum_{j=2}^{n-1}\sum_{l=0}^m a_{nj}^{lM}M_j>0. \tag{2.3}$$

Then system (1.1) *is permanent, that is,*

$$0<m_i\le\liminf_{t\to+\infty}x_i(t)\le\limsup_{t\to+\infty}x_i(t)\le M_i<+\infty,\ \ i=1,\dots,n.$$

Remark 2.1. For the case $n\ge 3$, if the size of time delay $\tau\ge 0$ is large enough, condition (2.3) is not satisfied, that is, the permanence of system (1.1) is not guaranteed (cf. [6, Example 3.2]).

Here, let us introduce the following result for the specific case $n=2$ for system (1.1).

Corollary 2.1. *For system* (1.1)*, consider the case $n=2$. Assume that*

$$a_{21}^m(t)\equiv 0,\ \ b_1^{l-L}\ge 0,\ \ b_2^{lL}\ge 0,\ \ l=0,1,\dots,m,\ \ \sum_{l=0}^m\sqrt{b_1^{l-L}b_2^{lL}}>0. \tag{2.4}$$

Then system (1.1) *is permanent, that is, there exist positive constants m_i' and M_i' $(i=1,2)$ such that*

$$0<m_i'\le\liminf_{t\to+\infty}x_i(t)\le\limsup_{t\to+\infty}x_i(t)\le M_i'<+\infty,\ \ i=1,2,$$

where

$$\begin{cases} M_1'=-\dfrac{a_{12}^{1M}P}{r_1^M}+\left\{\dfrac{a_{12}^{1M}P}{r_1^M}+\left(1/A_1^L\right)\left(r_1^M+\dfrac{a_{12}^{1M}P}{x_1^*}\right)\right\}\exp\left(r_1^M m\tau\right),\\ M_2'=\left\{\left(r_2^M+\sum_{l=0}^m a_{21}^{lM}M_1'\right)/A_2^L\right\}\exp\left(\left(r_2^M+\sum_{l=0}^m a_{21}^{lM}M_1'\right)m\tau\right),\\ m_i'=\left(r_i^L/A_i^M\right)\exp\left(\left(r_i^L-\sum_{l=0}^m a_{ii}^{lM}M_i'\right)m\tau\right),\ \ i=1,2. \end{cases}$$

Remark 2.2. For the case $n = 2$, system (1.1) becomes cooperative and is permanent for any finite size of time delays under the conditions of Corollary 2.1.

We note that we need some restrictions for $\tau \geq 0$ to be small enough such that condition (2.3) holds for the case $n \geq 3$ since M_i $(i = 1, \ldots, n)$ are monotonically increasing functions with respect to $\tau \geq 0$.

We now obtain the generalized result of Nakata and Muroya [8] for system (1.1) with delays. We improve the known results in Theorem B [6, Theorem 1.3] and Theorem C [8, Corollary 1.2] by using Corollary 2.1. Our result for system (1.4) becomes as follows.

Corollary 2.2. *For system* (1.4), *Assume that*

$$a_{11}^1 > a_{21}^0, \ a_{11}^2 \geq a_{21}^1, \ a_{22}^0 \geq a_{12}^1. \tag{2.5}$$

Then system (1.4) *is permanent.*

It is clear that condition (2.5) improves conditions (1.5) and (1.6) in the meaning that the permanence of system (1.4) is guaranteed for any finite size of time delays and either $a_{11}^2 = a_{21}^1$ or $a_{22}^0 = a_{12}^1$ is allowed, respectively.

3 Basic lemmas

Let us introduce the following basic lemmas. These lemmas play a really important role in this paper because one can show that $x_i(t)$ $(i = 1, \ldots, n)$ are also bounded above if $x_1(t)x_2(t - \tau)$ is bounded above.

Lemma 3.1. *(See [8, Lemma 2.2].)* *Assume that*

$$\underline{\mu} = \sum_{l=0}^{m} \underline{\mu}^l > 0, \ \overline{\lambda} > 0, \ \underline{\mu}^l \geq 0, \tag{3.1}$$

for $0 \leq l \leq m$. *For every positive solution* $y(t)$ *satisfying the following inequality:*

$$\frac{dy(t)}{dt} \leq y(t)\Big(\overline{\lambda} - \sum_{l=0}^{m} \underline{\mu}^l y(t - l\tau)\Big) + D, \tag{3.2}$$

with initial conditions $y(t) = \phi(t) \geq 0$ *for* $t \in [-m\tau, 0)$ *and* $\phi(0) > 0$, *it holds that*

$$\limsup_{t\to+\infty} y(t) \leq M \equiv -\frac{D}{\overline{\lambda}} + \Big\{\frac{D}{\overline{\lambda}} + \frac{1}{\underline{\mu}}\Big(\overline{\lambda} + \frac{D}{y^*}\Big)\Big\} \exp\big(\overline{\lambda} m\tau\big) < +\infty,$$

where $y = y^*$ *is a unique positive solution of the equation;*

$$y(\overline{\lambda} - \underline{\mu} y) + D = 0. \tag{3.3}$$

Lemma 3.2. *(See [8, Lemma 2.3].)* *Assume that*

$$\overline{\mu} = \sum_{l=0}^{m} \overline{\mu}^l > 0, \ \underline{\lambda} > 0, \ \overline{\mu}^l \geq 0, \tag{3.4}$$

for $0 \leq l \leq m$ *and all the hypotheses of Lemma 3.1 hold. Then for every positive solution* $y(t)$ *satisfying the following inequality:*

$$\frac{dy(t)}{dt} \geq y(t)\Big(\underline{\lambda} - \sum_{l=0}^{m} \overline{\mu}^l y(t - l\tau)\Big), \tag{3.5}$$

with initial conditions, $y(t) = \phi(t) \geq 0$ *for* $t \in [-m\tau, 0)$ *and* $\phi(0) > 0$, *it holds that*

$$\liminf_{t\to+\infty} y(t) \geq \frac{\underline{\lambda}}{\overline{\mu}} \exp\Big(\big[\underline{\lambda} - \overline{\mu} M\big] m\tau\Big) > 0.$$

Using Lemmas 3.1 and 3.2, we give a proof of Theorem 2.1. First, we show that $x_1(t)x_2(t - \tau)$ is bounded above and second we show that $x_i(t)$ $(i = 1, \ldots, n)$ are bounded above.

Proof of Theorem 2.1. Every solution of system (1.1) remains positive for all $t \geq 0$. We only need to prove the case $n \geq 3$ because it is similarly proved when $n = 2$. First, we consider the following inequality. There exists a sufficiently large T such that

$$\begin{aligned}
\frac{d}{dt}\left(x_1(t)x_2(t-\tau)\right) \leq \left(x_1(t)x_2(t-\tau)\right)\Bigg[& r_1(t) + r_2(t-\tau) + a_{23}^m(t-\tau)x_3(t-(m+1)\tau) \\
& - a_{11}^0(t)x_1(t) - \sum_{l=1}^{m}(a_{11}^l(t) + a_{21}^{l-1}(t-\tau))x_1(t-l\tau) \\
& - \sum_{l=0}^{m-1}(a_{22}^l(t-\tau) - a_{12}^{l+1}(t))x_2(t-(l+1)\tau) - a_{22}^m(t-\tau)x_2(t-(m+1)\tau) \\
& - a_{13}^0(t)x_3(t) - \sum_{l=1}^{m}(a_{13}^l(t) - a_{23}^{l-1}(t-\tau))x_3(t-l\tau)\Bigg] \\
\leq \left(x_1(t)x_2(t-\tau)\right) & \left[r_1^M + r_2^M - 2\sum_{l=0}^{m}\sqrt{b_1^{l+L}b_2^{lL}x_1(t-l\tau)x_2(t-(l+1)\tau)}\right], \quad t > T.
\end{aligned}$$

Let $u(t) = \sqrt{x_1(t)x_2(t-\tau)}$, then we have

$$\frac{du(t)}{dt} \leq u(t)\left[\frac{r_1^M + r_2^M}{2} - \sum_{l=0}^{m}\sqrt{b_1^{l+L}b_2^{lL}}u(t-l\tau)\right], \quad t > T.$$

From Lemma 3.1, it follows that

$$\limsup_{t\to+\infty} u(t) \leq \frac{r_1^M + r_2^M}{2\sum_{l=0}^{m}\sqrt{b_1^{l+L}b_2^{lL}}}\exp\big((r_1^M + r_2^M)m\tau/2\big) < +\infty,$$

which implies that

$$\limsup_{t\to+\infty} x_1(t)x_2(t-\tau) \leq P = \left(\frac{r_1^M + r_2^M}{2\sum_{l=0}^{m}\sqrt{b_1^{l+L}b_2^{lL}}}\right)^2 \exp\big((r_1^M + r_2^M)m\tau\big) < +\infty.$$

Second, we prove the boundedness of $x_1(t)$. From (1.1), for any positive $\bar{\epsilon}$, there exists a sufficiently large $\overline{T}_1 \geq T$ such that

$$\frac{dx_1(t)}{dt} \leq x_1(t)\left[r_1^M - \sum_{l=0}^{m}a_{11}^{lL}x_1(t-l\tau)\right] + a_{12}^{1M}(P + \bar{\epsilon}), \quad t > \overline{T}_1.$$

From Lemma 3.1 and arbitrarity of $\bar{\epsilon}$, we get

$$\limsup_{t\to+\infty} x_1(t) \leq -\frac{a_{12}^{1M}P}{r_1^M} + \left\{\frac{a_{12}^{1M}P}{r_1^M} + \left(1/A_1^L\right)\left(r_1^M + \frac{a_{12}^{1M}P}{x_1^*}\right)\right\}\exp\left(r_1^M m\tau\right) < +\infty.$$

Next, we show that $x_n(t)$ is also bounded above. From system (1.1), there exists a sufficiently large $\overline{T}_n \geq T$ for any positive $\bar{\epsilon}_1$ such that

$$\frac{dx_n(t)}{dt} \leq x_n(t)\left[r_n^M - \sum_{l=0}^{m}a_{nn}^{lL}x_n(t-l\tau) + \left(\sum_{l=0}^{m}a_{n1}^{lM}(M_1 + \bar{\epsilon}_1)\right)\right], \quad t > \overline{T}_n.$$

Since $\bar{\epsilon}_1$ is arbitrary, from Lemma 2.1, we obtain

$$\limsup_{t\to+\infty} x_n(t) \leq \left\{\left(r_n^M + \sum_{l=0}^{m}a_{n1}^{lM}M_1\right)/A_n^L\right\}\exp\left(\left(r_n^M + \sum_{l=0}^{m}a_{n1}^{lM}M_1\right)m\tau\right) < +\infty.$$

Similarly, for $i = 2, \ldots, n-1$, it holds that $\limsup_{t\to+\infty} x_i(t) \leq M_i < +\infty$.

Finally, we show that $x_i(t)$ $(i = 1, \ldots, n)$ are bounded below for sufficiently large $t > 0$. From (1.1), for any positive $\underline{\epsilon}_i$, there exists a sufficiently large $\underline{T}_i \geq \overline{T}_i$ such that

$$\frac{dx_i(t)}{dt} \geq x_i(t)\left[\left(r_i^L - \sum_{\substack{j=1 \\ j\neq i,i+1}}^{n}\sum_{l=0}^{m}a_{ij}^{lM}(M_j + \underline{\epsilon}_i)\right) - \sum_{l=0}^{m}a_{ii}^{lM}x_i(t-l\tau)\right], \quad t > \underline{T}_i,$$

for $i = 1, \ldots, n$. Since $\underline{\epsilon}_i$ is arbitrary, from our assumption and Lemma 3.2, we get

$$\liminf_{t\to+\infty} x_i(t) \geq \Big\{\Big(r_i^L - \sum_{\substack{j=1\\ j\neq i,i+1}}^{n} \sum_{l=0}^{m} a_{ij}^{lM} M_j\Big)/A_i^M\Big\} \exp\Big(\Big(r_i^L - \sum_{\substack{j=1\\ j\neq i+1}}^{n} \sum_{l=0}^{m} a_{ij}^{lM} M_j\Big) m\tau\Big),$$

for $i = 1, \ldots, n$. We obtain the conclusion of Theorem 2.1. The proof is completed. □

We now consider a 3-dimensional Lotka-Volterra cooperative system such that there is no undelayed intraspecific competition on the third equation of the following system:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) - a_{13}x_3(t-2\tau)], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_{21}x_1(t-\tau) - a_{22}x_2(t) + a_{23}x_3(t-\tau)], \\ \dfrac{dx_3(t)}{dt} = x_3(t)[r_3 + a_{31}x_1(t-\tau) - a_{32}x_2(t-\tau) - a_{33}x_3(t-2\tau)], \end{cases} \tag{3.6}$$

where $\tau \geq 0$, $a_{ij} > 0$ and $r_i > 0$ $(i, j = 1, 2, 3)$ with initial conditions $x_i(\theta) = \phi_i(\theta) \geq 0$, $\theta \in [-2\tau, 0)$, $\phi_i(0) > 0$ $(i = 1, 2, 3)$. Set

$$\begin{aligned} \hat{M}_1 &= -\frac{D}{r_1} + \Big\{\frac{D}{r_1} + \frac{1}{a_{11}}\Big(r_1 + \frac{D}{\hat{x}_1^*}\Big)\Big\} \exp(2r_1\tau), \ \hat{M}_2 = \frac{r_2 + a_{23}\hat{M}_3}{a_{22}}, \\ \hat{M}_3 &= \frac{r_3 + a_{31}\hat{M}_1}{a_{33}} \exp\Big(2(r_3 + a_{31}\hat{M}_1)\tau\Big), \\ \hat{m}_1 &= \frac{r_1 - a_{13}\hat{M}_3}{a_{11}} \exp\Big(2\{(r_1 - a_{13}\hat{M}_3) - a_{11}\hat{M}_1\}\tau\Big), \\ \hat{m}_2 &= \frac{r_2 - a_{21}\hat{M}_1}{a_{22}}, \ \hat{m}_3 = \frac{r_3 - a_{32}\hat{M}_2}{a_{33}^2} \exp\Big(2\{(r_3 - a_{32}\hat{M}_2) - a_{33}\hat{M}_3\}\tau\Big), \end{aligned}$$

where $x = \hat{x}_1^*$ is a unique positive solution of the following equation:

$$x(r_1 - a_{11}x) + a_{12}P' = 0, \ \ P' = \frac{(r_1 + r_2)^2}{4a_{11}(a_{22} - a_{12})} > 0.$$

We note that there is no undelayed intraspecific competition on the third equation of system (3.6). By Theorem 2.1, we have the following corollary.

Corollary 3.1. *System* (3.6) *is permanent if* $a_{13} \geq a_{23}$, $a_{22} > a_{12}$ *and*

$$r_1 - a_{13}\hat{M}_3 > 0, \ \ r_2 - a_{21}\hat{M}_1 > 0, \ \ r_3 - a_{32}\hat{M}_2 > 0, \tag{3.7}$$

that is,

$$0 < \hat{m}_i \leq \liminf_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_i(t) \leq \hat{M}_i < +\infty, \ \ i = 1, 2, 3.$$

As stated in Remark 1.1, we see that conditions (3.7) implies that the size of time delays restricts on lower bounds $\hat{m}_i > 0$ $(i = 1, 2, 3)$.

4 Discussion

Throughout the paper, we investigate the harmlessness of time delays for the permanence of the cooperative Lotka-Voltera-type system. We obtain sufficient conditions which improve the result of Lu and Lu [6] for 2-dimensional Lotka-Volterra cooperative systems without any restrictions of the size of time delays. Using Lemmas 3.1 and 3.2, we derive the upper bound of $x_1(t)x_2(t-\tau)$ to prove that $x_i(t)$ $(i = 1, \ldots, n)$ are bounded in Theorem 2.1 for system (1.1). For the system with particular discrete time delay, our condition holds even if there is no undelayed intraspecific competition. The results of this work indicate that the restriction on the size of time delays is not required for the case $n = 2$, but it is required for the case $n \geq 3$ to obtain lower bounds of solutions. Since Theorem 2.1 indicates the sufficient conditions of permanence, it is a future work to induce necessary and sufficient conditions.

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