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Stability analysis of delayed SIR epidemic models with a class of nonlinear incidence rates

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Abstract. We analyze stability of equilibria for a delayed SIR epidemic model, in which population growth is subject to logistic growth in absence of disease, with a nonlinear incidence rate satisfying suitable monotonicity conditions. The model admits a unique endemic equilibrium if and only if the basic reproduction number R_0 exceeds one, while the trivial equilibrium and the disease-free equilibrium always exist. First we show that the disease-free equilibrium is globally asymptotically stable if and only if $R_0 \leq 1$. Second we show that the model is permanent if and only if $R_0 > 1$. Moreover, using a threshold parameter $\overline{R}_0$ characterized by the nonlinear incidence function, we establish that the endemic equilibrium is locally asymptotically stable for $1 < R_0 \leq \overline{R}_0$ and it loses stability as the length of the delay increases past a critical value for $1 < \overline{R}_0 < R_0$. Our result is an extension of the stability results in [J-J. Wang, J-Z. Zhang, Z. Jin, Analysis of an SIR model with bilinear incidence rate, *Nonl. Anal. RWA.* **11** (2009) 2390-2402].

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1 Introduction

From an epidemiological viewpoint, it is important to investigate global dynamics of the disease transmission. In the literature, many authors have formulated various epidemic models, in which the stability analysis have been carried out extensively (see [1–15] and references therein). Recently, based on an SIR (Susceptible-Infected-Recovered) epidemic model, in order to investigate the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.),

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Takeuchi [11] formulated a delayed SIR epidemic model with a bilinear incidence rate. The global dynamics for the system has now been completely analyzed in McCluskey [8]. Later, Wang *et al.* [12] considered the asymptotic behavior of the following delayed SIR epidemic model:

$$\begin{cases} \dfrac{dS(t)}{dt} = r\left(1 - \dfrac{S(t)}{K}\right)S(t) - \beta S(t)I(t-\tau), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau) - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the fractions of susceptible, infective and recovered host individuals at time t, respectively. In system (1.1), it is assumed that the population growth in susceptible host individuals is governed by the logistic growth with a carrying capacity $K > 0$ as well as intrinsic birth rate constant $r > 0$. $\beta > 0$ is the average number of constants per infective per unit time and $\tau \geq 0$ is the incubation time, $\mu_1 > 0$ and $\mu_2 > 0$ represent the death rates of infective and recovered individuals, respectively. $\gamma > 0$ represents the recovery rate of infective individuals.

Wang *et al.* [12] obtained stability results of equilibria of (1.1) in terms of the basic reproduction number R_0: the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ while a unique endemic equilibrium can be unstable if $R_0 > 1$. More precisely, if $1 < R_0 \leq 3$, then the endemic equilibrium is asymptotically stable for any delay τ and if $R_0 > 3$, then there exists a critical length of delay such that the endemic equilibrium is asymptotically stable for delay which is less than the value while it is unstable for delay which is greater than the value. It is also shown that Hopf bifurcation at the endemic equilibrium occurs when the delay crosses a sequence of critical values.

Since nonlinearity in the incidence rates has been observed in disease transmission dynamics, it has been suggested that the standard bilinear incidence rate shall be modified into a nonlinear incidence rate by many authors (see, e.g., [2,7]). In this paper we replace the incidence rate in (1.1) by a nonlinear incidence rate of the form $\beta S(t)G(I(t-\tau))$. We assume that the function G is continuous on $[0, +\infty)$ and continuously differentiable on $(0, +\infty)$ satisfying the following hypotheses.

(H1) $G(I)$ is strictly monotone increasing on $[0, +\infty)$ with $G(0) = 0$,

(H2) $I/G(I)$ is monotone increasing on $(0, +\infty)$ with $\lim_{I \to +0} I/G(I) = 1$.

Then we obtain the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = r\left(1 - \dfrac{S(t)}{K}\right)S(t) - \beta S(t)G(I(t-\tau)), \\ \dfrac{dI(t)}{dt} = \beta S(t)G(I(t-\tau)) - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{cases} \tag{1.2}$$

The incidence function G includes some special incidence rates. For instance, if $G(I) = I$, then the incidence rate with a distributed delay is used in [8,11] and if $G(I) = \frac{I}{1+\alpha I}$, then the incidence rate, describing saturated effects of the prevalence of infectious diseases, is used in [9,13,15].

For simplicity, we nondimensionalize system (1.2) by defining

$$\tilde{S}(\tilde{t}) = \frac{S(t)}{K},\ \tilde{I}(\tilde{t}) = \frac{I(t)}{K},\ \tilde{R}(\tilde{t}) = \frac{R(t)}{K}$$

and

$$\tilde{t} = \beta K t,\ \tilde{r} = \frac{r}{\beta K},\ \tilde{h} = \beta K h,\ \tilde{\tau} = \beta K \tau,\ \tilde{G}(\tilde{I}(\tilde{t})) = \frac{G(I(t))}{K},\ \tilde{\mu}_1 = \frac{\mu_1}{\beta K},\ \tilde{\mu}_2 = \frac{\mu_2}{\beta K},\ \tilde{\gamma} = \frac{\gamma}{\beta K}.$$

We note that $\tilde{G}$ also satisfies the hypotheses (H1) and (H2). Dropping the " ˜ " for convenience of readers, system (1.2) can be rewritten into the following form:

$$\begin{cases} \dfrac{dS(t)}{dt} = r(1 - S(t))S(t) - S(t)G(I(t-\tau)), \\ \dfrac{dI(t)}{dt} = S(t)G(I(t-\tau)) - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{cases} \tag{1.3}$$

We hereafter restrict our attention to system (1.3). The initial conditions for system (1.3) take the following form

$$\begin{cases} S(\theta) = \phi_1(\theta), \ \ I(\theta) = \phi_2(\theta), \ \ R(\theta) = \phi_3(\theta), \\ \phi_i(\theta) \geq 0, \ \ \theta \in [-h, 0], \ \phi_i(0) > 0, \ \ \phi_i \in C([-h, 0], \mathbb{R}_{+0}), \ \ i = 1, 2, 3, \end{cases} \tag{1.4}$$

where $\mathbb{R}_{+0} = \{x \in \mathbb{R} : x \geq 0\}$. By the fundamental theory of functional differential equations, system (1.3) has a unique positive solution $(S(t), I(t), R(t))$ satisfying the initial conditions (1.4). We define the basic reproduction number by

$$R_0 = \frac{1}{\mu_1 + \gamma}. \tag{1.5}$$

In this paper we analyze the stability of equilibria by investigating location of the roots of associated characteristic equation and constructing a Lyapunov functional. System (1.3) always has a trivial equilibrium $E_0 = (0, 0, 0)$ and a disease-free equilibrium $E_1 = (1, 0, 0)$. If $R_0 > 1$, then system (1.3) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$, $S^* > 0$, $I^* > 0$, $R^* > 0$ (see Lemma 3.1).

The organization of this paper is as follows. In Section 2, we investigate the stability of the trivial equilibrium and the disease-free equilibrium. In Section 3, for $R_0 > 1$, we show the unique existence of the endemic equilibrium and the permanence of system (1.3). Moreover, we investigate the delay effect concerning the local asymptotic stability of endemic equilibrium. In Section 4, we introduce an example and visualize stability conditions for the disease-free equilibrium and the endemic equilibrium in a two-parameter plane. Finally, we offer concluding remarks in Section 5.

2 Stability of the trivial equilibrium and the disease-free equilibrium

In this section, we analyze the stability of the trivial equilibrium E_0. By constructing a Lyapunov functional, we further establish the global asymptotic stability of the disease-free equilibrium E_1 for $R_0 \leq 1$. At an arbitrary equilibrium $(\hat{S}, \hat{I}, \hat{R})$ of (1.3), the characteristic equation is given by

$$(\lambda + \mu_2)[\{\lambda + G(\hat{I}) - r(1 - 2\hat{S})\}\{\lambda + \mu_1 + \gamma - \hat{S}G'(\hat{I})e^{-\lambda\tau}\} + \hat{S}G'(\hat{I})e^{-\lambda\tau}G(\hat{I})] = 0. \tag{2.1}$$

Theorem 2.1. *The trivial equilibrium E_0 of system (1.3) is always unstable.*

Proof. For $(\hat{S}, \hat{I}, \hat{R}) = (0, 0, 0)$ the characteristic equation (2.1) becomes as follows.

$$(\lambda + \mu_2)(\lambda - r)(\lambda + \mu_1 + \gamma) = 0. \tag{2.2}$$

Since (2.2) has a positive root $\lambda = r$, E_0 is unstable. □

Constructing a Lyapunov functional, we prove that the global asymptotic stability of the disease-free equilibrium E_1 is determined by the basic reproduction number R_0.

Theorem 2.2. *The disease-free equilibrium E_1 of system (1.3) is globally asymptotically stable if and only if $R_0 \leq 1$ and it is unstable if and only if $R_0 > 1$.*

Proof. First we assume $R_0 \leq 1$. We define a Lyapunov functional by

$$V(t) = g(S(t)) + I(t) + \int_{t-\tau}^{t} G(I(s))ds, \tag{2.3}$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$ for $x > 0$. Then the time derivative of $V(t)$ along the solution of (1.3) becomes as follows.

$$\begin{aligned} \frac{dV(t)}{dt} &= \left(1 - \frac{1}{S(t)}\right)\{r(1 - S(t))S(t) - S(t)G(I(t - \tau))\} + S(t)G(I(t - \tau)) - (\mu_1 + \gamma)I(t) + G(I(t)) - G(I(t - \tau)) \\ &= -r(S(t) - 1)^2 + G(I(t)) - (\mu_1 + \gamma)I(t). \\ &= -r(S(t) - 1)^2 + \left\{\frac{G(I(t))}{I(t)} - (\mu_1 + \gamma)\right\} I(t). \end{aligned}$$

From the hypothesis (H2), noting that $0 < \frac{G(I)}{I} \leq 1$ for $I > 0$, we have

$$\frac{dV(t)}{dt} \leq -r(S(t) - 1)^2 + \left(1 - \frac{1}{R_0}\right) I(t) \leq 0. \tag{2.4}$$

By Lyapunov-LaSalle asymptotic stability theorem, we have that $\lim_{t\to+\infty} S(t) = 1$ if $R_0 \leq 1$. By the first and third equations of (1.3), $\lim_{t\to+\infty} S(t) = 1$ implies $\lim_{t\to+\infty} I(t) = 0$ and $\lim_{t\to+\infty} R(t) = 0$. Since it follows that E_1 is uniformly stable from the relation $V(t) \geq g(S(t)) + I(t)$, we obtain that E_1 is globally asymptotically stable.

Second we assume $R_0 > 1$. For $(\hat{S}, \hat{I}, \hat{R}) = (1, 0, 0)$ the characteristic equation (2.1) becomes as follows.

$$(\lambda + \mu_2)(\lambda + r)\left(\lambda + \mu_1 + \gamma - e^{-\lambda\tau}\right) = 0. \tag{2.5}$$

One can see that $\lambda = -r$ and $\lambda = -\mu_2$ are negative real roots of (2.5). Moreover, (2.5) has roots of

$$p(\lambda) := \lambda + \mu_1 + \gamma - e^{-\lambda\tau} = 0.$$

From $p(0) < 0$ and $\lim_{\lambda\to+\infty} p(\lambda) = +\infty$, $p(\lambda) = 0$ has at least one positive root. Hence E_1 is unstable. The proof is complete. □

3 Permanence of the system and local asymptotic stability of the endemic equilibrium for $R_0 > 1$

In this section, for $R_0 > 1$, we obtain the permanence of system (1.3). In addition, we establish local asymptotic stability of the endemic equilibrium E_* by investigating location of the roots of the characteristic equation.

3.1 Existence and uniqueness of the endemic equilibrium E_* for $R_0 > 1$

In this subsection, we give the result on the unique existence of the endemic equilibrium for $R_0 > 1$.

Lemma 3.1. *System (1.3) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$ if and only if $R_0 > 1$.*

Proof. We assume $R_0 > 1$. In order to find the endemic equilibrium of system (1.3), for $S > 0$, $I > 0$ and $R > 0$, we consider the following equations:

$$\begin{cases} r(1-S)S - SG(I) = 0, \\ SG(I) - (\mu_1 + \gamma)I = 0, \\ \gamma I - \mu_2 R = 0. \end{cases} \tag{3.1}$$

Substituting the second equation of (3.1) into the first equation of (3.1), we have

$$F(I) := r\left\{1 - \frac{(\mu_1 + \gamma)I}{G(I)}\right\} - G(I) = 0.$$

By the hypothesis (H2), we obtain

$$\lim_{I\to+0} F(I) = r\left\{1 - (\mu_1 + \gamma)\right\} = r\left(1 - \frac{1}{R_0}\right) > 0.$$

Since $F(I)$ is a strictly monotone decreasing function on $(0, +\infty)$, it sufficies to show that $F(I) < 0$ holds for I sufficiently large. From (H1), $G(I)$ is either unbounded above or bounded above on $[0, +\infty)$. First we suppose that $G(I)$ is unbounded above. Then there exists an $I_1 > 0$ such that $G(I_1) = r$, from which we have $F(I) < 0$ for $I > I_1$. Second we suppose that $G(I)$ is bounded above. Then, from (H2), $\frac{I}{G(I)}$ is unbounded above on $[0, +\infty)$, that is, there exists an $I_2 > 0$ such that such that $\frac{I_2}{G(I_2)} = \frac{1}{\mu_1+\gamma}$. This yields $F(I) < 0$ for $I > I_2$. Therefore, for the both cases, there exists a unique $I^* > 0$ such that $F(I^*) = 0$. By the second and third equations of (3.1), there exists a unique endemic equilibrium E_* of system (1.3) if $R_0 > 1$. Second we assume $R_0 \leq 1$. Then it is obvious that system (1.3) has no endemic equilibria. Hence the proof is complete. □

3.2 Permanence of the system for $R_0 > 1$

In this subsection, we obtain the permanence of the system (1.3). We introduce the following lemma without proof.

Lemma 3.2. *For system (1.3) with initial conditions (1.4),*

$$\limsup_{t\to+\infty}(S(t) + I(t) + R(t)) \leq \frac{1}{\underline{\mu}},$$

where $\underline{\mu} = \min(\mu_1, \mu_2, 1)$.

Similar as in the proof of Wang *et al.* [12, Theorem 3.2], we obtain the following theorem. We omit the proof.

Theorem 3.1. *There exist positive constants v_i $(i = 1, 2, 3)$ such that for any initial conditions of system (1.3),*

$$\liminf_{t\to+\infty} S(t) \geq v_1, \ \liminf_{t\to+\infty} I(t) \geq v_2, \ \liminf_{t\to+\infty} R(t) \geq v_3,$$

if and only if $R_0 > 1$.

Combining Lemma 3.2 and Theorem 3.1, we obtain the permanence of system (1.3) for $R_0 > 1$.

3.3 Local asymptotic stability of E_* for $R_0 > 1$

In this subsection, we study local asymptotic stability of the endemic equilibrium $E_* = (S^*, I^*, R^*)$ for system (1.3). Let us assume that $R_0 > 1$ holds. For $(\hat{S}, \hat{I}, \hat{R}) = (S^*, I^*, R^*)$ the characteristic roots of (2.1) are the root $\lambda = -\mu_2$ and the roots of

$$\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda + d) = 0, \tag{3.2}$$

where

$$a = S^*\left(\frac{G(I^*)}{I^*} + r\right), \ b = \frac{r(S^*)^2 G(I^*)}{I^*}, \ c = S^* G'(I^*), \ d = S^* G'(I^*)(rS^* - G(I^*)).$$

First we analyze the characterstic equation (3.2) with $\tau = 0$. We prove that all the roots of (3.2) have negative real part.

Proposition 3.1. *Assume $R_0 > 1$. Then all the roots of (3.2) have negative real part for $\tau = 0$.*

Proof. When $\tau = 0$, (3.2) yields

$$\lambda^2 + (a - c)\lambda + (b - d) = 0. \tag{3.3}$$

Noting from the hypotheses (H1) and (H2) that $G(I^*) - I^* G'(I^*) \geq 0$, we have

$$a - c = S^*\left(\frac{G(I^*)}{I^*} - G'(I^*) + r\right) > 0$$

and

$$b - d = r(S^*)^2\left(\frac{G(I^*)}{I^*} - G'(I^*)\right) + S^* G'(I^*) G(I^*) > 0,$$

which implies that all the roots of equation (3.3) have negative real part. The proof is complete. □

Next we analyze the characteristic equation (3.2) with $\tau > 0$. Let us define

$$\overline{R}_0 = 2\frac{I^*}{G(I^*)} + \frac{1}{G'(I^*)}. \tag{3.4}$$

Then we prove that $R_0 = \overline{R}_0$ is a threshold condition which determines the existence of purely imaginary roots of (3.2).

Proposition 3.2. *Assume $R_0 > 1$. Then the following statement holds true.*

(i) *If $R_0 \leq \overline{R}_0$, then all the roots of (3.2) have negative real part for any $\tau > 0$.*

(ii) *If $\overline{R}_0 < R_0$, then there exists a monotone increasing sequence $\{\tau_n\}_{n=0}^{\infty}$ with $\tau_0 > 0$ such that (3.2) has a pair of imaginary roots for $\tau = \tau_n$ $(n = 0, 1, \ldots)$.*

Proof. From Proposition 3.1, all the roots of equation (3.2) have negative real part for sufficiently small τ. Suppose that $\lambda = i\omega$, $\omega > 0$ is a root of (3.2). Substituting $\lambda = i\omega$ into the characteristic equation (3.2) yields equations, which split into its real and imaginary parts as follows:

$$\begin{cases} -\omega^2 + b = d\cos\omega\tau + c\omega\sin\omega\tau, \\ a\omega = c\omega\cos\omega\tau - d\sin\omega\tau. \end{cases} \tag{3.5}$$

Squaring and adding both equations in (3.5), we have

$$\omega^4 + (a^2 - 2b - c^2)\omega^2 + (b + d)(b - d) = 0. \tag{3.6}$$

By the relations $r(1-S^*) = G(I^*)$ and

$$2S^*G'(I^*) + \frac{1}{R_0} = \frac{2I^*G'(I^*)}{R_0 G(I^*)} + \frac{1}{R_0} = \frac{G'(I^*)}{R_0}\left(2\frac{I^*}{G(I^*)} + \frac{1}{G'(I^*)}\right) = \frac{\overline{R}_0 G'(I^*)}{R_0},$$

we obtain

$$a^2 - 2b - c^2 = \left(\frac{G(I^*)}{I^*} + r\right)^2 (S^*)^2 - 2r\frac{G(I^*)}{I^*}(S^*)^2 - (S^*)^2 G'(I^*)^2 = (S^*)^2\left\{\left(\frac{G(I^*)}{I^*}\right)^2 - G'(I^*)^2 + r^2\right\}$$

and

$$b + d = rS^*\left(2S^*G'(I^*) + \frac{1}{R_0} - G'(I^*)\right) = \frac{rS^*G'(I^*)}{R_0}(\overline{R}_0 - R_0).$$

First we assume $R_0 \le \overline{R}_0$. Then we have $a^2 - 2b - c^2 > 0$ and $b + d \ge 0$, that is, there is no positive real ω satisfying (3.6). This leads to a contradiction and all the roots of (3.2) have negative real part for any $\tau \ge 0$. Hence we obtain the first part of this proposition.

Second we assume $\overline{R}_0 < R_0$. Then it follows from the relations $a^2 - 2b - c^2 > 0$ and $b + d < 0$ that there is a unique positive real ω_0 satisfying (3.6), where

$$\omega_0 = \left\{\frac{-(a^2 - 2b - c^2) + \sqrt{(a^2 - 2b - c^2)^2 - 4(b+d)(b-d)}}{2}\right\}^{\frac{1}{2}}. \tag{3.7}$$

Noting from (3.5) that $\lambda = -i\omega_0$ is also a root of (3.2), this implies that (3.6) has a single pair of purely imaginary roots $\pm i\omega_0$. By the relation

$$(ac - d)\omega_0^2 + bd = (c^2\omega_0^2 + d^2)\cos\omega_0\tau,$$

τ_n corresponding to ω_0 can be obtained as follows:

$$\tau_n = \frac{1}{\omega_0}\arccos\frac{(ac-d)\omega_0^2 + bd}{c^2\omega_0^2 + d^2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots. \tag{3.8}$$

Hence we obtain the second part of this proposition. The proof is complete. □

The following proposition indicates that a conjugate pair of the characteristic roots $\lambda = \pm i\omega_0$ of (2.1) cross the imaginary axis from the left half complex plane to the right half complex plane when τ crosses τ_n $(n = 0, 1, \ldots)$ if $1 < \overline{R}_0 < R_0$.

Proposition 3.3. *Assume $R_0 > 1$. If $\overline{R}_0 < R_0$, then the transversality condition*

$$\left.\frac{d\mathrm{Re}(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_n} > 0$$

holds for $n = 0, 1, \ldots$.

Proof. Differentiating (3.2) with respect to τ, we obtain

$$(2\lambda + a)\frac{d\lambda}{d\tau} = \{e^{-\lambda\tau}c - \tau e^{-\lambda\tau}(c\lambda + d)\}\frac{d\lambda}{d\tau} - \lambda e^{-\lambda\tau}(c\lambda + d),$$

that is,

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{(2\lambda + a) - e^{-\lambda\tau}c + \tau e^{-\lambda\tau}(c\lambda + d)}{-\lambda e^{-\lambda\tau}(c\lambda + d)} \\
&= \frac{2\lambda + a}{-\lambda e^{-\lambda\tau}(c\lambda + d)} + \frac{c}{\lambda(c\lambda + d)} - \frac{\tau}{\lambda} \\
&= -\frac{\lambda(2\lambda + a)}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{c\lambda}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda} \\
&= -\frac{(\lambda^2 + a\lambda + b) + \lambda^2 - b}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{(c\lambda + d) - d}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda} \\
&= -\frac{\lambda^2 - b}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{-d}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda}.
\end{aligned}$$

By the relation

$$\frac{d\lambda}{d\tau} = \frac{d\mathrm{Re}(\lambda)}{d\tau} + i\frac{d\mathrm{Im}(\lambda)}{d\tau} = \left\{\left(\frac{d\mathrm{Re}(\lambda)}{d\tau}\right)^2 + \left(\frac{d\mathrm{Im}(\lambda)}{d\tau}\right)^2\right\}\left(\frac{d\mathrm{Re}(\lambda)}{d\tau} - i\frac{d\mathrm{Im}(\lambda)}{d\tau}\right)^{-1},$$

we have

$$\frac{d\mathrm{Re}(\lambda)}{d\tau} = \mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\left\{\left(\frac{d\mathrm{Re}(\lambda)}{d\tau}\right)^2 + \left(\frac{d\mathrm{Im}(\lambda)}{d\tau}\right)^2\right\}$$

and

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n} = \frac{(-\omega_0^2 - b)(b-\omega_0^2)}{\omega_0^2\{(b-\omega_0^2)^2 + a^2\omega_0^2\}} + \frac{d^2}{\omega_0^2(c^2\omega_0^2 + d^2)} = \frac{\omega_0^4 - b^2 + d^2}{\omega_0^2(c^2\omega_0^2 + d^2)} = \frac{\omega_0^4 - (b-d)(b+d)}{\omega_0^2(c^2\omega_0^2 + d^2)} > 0.$$

Hence we obtain $\frac{d\mathrm{Re}(\lambda)}{d\tau}|_{\tau=\tau_n} > 0$ for $n = 0, 1, \ldots$. The proof is complete. □

By Proposition 3.1 and the first part of Proposition 3.2, all the roots of (3.2) have negative real part for any $\tau \geq 0$ if $1 < R_0 \leq \overline{R}_0$. By Proposition 3.1, the second part of Proposition 3.2 and Proposition 3.3, all the roots of (3.2) have negative real part for $0 \leq \tau < \tau_0$ and there exists at least 2 roots having positive real part for $\tau > \tau_0$ if $1 < \overline{R}_0 < R_0$. We then establish the stability condition for the endemic equilibrium as follows.

Theorem 3.2. *Assume $R_0 > 1$. Then the following statement holds true.*

(i) *If $R_0 \leq \overline{R}_0$, then the endemic equilibrium E_* of system (1.3) is locally asymptotically stable for any $\tau \geq 0$.*

(ii) *If $\overline{R}_0 < R_0$, then the endemic equilibrium E_* of system (1.3) is locally asymptotically stable for $0 \leq \tau < \tau_0$ and it is unstable for $\tau > \tau_0$.*

Remark 3.1. *System (1.3) undergoes Hopf bifurcation at the endemic equilibrium E_* when τ crosses τ_n $(n = 0, 1, \ldots)$ for $1 < \overline{R}_0 < R_0$.*

4 Example

In this section, we consider the following model as an example.

$$\begin{cases} \dfrac{dS(t)}{dt} = r(1 - S(t))S(t) - S(t)\dfrac{I(t-\tau)}{1+\alpha I(t-\tau)}, \\ \dfrac{dI(t)}{dt} = S(t)\dfrac{I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t), \end{cases} \tag{4.1}$$

where $\alpha \geq 0$. Since $G(I) = \frac{I}{1+\alpha I}$ satisfies the hypotheses (H1) and (H2), system (4.1) always has the trivial equilibrium E_0 and the disease-free equilibrium E_1. Applying Theorems 2.1 and 2.2 we obtain the following results.

Corollary 4.1. *The trivial equilibrium E_0 of system (4.1) is always unstable.*

Corollary 4.2. *The disease-free equilibrium E_1 of system (4.1) is globally asymptotically stable if and only if $R_0 \leq 1$ and it is unstable if and only if $R_0 > 1$.*

By Lemma 3.1, system (4.1) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$ if and only if $R_0 > 1$. Applying Theorem 3.2, we obtain the following result.

Corollary 4.3. *Assume $R_0 > 1$. Then the following statement holds true.*

(i) *If $R_0 \leq \overline{R}_0$, then the endemic equilibrium E_* of system (4.1) is locally asymptotically stable for any $\tau \geq 0$.*

(ii) *If $\overline{R}_0 < R_0$, then the endemic equilibrium E_* of system (4.1) is locally asymptotically stable for $0 \leq \tau < \tau_0$ and it is unstable for $\tau > \tau_0$.*

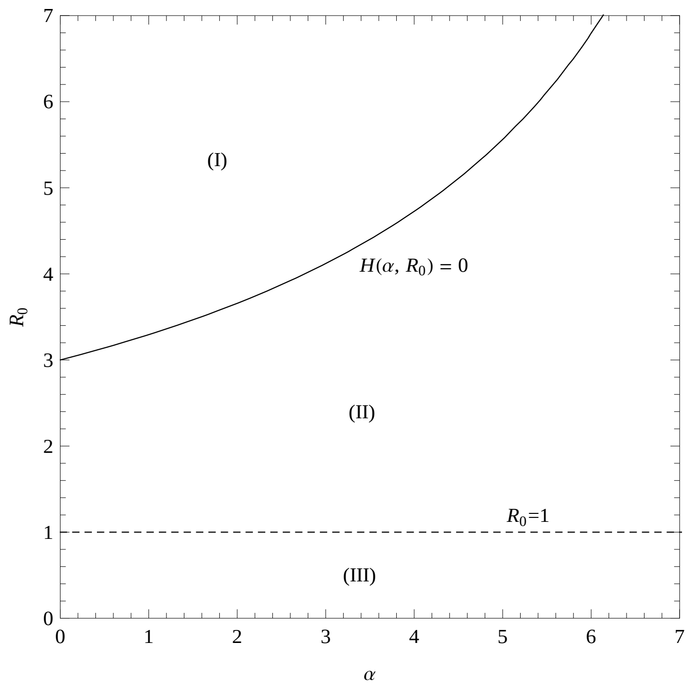

Figure 1: Delay dependent/independent stability boundary for the endemic equilibrium and the stability boundary for the disease-free equilibrium in (α, R_0) parameter plane. The dashed curve and the dotted line denotes $H(\alpha, R_0) = 0$ with $r = 0.1$ and $R_0 = 1$, respectively. In the region (I) there exists a $\tau_0 := \tau_0(\alpha, R_0)$ such that the endemic equilibrium E_* is asymptotically stable for $0 \le \tau < \tau_0$ and it is unstable for $\tau > \tau_0$. In the region (II) the endemic equilibrium E_* is asymptotically stable for any τ. In the region (III) the disease-free equilibrium E_1 is globally asymptotically stable.

The condition $R_0 = 1$ is a threshold condition which determines stability of the disease-free equilibrium and the existence of the endemic equilibrium. Moreover, if $R_0 > 1$ then the condition $R_0 = \overline{R}_0$ works as a condition which determines delay-dependent stability or delay-independent stability for the endemic equilibrium. In the following we visualize these conditions by plotting them in a two-parameter plane. We choose α and R_0 as free parameters and fix r. Since it is straightforward to plot the condition $R_0 = 1$ in (α, R_0) parameter plane, we explain how to visualize the condition $R_0 = \overline{R}_0$ in the same parameter plane.

Let us assume that $R_0 > 1$ holds. The component of the endemic equilibrium for I can be given as

$$I^*(\alpha, R_0) = \frac{\alpha r(R_0 - 2) - R_0 + \sqrt{\{\alpha r(R_0 - 2) - R_0\}^2 + 4\alpha^2 r^2 (R_0 - 1)}}{2\alpha^2 r} \tag{4.2}$$

for $\alpha > 0$ and

$$I^*(0, R_0) = r\left(1 - \frac{1}{R_0}\right). \tag{4.3}$$

We note that $\lim_{\alpha \to +0} I^*(\alpha, R_0) = I^*(0, R_0) > 0$. Then from the definition (3.4) $\overline{R}_0$ is computed as

$$\begin{aligned} \overline{R}_0(\alpha, R_0) &= 2(1 + \alpha I^*(\alpha, R_0)) + (1 + \alpha I^*(\alpha, R_0))^2 \\ &= (1 + \alpha I^*(\alpha, R_0))(3 + \alpha I^*(\alpha, R_0)). \end{aligned} \tag{4.4}$$

We define the following function.

$$H(\alpha, R_0) := R_0 - \overline{R}_0(\alpha, R_0). \tag{4.5}$$

If there exists (α, R_0) satisfying $H(\alpha, R_0) = 0$, then it expresses the condition $R_0 = \overline{R}_0(\alpha, R_0)$ in terms of two parameters (α, R_0). We note that $H(0, 3) = 0$ holds true. The following proposition indicates that $H(\alpha, R_0) = 0$ for $\alpha > 0$ has exactly one solution α for each $R_0 > 3$.

Proposition 4.1. *There exists a unique continuously differentiable function* $\tilde{\alpha} : (3, +\infty) \longrightarrow (0, +\infty)$ *such that* $H(\tilde{\alpha}(R_0), R_0) = 0$. *In addition, it holds that* $\lim_{R_0 \to 3+0} \tilde{\alpha}(R_0) = 0$.

The proof of Proposition 4.1 is given in A. In Figure 1 we plot the line $R_0 = 1$ and the curve $H(\alpha, R_0) = 0$ in (α, R_0) parameter plane for a fixed r. Figure 1 suggests that the parameter α has a positive effect for the stability of the endemic equilibrium: if α is large enough then the endemic equilibrium is stable for any delay. On the other hand, if R_0 is large enough then for small α there is a possibility that the stability of the endemic equilibrium depends on the delay.

5 Concluding remarks

In this paper we consider SIR epidemic model in which population growth is subject to logistic growth in absence of disease. The force of infection with a discrete delay is given by a general nonlinear incidence rate satisfying monotonicity conditions (H1) and (H2). We analyze stability of the trivial equilibrium, the disease-free equilibrium and the endemic equilibrium by investigating roots of the associated characteristic equations and constructing a Lyapunov functional. We show that the global asymptotic stability of the disease-free equilibrium is determined by the basic reproduction number as often in SIR epidemic models [4, 6–10]: the disease-free equilibrium is globally asymptotically stable if and only if the basic reproduction number is less than or equal to one and it is unstable if and only if the basic reproduction number exceeds one. The system admits a unique endemic equilibrium if and only if the basic reproduction number exceeds one. In order to investigate the stability of the endemic equilibrium we define $\overline{R}_0$, which is characterized by the nonlinear incidence. The condition $R_0 = \overline{R}_0$ is a threshold condition which determines delay-independent stability or delay-dependent stability of the endemic equilibrium: the endemic equilibrium is locally asymptotically stable for any delay if $R_0 \le \overline{R}_0$ and there exists a critical length of delay such that the endemic equilibrium is locally asymptotically stable when the delay is less than the value, whereas it is unstable when the delay is greater than the value if $R_0 > \overline{R}_0$.

Recently, [4, 6, 9, 10] has investigated the stability of equilibria for delayed SIR epidemic models with nonlinear incidence rates, where population growth is governed not by the logistic function but by the linear function under the condition that the incidence function satisfies the monotone properties in (H1) and (H2). It is proved that the endemic equilibrium is globally asymptotically stable for any delay if the basic reproduction number exceeds one. This implies that the logistic growth of population of susceptible individuals is responsible for the instability of the endemic equilibrium. On the other hand, Wang et al. [12] studied the stability of equilibria for (1.3) when the incidence function is a linear function $G(I) = I$ and proved that $R_0 = 3$ is the threshold condition for delay-independent stability or delay-dependent stability of the endemic equilibrium. Since (3.4) implies that $\overline{R}_0$ is reduced to 3 when $G(I) = I$, our results for the stability of endemic equilibrium extends the results in [12, Theorems 3.1 and 4.1].

In Section 4 we consider a special case that the incidence rate has saturation effect to visualize the threshold condition $R_0 = \overline{R}_0$ in a two-parameter plane. We choose α and R_0 as free parameters and fix other parameter. Since the threshold value $\overline{R}_0$ is given by an expression with variables α and R_0 in (4.4), the condition is expressed as $H(\alpha, R_0) = 0$. We further obtain the analytical result of the unique existence of $\alpha > 0$ satisfying $H(\alpha, R_0) = 0$ for any fixed $R_0 > 3$ (see Proposition 4.1). In Figure 1, the two conditions $R_0 = 1$ and $H(\alpha, R_0) = 0$ are depicted. Figure 1 suggests that the function $\tilde{\alpha}$ is a monotone increasing function on $(3, +\infty)$. This implies that the endemic equilibrium is locally stable for any delay if $1 < R_0 \le 3$.

For $R_0 > 3$, the parameter α which measures crowding effect of infective individuals, seems to have a positive effect for the stability of the endemic equilibrium. The endemic equilibrium is locally asymptotically stable for any delay if the crowding effect is large enough. On the other hand the stability of the endemic equilibrium depends on the delay if the crowding effect is small.

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A Proof of Proposition 4.1

In order to prove Proposition 4.1 by means of the implicit function theorem, we introduce the following lemma concerning the continuous differentiability of H.

Lemma A.1. $\frac{\partial H(\alpha,R_0)}{\partial\alpha} < 0$ *holds for all* $\alpha > 0$ *and* $R_0 > 1$. *Moreover,* H *is continuously differentiable on* $(0,+\infty) \times (1,+\infty)$.

Proof. We now show that $\frac{\partial \overline{R}_0(\alpha,R_0)}{\partial\alpha} > 0$ holds for all $\alpha > 0$. From (4.2) and the relation $-1 < \frac{x}{\sqrt{x^2+k}} < 1$ $(x \in \mathbb{R},\ k > 0)$ we have

$$\frac{\partial \alpha I^*(\alpha,R_0)}{\partial\alpha} = \frac{R_0}{2\alpha^2 r}\left\{1 + \frac{R_0 - 2 - \frac{R_0}{r\alpha}}{\sqrt{\left(R_0 - 2 - \frac{R_0}{r\alpha}\right)^2 + 4(R_0-1)}}\right\} > 0.$$

Hence it follows from (4.4) that

$$\frac{\partial \overline{R}_0(\alpha,R_0)}{\partial\alpha} = \frac{\partial}{\partial\alpha}(1 + \alpha I^*(\alpha,R_0))(3 + \alpha I^*(\alpha,R_0)) > 0$$

for all $\alpha > 0$. This implies that $\frac{\partial H(\alpha,R_0)}{\partial\alpha} = -\frac{\partial \overline{R}_0(\alpha,R_0)}{\partial\alpha} < 0$ holds for all $\alpha > 0$. We note that $\frac{\partial H(\alpha,R_0)}{\partial\alpha}$ is continuous on $(0,+\infty)\times(1,+\infty)$. In addition, since $\frac{\partial H(\alpha,R_0)}{\partial R_0}$ exists for all $\alpha > 0$ and $R_0 > 1$ and it is continuous on $(0,+\infty)\times(1,+\infty)$, H is continuously differentiable on $(0,+\infty) \times (1,+\infty)$. The proof is complete. □

Proof of Proposition 4.1. From (4.4) we obtain

$$H(0,R_0) = R_0 - \overline{R}_0(0,R_0) = R_0 - 3 > 0.$$

for any $R_0 > 3$. Moreover from (4.2) we have

$$\lim_{\alpha\to+\infty} \alpha I^*(\alpha,R_0) = \lim_{\alpha\to+\infty} \frac{R_0 - 2 + \sqrt{(R_0-2)^2 + 4(R_0-1)}}{2} = R_0 - 1,$$

which yields

$$\begin{aligned}\lim_{\alpha\to+\infty} H(\alpha, R_0) &= R_0 - \lim_{\alpha\to+\infty} \overline{R}_0(\alpha, R_0) = R_0 - \lim_{\alpha\to+\infty}(1+\alpha I^*(\alpha, R_0))(3+\alpha I^*(\alpha, R_0)) \\ &= R_0 - R_0(R_0+2) \\ &= -R_0(R_0+1) < 0\end{aligned}$$

for a fixed $R_0 > 3$. Therefore, by Lemma A.1 and the implicit function theorem, for any $R_0 > 3$ there exists a unique $\alpha > 0$ such that the following statement holds true.

(i) $H(\alpha, R_0) = 0$.

(ii) There exist neighborhood $\Omega \subseteq (3, +\infty)$ of R_0 and a unique C^1-function $\tilde{\alpha} : \Omega \longrightarrow (0, +\infty)$ such that $\alpha = \tilde{\alpha}(R_0)$ and $H(\tilde{\alpha}(R_0), R_0) = 0$.

Since the parameter $R_0 > 3$ can be arbitrarily chosen, the function $\tilde{\alpha}$ is continuously differentiable on $(3, +\infty)$. Hence we obtain the conclusion of the first part of this proposition.

Finally we prove $\lim_{R_0\to3+0} \tilde{\alpha}(R_0) = 0$. From (4.4) and (4.5) the following equation holds for $R_0 > 3$.

$$R_0 - (1+\tilde{\alpha}(R_0)I^*(\tilde{\alpha}(R_0), R_0))(3+\tilde{\alpha}(R_0)I^*(\tilde{\alpha}(R_0), R_0)) = 0.$$

Since it follows from (4.2) and (4.3) that

$$(1+\tilde{\alpha}(R_0)I^*(\tilde{\alpha}(R_0), R_0))(3+\tilde{\alpha}(R_0)I^*(\tilde{\alpha}(R_0), R_0)) \geq 3$$

holds with equality if and only if $\tilde{\alpha}(R_0) = 0$, $\tilde{\alpha}$ has a right-hand limit 0 as R_0 approaches 3. Hence we obtain the conclusion of the second part of this proposition. The proof is complete. □