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Global stability of extended multi-group SIR epidemic models with patches through migration and cross patch infection

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Abstract. In this article, we establish the global stability of an endemic equilibrium of multi-group SIR epidemic models, which have not only an exchange of individuals between patches through migration but also cross patch infection between different groups. As a result, we partially generalize the recent result in the article [16].

Keywords: Multi-group SIR epidemic model, patch, global asymptotic stability, Lyapunov function.
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1 Introduction

Recently, how to clarify transportation effects on the spreading pattern of global pandemic of diseases such as influenza, is one of the important problems on the multi-group epidemic models. In particular, for the geographical spread of mild infectious disease, individuals do not have so severe symptoms that they retain from traveling abroad during their infectious period.

In 1956, Bartlett [1] firstly used a patch approach to the following model (see Arino [2]):

$$\begin{cases} S_1' = -(\beta_1 I_1 + \beta_2 I_2)S_1 + b + m_S(S_2 - S_1), \\ I_1' = (\beta_1 I_1 + \beta_2 I_2)S_1 - (d+\rho)\mu I_1 + m_I(I_2 - I_1), \\ S_2' = -(\beta_1 I_1 + \beta_2 I_2)S_2 + b + m_S(S_1 - S_2), \\ I_2' = (\beta_1 I_1 + \beta_2 I_2)S_2 - (d+\rho)I_2 + m_I(I_1 - I_2). \end{cases} \tag{1.1}$$

Note that in this group model, there is an exchange of individuals between two patches through migration but there is also cross patch infection. For patch models, there are several works (see for example, Arino [2], Wang and Zhao [3] and the references therein).

By making use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov functions guided by graph theory, Guo *et al.* [4] have firstly succeeded in the proof, and after this paper published, almost all researchers commonly used this research approach by citing the result of estimation by graph theory in [4] to analyze the global stability of various multi-group epidemic models, (see for example, [5-12]). However, all of such multi-group models cited above, did not take into account the phenomenon of population movement between different groups.

On the other hand, for the case of two patches, Liu and Takeuchi [13] and Nakata [14] studied the effect of transport-related infection and entry screening, and Liu and Zhou [15] proposed an SIRS epidemic model and investigated the global dynamics of an SIRS epidemic model with transport-related infection, and for the endemic equilibrium, they

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established a sufficient condition for the global asymptotic stability which implies that the effect of exchange-related infection makes the disease endemic even if both the isolated regions are disease-free, but their sufficient conditions are applicable too restrictive cases.

Recently, there are some interesting papers on construction of Lyapunov functions to prove the global stability of models, for example, Li *et al.* [16], Kajiwara *et al.* [17] and Vargas-De-León [18] (see also Muroya *et al.* [19]). Moreover, Guo *et al.* [20, 21] consider the stage-progression models for HIV/AIDS with amelioration, and Li *et al.* [16] proposed very interesting approach on the proof techniques for the global stability of a class of epidemic models.

Motivated by the above results, in this paper, we consider the global stability of the following group model which has not only an exchange of individuals between patches through migration but also cross patch infection between different groups:

$$\begin{cases} \dfrac{dS_k}{dt} = b_k - \left(\mu_{k1} + \displaystyle\sum_{j=1}^{n}(1-\delta_{jk})l_{jk}\right)S_k - S_k\left(\displaystyle\sum_{j=1}^{n}\beta_{kj}I_j\right) + \displaystyle\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j, \\ \dfrac{dI_k}{dt} = S_k\left(\displaystyle\sum_{j=1}^{n}\beta_{kj}I_j\right) - \left(\mu_{k2} + \gamma_k + \displaystyle\sum_{j=1}^{n}(1-\delta_{jk})m_{jk}\right)I_k + \displaystyle\sum_{j=1}^{n}(1-\delta_{kj})m_{kj}I_j, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \cdots, n, \end{cases} \tag{1.2}$$

and

$$\frac{dR_k}{dt} = \gamma_k I_k - \left(\mu_{k3} + \sum_{j=1}^{n}(1-\delta_{jk})n_{jk}\right)R_k + \sum_{j=1}^{n}(1-\delta_{kj})n_{kj}R_j, \quad k = 1, 2, \cdots, n, \tag{1.3}$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j, \end{cases} \tag{1.4}$$

and in each city k, we refer three groups $S_k(t)$, $I_k(t)$ and $R_k(t)$, $(k = 1, 2, \ldots, n)$ denoting the numbers of susceptible, infected and recovered individuals in city k at time t, respectively. b_k $(k = 1, 2, \ldots, n)$ is the recruitment rate of the population, μ_{kj} $(k = 1, 2, \ldots, n,\ j = 1, 2, 3)$ is the natural death rates of each population of susceptible, infected and recovered individuals in city k, and γ_k $(k = 1, 2, \ldots, n)$ denotes the natural recovery rate of the infected individuals in city k. Functions describing the dynamics within city k of each population of individuals, might involve all populations of individuals that are present in the city, and we suppose that there are no between-city interactions, though. We assume that two cities are connected by the direct transport such as airplanes or trains, etc.. Furthermore, we assume that susceptible, infected and recovered individuals in every city k leave toward other city $j \neq k$ at a per capita rate l_{jk}, m_{jk} and n_{jk} $(j \neq k)$, respectively. Once an individual from patch j arrives patch k, then the individual homogeneously mixes with individuals in patch k and is counted as an individual in patch k as there is no track for each individual.

Then, the term $\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j$ describes the inflow of individuals of susceptible from all other cities $j \neq k$ into city k. The term $\sum_{j=1}^{n}(1-\delta_{jk})l_{jk}S_k$ is the outflow of individuals of susceptible from city k towards all other cities $j \neq k$, and it is the same for infected individuals.

Therefore, for the model (1.2), the total outflow population $\sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{jk})l_{jk}S_k$ is in balance with the total inflow population $\sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j$, since the only input is the recruitment and $\sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{jk})l_{jk}S_k = \sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j$ holds true. Moreover, we assume that there are many communications by travel during groups in different patches each other, and hence we consider not only for infective individuals I_k in city k, disease is transmitted to the susceptible individuals S_k by the incidence rate $\beta_{kk}S_kI_k$ with a transmission rate β_{kk}, but also cross patch infection between groups of different patches such that for each I_j, $j \neq k$ who travel shortly from other city j into city k, disease is transmitted by the incidence rate $\beta_{kj}S_kI_j$ with a transmission rate β_{kj} and return to the original city j.

Observe that the variable R_k, $k = 1, 2, \ldots, n$ does not appear in (1.2). This allows us hereafter to consider the reduced system (1.2) for S_k and I_k, $k = 1, 2, \ldots, n$.

The initial conditions of system (1.2) take the form

$$\begin{cases} S_k(0) = \phi_1^k, \quad I_k(0) = \phi_2^k, \quad k = 1, 2, \ldots, n, \\ (\phi_1^1, \phi_2^1, \phi_1^2, \phi_2^2, \ldots, \phi_1^n, \phi_2^n) \in \mathbb{R}_{+0}^{2n}, \end{cases} \tag{1.5}$$

where $\mathbb{R}_{+0}^{2n} = \{(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n} : x_k,\ y_k \geq 0,\ k = 1, 2, \ldots, n\}$.

Put

$$\mathbf{B} = (\beta_{kj})_{n\times n}. \tag{1.6}$$

Hereafter, for the reader's convenience, we put

$$l_{kk} = \sum_{j=1}^{n}(1-\delta_{jk})l_{jk},\ m_{kk} = \sum_{j=1}^{n}(1-\delta_{jk})m_{jk},\ n_{kk} = \sum_{j=1}^{n}(1-\delta_{jk})n_{jk}, \tag{1.7}$$

and for an $n \times n$ matrix $\mathbf{H}$ and a positive n column vector $\mathbf{b}$ defined by

$$\mathbf{H} = \begin{bmatrix} \mu_{11}+l_{11} & -l_{12} & \cdots & -l_{1n} \\ -l_{21} & \mu_{21}+l_{22} & \cdots & -l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -l_{n1} & -l_{n2} & \cdots & \mu_{n1}+l_{nn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \tag{1.8}$$

we consider the positive n column vector $\mathbf{S}^0 = (S_1^0, S_2^0, \dots, S_n^0)^T$ such that

$$\mathbf{S}^0 = \mathbf{H}^{-1}\mathbf{b}. \tag{1.9}$$

We here note that $\mathbf{H}$ is an M-matrix by (1.7) (see for example, Berman and Plemmons [22] or Varga [23]), and $\mathbf{S}^0$ depends on l_{kj}, $k, j = 1, 2, \dots, n$.

Let $\mathbf{S} = (S_1, S_2, \cdots, S_n)^T$, and $\mathbf{S}^0 = (S_1^0, S_2^0, \cdots, S_n^0)^T$ be defined by (1.9), and

$$\tilde{\mathbf{V}} = \begin{bmatrix} \mu_{12}+\gamma_1+m_{11} & 0 & \cdots & 0 \\ 0 & \mu_{22}+\gamma_2+m_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{n2}+\gamma_n+m_{nn} \end{bmatrix} = \mathrm{diag}(\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_n), \tag{1.10}$$

and $\tilde{\mathbf{F}}(\mathbf{S})$ be an $n \times n$ matrix such that

$$\tilde{\mathbf{F}}(\mathbf{S}) = \begin{bmatrix} S_1\beta_{11} & S_1\beta_{12}+m_{12} & \cdots & S_1\beta_{1n}+m_{1n} \\ S_2\beta_{21}+m_{21} & S_2\beta_{22} & \cdots & S_2\beta_{2n}+m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_n\beta_{n1}+m_{n1} & S_n\beta_{n2}+m_{n2} & \cdots & S_n\beta_{nn} \end{bmatrix} = \begin{bmatrix} \tilde{F}_{11}(S_1) & \tilde{F}_{12}(S_1) & \cdots & \tilde{F}_{1n}(S_1) \\ \tilde{F}_{21}(S_2) & \tilde{F}_{22}(S_2) & \cdots & \tilde{F}_{2n}(S_2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{F}_{n1}(S_n) & \tilde{F}_{n2}(S_n) & \cdots & \tilde{F}_{nn}(S_n) \end{bmatrix}. \tag{1.11}$$

We define

$$\tilde{\mathbf{M}}(\mathbf{S}) = \tilde{\mathbf{V}}^{-1}\tilde{\mathbf{F}}(\mathbf{S}) = (\tilde{M}_{kj})_{n\times n}, \quad \tilde{M}_{kj} = \frac{S_k\beta_{kj} + (1-\delta_{kj})m_{kj}}{\mu_{k2}+\gamma_k+m_{kk}}, \ k, j = 1, 2, \dots, n. \tag{1.12}$$

Consider the following threshold parameter

$$\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)). \tag{1.13}$$

In this paper, under the condition that

$$\mathbf{B} \text{ is irreducible}, \tag{1.14}$$

we establish the global stability for $n \geq 2$ and not only for the special case $l_{kj} = m_{kj} = 0$, $k, j = 1, 2, \cdots, n$ of (1.2) but also for a class of more complicated multi-group epidemic model (1.2), applying extended Lyapunov function techniques than those in Guo *et al.* [4] and McCluskey [24], and no longer need such a grouping technique by graph theory in Guo *et al.* [4] and also Li *et al.* [16].

To extend our techniques to a class of the case $l_{kj} \geq 0$, $k, j = 1, 2, \cdots, n$ compared to Guo *et al.* [4] and Li *et al.* [8], we need some special techniques (see Lemma 4.1 and its proof before this lemma), and obtain sufficient conditions (see (1.17)). In particular, for a special case $l_{kj} = 0$, $k \neq j$, we establish the complete global stability for (1.2).

The main theorem in this paper is as follows.

Theorem 1.1. *For $\tilde{R}_0 \leq 1$, the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, S_2^0, 0, \dots, S_n^0, 0)$ of (1.2) is globally asymptotically stable in $\mathbf{\Gamma}$, where*

$$\mathbf{\Gamma} = \left\{ (S_1, I_1, S_2, I_2, \dots, S_n, I_n) \in \mathbb{R}_+^{2n} \mid S_k \leq S_k^0, \ \sum_{k=1}^n (S_k + I_k) \leq \frac{\bar{b}}{\underline{\mu}}, \ k = 1, 2, \dots, n \right\}, \tag{1.15}$$

and

$$\bar{b} = \sum_{k=1}^n b_k, \quad \underline{\mu} = \min_{1\leq k\leq n} (\mu_{k1}, \mu_{k2}+\gamma_k). \tag{1.16}$$

For $\tilde{R}_0 > 1$, system (1.2) is uniformly persistent in $\mathbf{\Gamma}^0$ and there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_n^*, I_n^*)$ in $\mathbf{\Gamma}^0$ (see Diekmann and Heesterbeek [25]), where $\mathbf{\Gamma}^0$ is the interior of the feasible region $\mathbf{\Gamma}$. Moreover, if there exists a positive n column vectors $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$ such that

$$\begin{cases} v_k(\beta_{kk}I_k^* + (\mu_{k1}+l_{kk})) - \displaystyle\sum_{j=1}^n v_j(1-\delta_{jk})l_{jk} \geq 0, \\ \displaystyle\sum_{j=1}^n v_j\{\beta_{jk}S_j^* + (1-\delta_{jk})m_{jk}\} \leq v_k(\mu_{k2}+\gamma_k+m_{kk}), \quad \textit{for any } k = 1, 2, \dots, n, \end{cases} \tag{1.17}$$

then $\mathbf{E}^*$ *is globally asymptotically stable in* $\mathbf{\Gamma}^0$.

Corollary 1.1. *Assume* (1.14). *Then, for* $\tilde{R}_0 > 1$, *there exists a positive* n *column vector* $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ *such that*

$$\sum_{j=1}^{n} v_j\{\beta_{jk}S_j^* + (1-\delta_{jk})m_{jk}\} = v_k(\mu_{k2} + \gamma_k + m_{kk}), \quad k = 1, 2, \ldots, n. \tag{1.18}$$

Moreover, for this $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$, *if*

$$v_k(\beta_{kk}I_k^* + (\mu_{k1} + l_{kk})) - \sum_{j=1}^{n} v_j(1-\delta_{jk})l_{jk} \geq 0, \text{ for any } k = 1, 2, \ldots, n, \tag{1.19}$$

then $\mathbf{E}^*$ *of* (1.2) *is globally asymptotically stable in* $\mathbf{\Gamma}^0$. *In particular, if*

$$l_{jk} = 0, \quad \text{for any } j, k = 1, 2, \ldots, n \text{ and } j \neq k, \tag{1.20}$$

then (1.19) *holds.*

The organization of this paper is as follows. To prove Theorem 1.1, we only consider the reduced system (1.2). In Section 2, we offer the positiveness and eventual boundedness of solutions for system (1.2). In Section 3, following the proof techniques in Guo *et al.* [4], we similarly prove the global asymptotic stability of the disease-free equilibrium for $\tilde{R}_0 \leq 1$ and the uniform persistence of system (1.2) and the existence of the endemic equilibrium $\mathbf{E}^*$ of system (1.2) for $\tilde{R}_0 > 1$ (see Proposition 3.1 and Corollary 3.1). In Section 4, for $\tilde{R}_0 > 1$, using Lyapunov function techniques to the system (1.2), under the condition (1.17), we derive an important lemma (see Lemma 4.1) and prove the global asymptotic stability for the endemic equilibrium of (1.2). Moreover, in Section 5, we investigate more wider conditions for $n = 2$ and give Theorem 5.1. Finally, in Section 6, we provide three examples of (1.2) for applications.

2 Positiveness and eventual boundedness of solutions of (1.2)

We have the following lemma on the positiveness and eventual boundedness of $S_k,\ I_k,\ k = 1, 2, \ldots, n$ of (1.2).

Lemma 2.1. *For system* (1.2), *it holds that*

$$S_k(t) > 0,\ I_k(t) > 0, \text{ for any } k = 1, 2, \ldots, n \text{ and } t > 0, \tag{2.1}$$

and

$$\begin{cases} \limsup_{t\to+\infty} \sum_{k=1}^{n}\{S_k(t) + I_k(t)\} \leq \dfrac{\bar{b}}{\underline{\mu}}, \\ \limsup_{t\to+\infty} S_k(t) \leq S_k^0, \quad k = 1, 2, \ldots, n. \end{cases} \tag{2.2}$$

Proof. By (1.2), we have that $\frac{d}{dt}S_k(+0) \geq b_k > 0$ and $S_k(0) \geq 0$ for any $k = 1, 2, \ldots, n$, which implies that there exist positive constants $t_{k0},\ k = 1, 2, \ldots, n$ such that $S_k(t) > 0$ for any $0 < t < t_{k0},\ k = 1, 2, \ldots, n$. First, we prove that $S_k(t) > 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$. On the contrary, suppose that there exist a positive t_1 and a positive integer $k_1 \in \{1, 2, \ldots, n\}$ such that $S_{k_1}(t_1) = 0$ and $S_{k_1}(t) > 0$ for any $0 < t < t_1$. But by (1.2), we have that $\frac{d}{dt}S_{k_1}(t_1) \geq b_{k_1} > 0$ which is a contradiction to the fact that $S_{k_1}(t) > 0 = S_{k_1}(t_1)$ for any $0 < t < t_1$. Hence, we obtain that $S_k(t) > 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$.

Moreover, by (1.2) and (1.7), we have that

$$I_k(t) = e^{-(\mu_{k2}+\gamma_k+m_{kk})t}\left[I_k(0) + \int_0^t e^{(\mu_{k2}+\gamma_k+m_{kk})u}\left\{S_k(u)\left(\sum_{j=1}^{n}\beta_{kj}I_j(u)\right) + \sum_{j=1}^{n}(1-\delta_{kj})m_{kj}I_j(u)\right\}du\right]$$

for $k = 1, 2, \ldots, n$ and $t > 0$, from which we obtain that $I_k(t) > 0$ for any $k = 1, 2, \ldots, n$ and $t > 0$. Thus, we obtain (2.1).

Next, we prove that $I_k(t) \geq 0$ for any $0 < t < +\infty$ and $k = 1, 2, \ldots, n$. On the contrary, suppose that there exist a positive t_2 and a positive integer $k_2 \in \{1, 2, \ldots, n\}$ such that $I_{k_2}(t_2) < 0$. Set $t_{k_2} = \inf\{0 < t < t_2 : I_{k_2}(t) < 0\}$. Then, $0 \leq t_{k_2} < t_2$ and $I_{k_2}(t_{k_2}) = 0$. But by (1.2), we have that $\frac{d}{dt}I_{k_2}(+t_{k_2}) \geq 0$ which is a contradiction to the fact that $I_{k_2}(t) < 0 = I_{k_2}(t_{k_2})$ for any $t_{k_2} < t < t_2$. Thus, we obtain (2.1).

Since by (1.7), we have

$$\sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j(t) - \sum_{k=1}^{n} l_{kk}S_k(t) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n}(1-\delta_{kj})l_{kj} - l_{jj}\right)S_j(t) = 0,$$

and similarly,

$$\sum_{k=1}^{n}\sum_{j=1}^{n}(1-\delta_{kj})m_{kj}I_j(t)-\sum_{k=1}^{n}m_{kk}I_k(t)=0,$$

therefore, by (1.2), we have that

$$\begin{aligned}\frac{d}{dt}\Big[\sum_{k=1}^{n}\{S_k(t)+I_k(t)\}\Big] &=\sum_{k=1}^{n}\Big[b_k-(\mu_{k1}+l_{kk})S_k(t)-(\mu_{k2}+\gamma_k+m_{kk})I_k(t)+\sum_{j=1}^{n}(1-\delta_{kj})\{l_{kj}S_j(t)+m_{kj}I_j(t)\}\Big]\\ &=\sum_{k=1}^{n}\{b_k-\mu_{k1}S_k(t)-(\mu_{k2}+\gamma_k)I_k(t)\}\\ &\le\sum_{k=1}^{n}b_k-\min_{1\le k\le n}(\mu_{k1},\mu_{k2}+\gamma_k)\sum_{k=1}^{n}\{S_k(t)+I_k(t)\},\end{aligned}$$

from which we obtain the first equation of (2.2). On the other hand, we have

$$\frac{dS_k}{dt}\le b_k-(\mu_{k1}+l_{kk})S_k+\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j,\quad k=1,2,\dots,n.$$

Then, by (1.9) and theory of linear differential equations and the comparison theorem, we have that for $\mathbf{S}=(S_1,S_2,\dots,n)^T$,

$$\frac{d\mathbf{S}}{dt}\le(\mathbf{S}(0)-\mathbf{S}^0)\exp(-\mathbf{H}t)+\mathbf{S}^0.$$

By the fact that $\mathbf{H}$ defined by (1.8) is an M-matrix, all the eigenvalue of H have negative real part. Hence, we have $\limsup_{t\to+\infty}\exp(-\mathbf{H}t)=\emptyset$. Thus, we obtain

$$\limsup_{t\to+\infty}S_k(t)\le S_k^0,\quad k=1,2,\dots,n,$$

from which we obtain the remaining equations of (2.2). $\square$

3 Global stability of the disease-free equilibrium $\mathbf{E}^0$ for $\tilde{R}_0\le 1$

Since we assume (1.14) and $\mathbf{H}$ and $\mathbf{V}$ defined by (1.8) and (1.10), respectively, are M-matrices (see for example, [22] or [23]), we can obtain the following Proposition, whose proof is similar to that of Guo *et al.* [4, Proposition 3.1] but for the reader's convenience, we give a proof of the following proposition:

Proposition 3.1. *(1) If $\tilde{R}_0\le 1$, then the disease-free equilibrium $\mathbf{E}^0=(S_1^0,0,S_2^0,0,\dots,S_n^0,0)$ is the unique equilibrium of (1.2) and it is globally asymptotically stable in $\mathbf{\Gamma}$.*
(2) If $\tilde{R}_0>1$, then $\mathbf{E}^0$ is unstable and system (1.2) is uniformly persistent in $\mathbf{\Gamma}^0$.

Proof. Let $\mathbf{S}=(S_1,S_2,\cdots,S_n)^T$ and $\mathbf{S}^0=(S_1^0,S_2^0,\cdots,S_n^0)^T$. Since in $\mathbf{\Gamma}$, it holds that $0\le S_k\le S_k^0$ for $1\le k\le n$ and $\emptyset\le\tilde{\mathbf{M}}(\mathbf{S})\le\tilde{\mathbf{M}}(\mathbf{S}^0)$. Since $\mathbf{M}$ is irreducible, we know $\tilde{\mathbf{M}}(\mathbf{S})$ and $\tilde{\mathbf{M}}(\mathbf{S}^0)$ are irreducible. Therefore, $\rho(\tilde{\mathbf{M}}(\mathbf{S}))<\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$, provided $\mathbf{S}\ne\mathbf{S}^0$ (see, for example, [23, Lemma 2.3]).

If $\tilde{R}_0=\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))\le 1$, then for $\mathbf{S}\ne\mathbf{S}^0$, by the above, $\rho(\tilde{\mathbf{M}}(\mathbf{S}))<1$, and

$$\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I}=\mathbf{I}$$

has only the trivial solution $\mathbf{I}=\mathbf{0}$. Thus, $\mathbf{E}^0$ is the only equilibrium of system (1.2) in $\mathbf{\Gamma}$.

Let $(\omega_1,\omega_2,\dots,\omega_n)$ be a left eigenvector of $\tilde{\mathbf{M}}(\mathbf{S}^0)$ corresponding to $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$, i.e.,

$$(\omega_1,\omega_2,\dots,\omega_n)\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))=(\omega_1,\omega_2,\dots,\omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0).$$

Since $\tilde{\mathbf{M}}(\mathbf{S}^0)$ is irreducible, we know $\omega_k>0$ for $k=1,2,\dots,n$. Set

$$L=(\omega_1,\omega_2,\dots,\omega_n)\begin{bmatrix}\mu_{12}+\gamma+m_{11}&0&\cdots&0\\0&\mu_{22}+\gamma_2+m_{22}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\mu_{n2}+\gamma_n+m_{nn}\end{bmatrix}^{-1}\begin{bmatrix}I_1\\I_2\\\vdots\\I_n\end{bmatrix}.$$

Differentiations gives

$$\begin{aligned} L' &= (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} - \mathbf{I}] \le (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S}^0)\mathbf{I} - \mathbf{I}] \\ &= \{\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1\}(\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I} \le 0, \quad \text{if } \tilde{R}_0 \le 1. \end{aligned}$$

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) < 1$, then $L' = 0 \iff \mathbf{I} = \mathbf{0}$. If $\tilde{R}_0 = 1$, then $L' = 0$ implies

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} = (\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I}. \tag{3.1}$$

If $\mathbf{S} \neq \mathbf{S}^0$, then

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}) < (\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0) = (\omega_1, \omega_2, \ldots, \omega_n).$$

Thus, (3.1) has only the trivial solution $\mathbf{I} = \mathbf{0}$. Therefore, $L' = 0 \iff I = \mathbf{0}$ or $\mathbf{S} = \mathbf{S}^0$ provided $\tilde{R}_0 \le 1$. It can be verified that the only compact invariant subset of the set, where $L' = 0$ is the singleton $\{\mathbf{E}^0\}$. By LaSalle's Invariance Principle (see [26]), $\mathbf{E}^0$ is globally asymptotically stable in $\mathbf{\Gamma}$ if $\tilde{R}_0 \le 1$.

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) > 1$ and $\mathbf{I} \neq \mathbf{0}$, we know that

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0) - (\omega_1, \omega_2, \ldots, \omega_n) = \{\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1\}(\omega_1, \omega_2, \ldots, \omega_n) > 0.$$

and thus $L' = (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} - \mathbf{I}] > 0$ in a neighborhood of $\mathbf{E}^0$ in $\mathbf{\Gamma}^0$, by continuity. This implies $\mathbf{E}^0$ is unstable.

Using a uniform persistence result from Freedman *et al.* [27] and a similar argument as in the proof of Li *et al.* [28, Proposition 3.3], we can show that, when $\tilde{R}_0 > 1$, the instability of $\mathbf{E}^0$ implies the uniform persistence of (1.2). This completes the proof of Proposition 3.1. □

Uniform persistence of system (1.2), together with uniform boundedness of solutions in $\mathbf{\Gamma}^0$ (follows from the positive invariance of the bounded region $\mathbf{\Gamma}$), implies the existence of an equilibrium of (1.2) in $\mathbf{\Gamma}^0$ (see Smith and Waltman [29, Theorem D.3] or Bhatia *et al.* [30, Theorem 2.8.6]).

Corollary 3.1. *Assume* (1.14). *If $\tilde{R}_0 > 1$, then (1.2) has at least one endemic equilibrium* $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ *such that*

$$\tilde{\mathbf{F}}(\mathbf{S}^*) - \tilde{\mathbf{V}} = \mathbf{0}, \quad \mathbf{S}^* = (S_1^*, S_2^*, \ldots, S_n^*)^T. \tag{3.2}$$

Now, we investigate the relation between the reproduction number R_0 and $\tilde{R}_0$ in (1.13).

Let

$$\mathbf{V} = \begin{bmatrix} \mu_{12} + \gamma_1 + m_{11} & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \mu_{22} + \gamma_2 + m_{22} & \cdots & -m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -m_{n1} & -m_{n2} & \cdots & \mu_{n2} + \gamma_n + m_{nn} \end{bmatrix}. \tag{3.3}$$

Then, $\mathbf{V}$ is an M-matrix. For $\mathbf{S} = (S_1, S_2, \ldots, S_n)^T$, we put

$$\mathbf{F}(S) = \begin{bmatrix} S_1\beta_{11} & S_1\beta_{12} & \cdots & S_1\beta_{1n} \\ S_2\beta_{21} & S_2\beta_{22} & \cdots & S_2\beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_n\beta_{n1} & S_n\beta_{n2} & \cdots & S_n\beta_{nn} \end{bmatrix}, \tag{3.4}$$

and $\mathbf{M}(\mathbf{S}) = \mathbf{F}(\mathbf{S})\mathbf{V}^{-1}$. Then the basic reproduction number R_0 of system (1.2) is defined (see, for example, van den Driessche and Watmough [**?**]) as follows.

$$R_0 = \rho(\mathbf{M}(\mathbf{S}^0)), \quad \mathbf{S}^0 = (S_1^0, S_2^0, \ldots, S_n^0)^T. \tag{3.5}$$

Then, by

$$\mathbf{F}(\mathbf{S}^*) - \mathbf{V} = \tilde{\mathbf{F}}(\mathbf{S}^*) - \tilde{\mathbf{V}} = \mathbf{0}, \tag{3.6}$$

we have that for the $n \times n$ unit matrix $\mathbf{E}$,

$$\mathbf{F}(\mathbf{S}^*)\mathbf{V}^{-1} = \tilde{\mathbf{V}}^{-1}\tilde{\mathbf{F}}(\mathbf{S}^*) = \mathbf{E} = \mathrm{diag}(1, 1, \ldots, 1), \quad \rho(\mathbf{F}(\mathbf{S}^*)\mathbf{V}^{-1}) = \rho(\tilde{\mathbf{V}}^{-1}\tilde{\mathbf{F}}(\mathbf{S}^*)) = 1.$$

Then,

$$\rho(\mathbf{M}(\mathbf{S}^*)) = \rho(\tilde{\mathbf{M}}(\mathbf{S}^*)) = 1, \tag{3.7}$$

and by (1.13), (3.5), Lemma 2.1 and the above discussions on irreducible non-negative matrices theory (see for example, Varga [23, Chapter 2]), we can easily obtain that

$$\begin{cases} R_0 < 1, & \text{if and only if, } \tilde{R}_0 < 1, \\ R_0 = 1, & \text{if and only if, } \tilde{R}_0 = 1, \\ R_0 > 1, & \text{if and only if, } \tilde{R}_0 > 1. \end{cases} \tag{3.8}$$

Therefore, for convenience, we may use $\tilde{R}_0$ defined by (1.13) as a threshold parameter (see Guo *et al.* [4]) in place of the reproduction number R_0 defined by (3.5).

4 Global stability of the endemic equilibrium $\mathbf{E}^*$ for $\tilde{R}_0 > 1$

In this section, we restrict our attention to the special case that $\mathbf{B}$ defined in (1.6) is irreducible and $\tilde{R}_0 > 1$. Then, $\tilde{\mathbf{M}}(\mathbf{S})$ in (1.12) is irreducible and by Corrollary 3.1, there exists an endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ of (1.2) in $\mathbf{\Gamma}^0$ such that

$$\begin{cases} b_k = (\mu_{k1} + l_{kk})S_k^* + \sum_{j=1}^n \{\beta_{kj} S_k^* I_j^* - (1-\delta_{kj}) l_{kj} S_j^*\}, \\ (\mu_{k2} + \gamma_k + m_{kk}) I_k^* = \sum_{j=1}^n \{\beta_{kj} S_k^* I_j^* + (1-\delta_{kj}) m_{kj} I_j^*\}, \quad k = 1, 2, \ldots, n. \end{cases} \tag{4.1}$$

We rewrite (1.2) as

$$\begin{cases} \dfrac{dS_k}{dt} = b_k - (\mu_{k1} + l_{kk})S_k - \sum_{j=1}^n \{\beta_{kj} S_k I_j - (1-\delta_{kj}) l_{kj} S_j\}, \\ \dfrac{dI_k}{dt} = \sum_{j=1}^n \{\beta_{kj} S_k I_j + (1-\delta_{kj}) m_{kj} I_j\} - (\mu_{k2} + \gamma_k + m_{kk}) I_k, \quad k = 1, 2 \ldots, n. \end{cases} \tag{4.2}$$

Set

$$U = \sum_{k=1}^n v_k \left\{ S_k^* g\left(\frac{S_k}{S_k^*}\right) + I_k^* g\left(\frac{I_k}{I_k^*}\right) \right\}, \tag{4.3}$$

where $v_1, v_2, \cdots, v_n$ will be appropriately chosen later (see (1.17)) and

$$g(x) = x - 1 - \ln x \geq g(1) = 0, \quad \text{for any } x > 0. \tag{4.4}$$

Differentiating U, we have

$$\frac{dU}{dt} = \sum_{k=1}^n v_k \left\{ \left(1 - \frac{S_k^*}{S_k}\right) \frac{dS_k}{dt} + \left(1 - \frac{I_k^*}{I_k}\right) \frac{dI_k}{dt} \right\}.$$

Put

$$x_k = \frac{S_k}{S_k^*}, \quad y_k = \frac{I_k}{I_k^*}, \quad k = 1, 2, \ldots, n. \tag{4.5}$$

By (4.1) and (4.2), we have that

$$\begin{aligned} \frac{dS_k}{dt} &= b_k - (\mu_{k1} + l_{kk})S_k - \sum_{j=1}^n \{\beta_{kj} S_k I_j - (1-\delta_{kj}) l_{kj} S_j\} \\ &= -(\mu_{k1} + l_{kk})(S_k - S_k^*) - \sum_{j=1}^n \{\beta_{kj}(S_k I_j - S_k^* I_j^*) - (1-\delta_{kj}) l_{kj}(S_j - S_j^*)\} \\ &= -(\mu_{k1} + l_{kk}) S_k^* (x_k - 1) - \sum_{j=1}^n \{\beta_{kj} S_k^* I_j^* (x_k y_j - 1) - (1-\delta_{kj}) l_{kj} S_j^* (x_j - 1)\}, \end{aligned}$$

and

$$\begin{aligned} \frac{dI_k}{dt} &= \sum_{j=1}^n \{\beta_{kj} S_k I_j + (1-\delta_{kj}) m_{kj} I_j\} - (\mu_{k2} + \gamma_k + m_{kk}) I_k \\ &= \sum_{j=1}^n \{\beta_{kj} S_k^* I_j^* x_k y_j + (1-\delta_{kj}) m_{kj} I_j^* y_j\} - (\mu_{k2} + \gamma_k + m_{kk}) I_k^* y_k \\ &= \sum_{j=1}^n \{\beta_{kj} S_k^* I_j^* (x_k y_j - y_k) + (1-\delta_{kj}) m_{kj} I_j^* (y_j - y_k)\}. \end{aligned}$$

Then,

$$
\begin{aligned}
\frac{dU}{dt} =& \sum_{k=1}^{n} v_k \Bigg[\left(1-\frac{1}{x_k}\right) \Bigg\{ -(\mu_{k1}+l_{kk})S_k^*(x_k-1) - \sum_{j=1}^{n} \{\beta_{kj}S_k^*I_j^*(x_ky_j-1) - (1-\delta_{kj})l_{kj}S_j^*(x_j-1)\} \Bigg\} \\
&+ \left(1-\frac{1}{y_k}\right) \Bigg\{ \sum_{j=1}^{n} \{\beta_{kj}S_k^*I_j^*(x_ky_j-y_k) + (1-\delta_{kj})m_{kj}I_j^*(y_j-y_k)\} \Bigg\} \Bigg] \\
=& \sum_{k=1}^{n} v_k \Bigg[-(\mu_{k1}+l_{kk})S_k^* \left(1-\frac{1}{x_k}\right)(x_k-1) \sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j^* \left(1-\frac{1}{x_k}\right)(x_j-1) \\
&+ \sum_{j=1}^{n} \beta_{kj}S_k^*I_j^* \Bigg\{ \left(1-\frac{1}{x_k}\right)(1-x_ky_j) + \left(1-\frac{1}{y_k}\right)(x_ky_j-y_k) \Bigg\} + \sum_{j=1}^{n}(1-\delta_{kj})m_{kj}I_j^* \left(1-\frac{1}{y_k}\right)(y_j-y_k) \Bigg]. \quad (4.6)
\end{aligned}
$$

Now, consider the first part of the last equation in (4.6). Since

$$
\begin{aligned}
\left(1-\frac{1}{x_k}\right)(x_k-1) &= x_k + \frac{1}{x_k} - 2 = g(x_k) + g\left(\frac{1}{x_k}\right), \\
\left(1-\frac{1}{x_k}\right)(x_j-1) &= x_j - \frac{x_j}{x_k} + \frac{1}{x_j} - 1 = g(x_j) - g\left(\frac{x_j}{x_k}\right) + g\left(\frac{1}{x_k}\right),
\end{aligned}
$$

it follows from the definition of l_{kk} $(k = 1, 2, \ldots, n)$ in (1.7) that

$$
(\mu_{k1}+l_{kk})S_k^* \left(1-\frac{1}{x_k}\right)(x_k-1) = \left(\mu_{k1} + \sum_{j=1}^{n}(1-\delta_{jk})l_{jk}\right) S_k^* \left\{ g(x_k) + g\left(\frac{1}{x_k}\right) \right\} \quad (4.7)
$$

and

$$
\sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j^* \left(1-\frac{1}{x_k}\right)(x_j-1) = \sum_{j=1}^{n}(1-\delta_{kj})l_{kj}S_j^* \left\{ g(x_j) - g\left(\frac{x_j}{x_k}\right) + g\left(\frac{1}{x_k}\right) \right\} \quad (4.8)
$$

hold for $k = 1, 2, \ldots, n$. Next, we consider the remaining parts of the last equation in (4.6):

$$
\begin{aligned}
\left(1-\frac{1}{x_k}\right)(1-x_ky_j) + \left(1-\frac{1}{y_k}\right)(x_ky_j-y_k) =& \left(1 - \frac{1}{x_k} - x_ky_j + y_j\right) + \left(x_ky_j - \frac{x_ky_j}{y_k} - y_k + 1\right) \\
=& 2 - \frac{1}{x_k} + y_j - \frac{x_ky_j}{y_k} - y_k \\
=& -g\left(\frac{1}{x_k}\right) - g\left(\frac{x_ky_j}{y_k}\right) + \{g(y_j) - g(y_k)\},
\end{aligned}
$$

and

$$
\left(1-\frac{1}{y_k}\right)(y_j-y_k) = y_j - \frac{y_j}{y_k} - y_k + 1 = -g\left(\frac{y_j}{y_k}\right) + \{g(y_j) - g(y_k)\}.
$$

Thus,

$$
\begin{aligned}
& \sum_{k=1}^{n} v_k \Bigg[\sum_{j=1}^{n} \beta_{kj}S_k^*I_j^* \Bigg\{ \left(1-\frac{1}{x_k}\right)(1-x_ky_j) + \left(1-\frac{1}{y_k}\right)(x_ky_j-y_k) \Bigg\} + \left(1-\frac{1}{y_k}\right) \sum_{j=1}^{n}(1-\delta_{kj})m_{kj}I_j^*(y_j-y_k) \Bigg] \\
=& -\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \Bigg[\beta_{kj}S_k^*I_j^* \left\{ g\left(\frac{1}{x_k}\right) + g\left(\frac{x_ky_j}{y_k}\right) \right\} + (1-\delta_{kj})m_{kj}I_j^* g\left(\frac{y_j}{y_k}\right) \Bigg] \\
& + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (\beta_{kj}S_k^* + (1-\delta_{kj})m_{kj})I_j^* \{g(y_j) - g(y_k)\}, \quad (4.9)
\end{aligned}
$$

and by (4.1), we have that

$$
\begin{aligned}
&\sum_{k=1}^{n} v_k \sum_{j=1}^{n} (\beta_{kj} S_k^* + (1-\delta_{kj}) m_{kj}) I_j^* \{g(y_j) - g(y_k)\} \\
&= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (\beta_{kj} S_k^* + (1-\delta_{kj}) m_{kj}) I_j^* g(y_j) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (\beta_{kj} S_k^* + (1-\delta_{kj}) m_{kj}) I_j^* g(y_k) \\
&= \sum_{j=1}^{n} v_j \sum_{k=1}^{n} (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) I_k^* g(y_k) - \sum_{k=1}^{n} v_k (\mu_{k2} + \gamma_k + m_{kk}) I_k^* g(y_k) \\
&= \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) - v_k (\mu_{k2} + \gamma_k + m_{kk}) \right\} I_k^* g(y_k). \qquad (4.10)
\end{aligned}
$$

Therefore, from (4.3)-(4.10), we have that

$$
\begin{aligned}
\frac{dU}{dt} =& -\sum_{k=1}^{n} v_k (\mu_{k1} + l_{kk}) S_k^* \left\{ g(x_k) + g\left(\frac{1}{x_k}\right) \right\} + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* \left\{ g(x_j) - g\left(\frac{x_j}{x_k}\right) + g\left(\frac{1}{x_k}\right) \right\} \\
&- \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* I_j^* \left\{ g\left(\frac{1}{x_k}\right) + g\left(\frac{x_k y_j}{y_k}\right) \right\} + (1-\delta_{kj}) m_{kj} I_j^* g\left(\frac{y_j}{y_k}\right) \right] \\
&+ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) - v_k (\mu_{k2} + \gamma_k + m_{kk}) \right\} I_k^* g(y_k) \\
=& -\sum_{k=1}^{n} \left\{ v_k (\beta_{kk} I_k^* + (\mu_{k1} + l_{kk})) - \sum_{j=1}^{n} v_j (1-\delta_{jk}) l_{jk} \right\} S_k^* g(x_k) \\
&- \sum_{k=1}^{n} v_k \left\{ \left(\sum_{j=1}^{n} \beta_{kj} I_j^* + (\mu_{k1} + l_{kk}) \right) S_k^* - \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* \right\} g\left(\frac{1}{x_k}\right) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* g\left(\frac{x_j}{x_k}\right) \\
&- \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left\{ (1-\delta_{kj}) \beta_{kj} S_k^* I_j^* g\left(\frac{x_k y_j}{y_k}\right) + (1-\delta_{kj}) m_{kj} I_j^* g\left(\frac{y_j}{y_k}\right) \right\} \\
&+ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) - v_k (\mu_{k2} + \gamma_k + m_{kk}) \right\} I_k^* g(y_k), \qquad (4.11)
\end{aligned}
$$

where we used the following relations:

$$
\begin{cases}
\displaystyle \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* g(x_j) = \sum_{j=1}^{n} v_j \sum_{k=1}^{n} (1-\delta_{jk}) l_{jk} S_k^* g(x_k) = \sum_{k=1}^{n} \sum_{j=1}^{n} v_j (1-\delta_{jk}) l_{jk} S_k^* g(x_k), \\
\displaystyle \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* g\left(\frac{x_k y_j}{y_k}\right) = \beta_{kk} S_k^* I_k^* g(x_k) + \sum_{j=1}^{n} (1-\delta_{kj}) \beta_{kj} S_k^* I_j^* g\left(\frac{x_k y_j}{y_k}\right), \quad k = 1, 2, \dots, n.
\end{cases}
$$

Moreover, by (4.1), we have

$$
\begin{cases}
\displaystyle \sum_{j=1}^{n} \{\beta_{kj} I_j^* + (\mu_{k1} + l_{kk})\} S_k^* - \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* = b_k, \\
\displaystyle \sum_{j=1}^{n} v_j (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) - v_k (\mu_{k2} + \gamma_k + m_{kk}) = 0, \quad k = 1, 2, \dots, n.
\end{cases} \qquad (4.12)
$$

Hence, we obtain the following lemma.

Lemma 4.1. *Assume $\tilde{R}_0 > 1$. Then,*

$$
\begin{aligned}
\frac{dU}{dt} =& -\sum_{k=1}^{n} \left\{ v_k (\beta_{kk} I_k^* + (\mu_{k1} + l_{kk})) - \sum_{j=1}^{n} v_j (1-\delta_{jk}) l_{jk} \right\} S_k^* g(x_k) - \sum_{k=1}^{n} v_k b_k g\left(\frac{1}{x_k}\right) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} (1-\delta_{kj}) l_{kj} S_j^* g\left(\frac{x_j}{x_k}\right) \\
&- \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left\{ (1-\delta_{kj}) g\left(\frac{x_k y_j}{y_k}\right) + (1-\delta_{kj}) m_{kj} I_j^* g\left(\frac{y_j}{y_k}\right) \right\} \\
&+ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j (\beta_{jk} S_j^* + (1-\delta_{jk}) m_{jk}) - v_k (\mu_{k2} + \gamma_k + m_{kk}) \right\} I_k^* g(y_k). \qquad (4.13)
\end{aligned}
$$

Moreover, if there exists a positive n column vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ such that (1.17) holds, then $\frac{dU}{dt} \leq 0$.

Proof of Theorem 1.1. If $\tilde{R}_0 \leq 1$, then by Proposition 3.1, we can obtain the first part $\tilde{R}_0 \leq 1$ of Theorem 1.1.

We now consider the case $\tilde{R}_0 > 1$. Then, by Proposition 3.1, system (1.2) is uniformly persistent in $\mathbf{\Gamma}^0$, and by Corollary 3.1, there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$. Moreover, suppose that there exists a positive n column vector $\mathbf{v}$ such that (1.17) holds. By Lemma 4.1, we have (4.13) for (4.3) and $\frac{dU(t)}{dt} = 0$ if and only if

$$x_k = 1, \text{ and } y_k = y_j, \text{ for any } t > 0, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n. \tag{4.14}$$

Then, there exists a positive constant c such that

$$\frac{I_k(t)}{I_k^*} = c, \text{ for any } t > 0, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n.$$

Thus, substituting

$$S_k(t) = S_k^*, \text{ and } I_k(t) = cI_k^*, \text{ for any } t > 0, \ k = 1, 2, \ldots, n,$$

into the first equation of system (1.2), we obtain that

$$0 = b_k - (\mu_{k1} + l_{kk}) + c\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - (1 - \delta_{kj}) l_{kj} S_j^*, \quad \text{for any } k = 1, 2, \ldots, n. \tag{4.15}$$

Since the right-hand side of (4.15) is strictly decreasing in c, (4.15) holds if and only if $c = 1$, namely at $\mathbf{E}^*$. Therefore, the only compact invariant subset where $\frac{dU(t)}{dt} = 0$ is the singleton $\{\mathbf{E}^*\}$. By Proposition 3.1 and a similar argument as in Section 3, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$, if $\tilde{R}_0 > 1$. Hence, the proof of this theorem is complete. $\square$

Lemma 4.2. *The following system*

$$\sum_{j=1}^{n} v_j \{\beta_{jk} S_j^* + (1 - \delta_{jk}) m_{jk}\} = v_k(\mu_{k2} + \gamma_k + m_{kk}), \quad k = 1, 2, \ldots, n \tag{4.16}$$

has a positive solution $(v_1, v_2, \ldots, v_n)$ defined by

$$(v_1, v_2, \cdots, v_n) = (C_{11}, C_{22}, \ldots, C_{nn}). \tag{4.17}$$

Here, C_{kk} $k = 1, 2, \ldots, n$ denote the cofactor of the k-th diagonal entry of $\tilde{\mathbf{B}}$, where

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{j\neq 1} \tilde{\beta}_{1j} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{j\neq 2} \tilde{\beta}_{2j} & \cdots & -\tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{j\neq n} \tilde{\beta}_{nj} \end{bmatrix}, \quad \tilde{\beta}_{kj} = (\beta_{kj} S_k^* + (1 - \delta_{kj}) m_{kj}) I_j^*$$

for $1 \leq k, j \leq n$.

Proof. Consider a basis for the solution space of the linear system

$$\tilde{\mathbf{B}}\mathbf{v} = 0, \tag{4.18}$$

which can be written as (4.17) (see for example, Berman and Plemmons [22]). By the irreducibility of $\mathbf{B}$, we know that $(\tilde{\beta}_{kj})_{n\times n}$ is irreducible and $v_k = C_{kk} > 0$, $k = 1, 2, \ldots, n$. Then, by (4.18), we have that

$$\begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{21} & \cdots & \tilde{\beta}_{n1} \\ \tilde{\beta}_{12} & \tilde{\beta}_{22} & \cdots & \tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\beta}_{1n} & \tilde{\beta}_{2n} & \cdots & \tilde{\beta}_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \left(\sum_{j=1}^{n} \tilde{\beta}_{1j}\right) v_1 \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{2j}\right) v_2 \\ \vdots \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{nj}\right) v_n \end{bmatrix},$$

from which we have that

$$\sum_{j=1}^{n} v_j\tilde{\beta}_{jk} = v_k\sum_{j=1}^{n}\tilde{\beta}_{kj}, \quad k=1,2,\ldots,n.$$

This yields

$$\sum_{j=1}^{n} v_j\{\beta_{jk}S_j^* + (1-\delta_{jk})m_{jk}\}I_k^* = v_k\sum_{j=1}^{n}\{\beta_{kj}S_k^* + (1-\delta_{kj})m_{kj}\}I_j^* = v_k(\mu_{k2}+\gamma_k+m_{kk})I_k^*$$

for any $k=1,2,\ldots,n$. Since $I_k^*>0$, we obtain that (4.16) has a positive solution $(v_1,v_2,\ldots,v_n)$ defined by (4.17). $\square$

Proof of Corollary 1.1. For system (1.2), by Lemma 4.2, it is evident that there exists a positive n column vectors $\mathbf{v}=(v_1,v_2,\ldots,v_n)^T$ such that (1.18) holds. Hence, by Theorem 1.1, we obtain the conclusion of this corollary. $\square$

5 The case that B is reducible for $n=2$

In this section, we consider more wider class such that $\tilde{\mathbf{M}}$ is irreducible but we may admit the case that $\mathbf{B}$ is reducible in (1.2). For simplicity to illustrate this, we only consider the following case $n=2$ that $\beta_{11}\neq 0$ or $\beta_{12}\neq 0$, but we may admit $\beta_{21}=\beta_{22}=0$. Set

$$U=\left\{S_1^* g\left(\frac{S_1}{S_1^*}\right)+I_1^* g\left(\frac{I_1}{I_1^*}\right)\right\}+\left\{a_2S_2^* g\left(\frac{S_2}{S_2^*}\right)+a_4I_2^* g\left(\frac{I_2}{I_1^*}\right)\right\}, \tag{5.1}$$

where the positive constants a_2 and a_4 will be appropriately chosen later (see (5.6)). Differentiating U along the solutions of (1.2) for $n=2$, we have

$$\frac{dU(t)}{dt}=\left(1-\frac{S_1^*}{S_1}\right)\frac{dS_1}{dt}+\left(1-\frac{I_1^*}{I_1}\right)\frac{dI_1}{dt}+a_2\left(1-\frac{S_2^*}{S_2}\right)\frac{dS_2}{dt}+a_4\left(1-\frac{I_2^*}{I_2}\right)\frac{dI_2}{dt}. \tag{5.2}$$

Then, similar to the discussion in Section 4, by Lemma 4.1, we have that

$$\begin{aligned}
\frac{dU(t)}{dt} =& -\left[(\mu_{11}+l_{11})S_1^*\left(1-\frac{1}{x_1}\right)(x_1-1)-l_{12}S_2^*\left(1-\frac{1}{x_1}\right)(x_2-1)\right]\\
&+\sum_{j=1}^{2}\beta_{1j}S_1^*I_j^*\left\{\left(1-\frac{1}{x_1}\right)(1-x_1y_j)+\left(1-\frac{1}{y_1}\right)(x_1y_j-y_1)\right\}+m_{12}I_2^*\left(1-\frac{1}{y_1}\right)(y_2-y_1)\\
&-a_2\left[(\mu_{21}+l_{22})S_2^*\left(1-\frac{1}{x_2}\right)(x_2-1)-l_{21}S_1^*\left(1-\frac{1}{x_2}\right)(x_1-1)\right]\\
&+\sum_{j=1}^{2}\beta_{2j}S_2^*I_j^*\left\{a_2\left(1-\frac{1}{x_2}\right)(1-x_2y_j)+a_4\left(1-\frac{1}{y_2}\right)(x_2y_j-y_2)\right\}+a_4m_{21}I_1^*\left(1-\frac{1}{y_2}\right)(y_1-y_2)\\
=& -\left[(\mu_{11}+l_{11})S_1^*\left\{g(x_1)+g\left(\frac{1}{x_1}\right)\right\}+l_{12}S_2^*\left\{g\left(\frac{x_2}{x_1}\right)-g(x_2)-g\left(\frac{1}{x_1}\right)\right\}\right]\\
&-\sum_{j=1}^{2}\beta_{1j}S_1^*I_j^*\left\{g\left(\frac{1}{x_1}\right)+g(x_1y_j)-g(y_j)+g\left(\frac{x_1y_j}{y_1}\right)+g(y_1)-g(x_1y_j)\right\}-m_{12}I_2^*\left\{g\left(\frac{y_2}{y_1}\right)+g(y_1)-g(y_2)\right\}\\
&-a_2\left[(\mu_{21}+l_{22})S_2^*\left\{g(x_2)+g\left(\frac{1}{x_2}\right)\right\}+l_{21}S_1^*\left\{g\left(\frac{x_1}{x_2}\right)-g(x_1)-g\left(\frac{1}{x_2}\right)\right\}\right]\\
&-\sum_{j=1}^{2}\beta_{2j}S_2^*I_j^*\left[a_2\left\{g\left(\frac{1}{x_2}\right)+g(x_2y_j)-g(y_j)\right\}+a_4\left\{g\left(\frac{x_2y_j}{y_2}\right)+g(y_2)-g(x_2y_j)\right\}\right]\\
&-a_4m_{21}I_1^*\left\{g\left(\frac{y_1}{y_2}\right)+g(y_2)-g(y_1)\right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}\frac{dU(t)}{dt} &= -\{(\mu_{11}+l_{11})+\beta_{11}I_1^*-a_2l_{21}\}S_1^*g(x_1)-\{a_2(\mu_{21}+l_{22})+a_4\beta_{22}I_2^*-l_{12}\}S_2^*g(x_2)\\
&-(a_2-a_4)\beta_{21}S_2^*I_1^*g(x_2y_1)-(a_2-a_4)\beta_{22}S_2^*I_2^*g(x_2y_2)\\
&-\{(\beta_{12}S_1^*+m_{12})I_2^*-(a_2\beta_{21}S_2^*+a_4m_{21})I_1^*\}g(y_1)\\
&+[(a_2-a_4)S_2^*(\beta_{21}I_1^*+\beta_{22}I_2^*)+(\beta_{12}S_1^*+m_{12})I_2^*-(a_2\beta_{21}S_2^*+a_4m_{21})I_1^*]g(y_2)\\
&-[\{(\mu_{11}+l_{11})+(\beta_{11}I_1^*+\beta_{12}I_2^*)\}S_1^*-l_{12}S_2^*]g\left(\frac{1}{x_1}\right)\\
&-a_2[\{(\mu_{21}+l_{22})+(\beta_{21}I_1^*+\beta_{22}I_2^*)\}S_2^*-l_{21}S_1^*]g\left(\frac{1}{x_2}\right)\\
&-\left\{l_{12}S_2^*g\left(\frac{x_2}{x_1}\right)+a_2l_{21}S_1^*g\left(\frac{x_1}{x_2}\right)\right\}-\left\{\beta_{12}S_1^*I_2^*g\left(\frac{x_1y_2}{y_1}\right)+m_{12}I_2^*g\left(\frac{y_2}{y_1}\right)\right\}\\
&-a_4\left\{\beta_{21}S_2^*I_1^*g\left(\frac{x_2y_1}{y_2}\right)+m_{21}I_1^*g\left(\frac{y_1}{y_2}\right)\right\}.\end{aligned}$$

Noting that the endemic equilibrium of (1.2) satisfy the following equations:

$$\begin{cases}\{(\mu_{11}+l_{11})+(\beta_{11}I_1^*+\beta_{12}I_2^*)\}S_1^*-l_{12}S_2^*=b_1\geq 0,\\ \{(\mu_{21}+l_{22})+(\beta_{21}I_1^*+\beta_{22}I_2^*)\}S_2^*-l_{21}S_1^*=b_2\geq 0,\\ S_1^*(\beta_{11}I_1^*+\beta_{12}I_2^*)-(\mu_{12}+\gamma_1+m_{21})I_1^*+m_{12}I_2^*=0,\\ S_2^*(\beta_{21}I_1^*+\beta_{22}I_2^*)-(\mu_{22}+\gamma_2+m_{12})I_2^*+m_{21}I_1^*=0,\end{cases} \tag{5.3}$$

we therefore obtain

$$\begin{aligned}&-[\{(\mu_{11}+l_{11})+(\beta_{11}I_1^*+\beta_{12}I_2^*)\}S_1^*-l_{12}S_2^*]g\left(\frac{1}{x_1}\right)-a_2[\{(\mu_{21}+l_{22})+(\beta_{21}I_1^*+\beta_{22}I_2^*)\}S_2^*-l_{21}S_1^*]g\left(\frac{1}{x_2}\right)\\
=&-b_1g\left(\frac{1}{x_1}\right)-a_2b_2g\left(\frac{1}{x_2}\right)\leq 0.\end{aligned}$$

Moreover,

$$\begin{aligned}&-\left\{l_{12}S_2^*g\left(\frac{x_2}{x_1}\right)+a_2l_{21}S_1^*g\left(\frac{x_1}{x_2}\right)\right\}-\left\{\beta_{12}S_1^*I_2^*g\left(\frac{x_1y_2}{y_1}\right)+m_{12}I_2^*g\left(\frac{y_2}{y_1}\right)\right\}\\
&-a_4\left\{\beta_{21}S_2^*I_1^*g\left(\frac{x_2y_1}{y_2}\right)+m_{21}I_1^*g\left(\frac{y_1}{y_2}\right)\right\}\leq 0.\end{aligned} \tag{5.4}$$

Thus,

$$\begin{aligned}\frac{dU(t)}{dt} \leq &-\{(\mu_{11}+l_{11})+\beta_{11}I_1^*-a_2l_{21}\}S_1^*g(x_1)-\{a_2(\mu_{21}+l_{22})+a_4\beta_{22}I_2^*-l_{12}\}S_2^*g(x_2)\\
&-(a_2-a_4)\beta_{21}S_2^*I_1^*g(x_2y_1)-(a_2-a_4)\beta_{22}S_2^*I_2^*g(x_2y_2)\\
&-\{(\beta_{12}S_1^*+m_{12})I_2^*-(a_2\beta_{21}S_2^*+a_4m_{21})I_1^*\}g(y_1)\\
&+[(a_2-a_4)S_2^*(\beta_{21}I_1^*+\beta_{22}I_2^*)+\{(\beta_{12}S_1^*+m_{12})I_2^*-(a_2\beta_{21}S_2^*+a_4m_{21})I_1^*\}]g(y_2).\end{aligned} \tag{5.5}$$

Lemma 5.1. *For system* (1.2)*, assume* $\tilde{R}_0>1$ *and consider a Lyapunov function* (5.1) *such that the positive constants* a_2 *and* a_4 *satisfy the following condition:*

$$\begin{cases}\text{(i) } \textit{if } \beta_{21}\neq 0 \textit{ or } \beta_{22}\neq 0,\ \textit{then}\\ \dfrac{l_{12}}{(\mu_{21}+l_{22})+\beta_{22}I_2^*}\leq a_2=a_4=\dfrac{(\beta_{12}S_1^*+m_{12})I_2^*}{(\beta_{21}S_2^*+m_{21})I_1^*}\leq\dfrac{(\mu_{11}+l_{11})+\beta_{11}I_1^*}{l_{21}},\\ \text{(ii) } \textit{if } \beta_{21}=\beta_{22}=0,\ \textit{then}\\ \dfrac{l_{12}}{(\mu_{21}+l_{22})}\leq a_2\leq\dfrac{(\mu_{11}+l_{11})+\beta_{11}I_1^*}{l_{21}},\ \textit{and } a_4=\dfrac{(\beta_{12}S_1^*+m_{12})I_2^*}{\beta_{21}S_2^*+m_{21}I_1^*},\end{cases} \tag{5.6}$$

where

$$\frac{(\beta_{12}S_1^*+m_{12})I_2^*}{(\beta_{21}S_2^*+m_{21})I_1^*}=\frac{(\mu_{12}+\gamma_1+m_{21})-\beta_{11}S_1^*}{\beta_{21}S_2^*+m_{21}}=\frac{\beta_{12}S_1^*+m_{12}}{(\mu_{22}+\gamma_2+m_{12})-\beta_{22}S_2^*}. \tag{5.7}$$

Then, $\frac{dU}{dt}\leq 0$.

Proof. For system (1.2), assume $\tilde{R}_0 > 1$ and consider a Lyapunov function (5.1) such that the positive constants a_2 and a_4 satisfy the following condition that

$$\begin{cases} (a_2 - a_4)\beta_{21} = 0, \ (a_2 - a_4)\beta_{22} = 0, \\ (\beta_{12} S_1^* + m_{12}) I_2^* - (a_2 \beta_{21} S_2^* + a_4 m_{21}) I_1^* = 0, \\ (\mu_{11} + l_{11}) + \beta_{11} I_1^* - a_2 l_{21} \geq 0, \ \text{and} \\ a_2(\mu_{21} + l_{22}) + a_4 \beta_{22} I_2^* - l_{12} \geq 0, \end{cases} \tag{5.8}$$

then by (5.5) and conditions (5.8), we obtain that $\frac{dU}{dt} \leq 0$. Moreover, by (5.3), one can see that (5.7) holds and (5.8) is equivalent to (5.6). Hence, we obtain the conclusion of this lemma. $\square$

Theorem 5.1. *For system* (1.2) *with* $n = 2$ *such that* $\beta_{11} \neq 0$ *or* $\beta_{12} \neq 0$ *(we may admit* $\beta_{21} = \beta_{22} = 0$*), if* $\tilde{\mathbf{M}}$ *is irreducible,* $\tilde{R}_0 > 1$ *and* (5.6) *holds, then* $\mathbf{E}^*$ *is globally asymptotically stable in* $\mathbf{\Gamma}^0$.

Proof. By Lemma 5.1, we obtain $\frac{dU}{dt} \leq 0$ for the Lyapunov function (5.1) with (5.6). Moreover, $\frac{dU(t)}{dt} = 0$ holds if and only if

$$x_k = 1, \ \text{and} \ y_k = y_j, \ \text{for any} \ t > 0, \ j = 1, 2, \ k = 1, 2. \tag{5.9}$$

Then, there exists a positive constant c such that

$$\frac{I_k(t)}{I_k^*} = c, \ \text{for any} \ t > 0, \ j = 1, 2, \ k = 1, 2.$$

Thus, substituting

$$S_k(t) = S_k^*, \ \text{and} \ I_k(t) = c I_k^*, \ \text{for any} \ t > 0, \ k = 1, 2,$$

into the first equation of system (1.2), we obtain that

$$0 = b_k - (\mu_{k1} + l_{kk}) + c \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* - (1 - \delta_{kj}) l_{kj} S_j^*, \ \text{for any} \ k = 1, 2. \tag{5.10}$$

Since the right-hand side of (5.10) is strictly decreasing in c, (4.15) holds if and only if $c = 1$, namely at $\mathbf{E}^*$. Therefore, the only compact invariant subset where $\frac{dU(t)}{dt} = 0$ is the singleton $\{\mathbf{E}^*\}$. By Proposition 3.1 and a similar argument as in Section 3, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$, if $\tilde{R}_0 > 1$. Hence, the proof of this theorem is complete. $\square$

For the case that $\tilde{\mathbf{M}}$ is irreducible in (1.12) but $\mathbf{B}$ in (1.6) is reducible for $n \geq 2$, one can similarly investigate by the above discussions. We leave this to the future work for the readers.

6 Applications

In this section, we give three epidemic models with patches through migration and cross patch infection which satisfies the sufficient conditions in Theorem 1.1 for $n \geq 2$ or Theorem 5.1 for $n = 2$.

Example 6.1

$$\begin{cases} \frac{dS_1}{dt} = b - \beta S_1 I_1 - (\mu + \alpha) S_1 + \alpha S_2 - \kappa\alpha S_1 I_2, \\ \frac{dI_1}{dt} = \beta S_1 I_1 - (\mu + \gamma + \alpha) I_1 + \alpha I_2 + \kappa\alpha S_1 I_2, \\ \frac{dR_1}{dt} = \gamma I_1 - (\mu + \alpha) R_1 + \alpha R_2, \\ \frac{dS_2}{dt} = b - \beta S_2 I_2 - (\mu + \alpha) S_2 + \alpha S_1 - \kappa\alpha S_2 I_1, \\ \frac{dI_2}{dt} = \beta S_2 I_2 - (\mu + \gamma + \alpha) I_2 + \alpha I_1 + \kappa\alpha S_2 I_1, \\ \frac{dR_2}{dt} = \gamma I_2 - (\mu + \alpha) R_2 + \alpha R_1, \end{cases} \tag{6.1}$$

with initial conditions:

$$\begin{cases} S_i(0) = \phi_1^i, \ \ I_i(0) = \phi_2^i, \ \ R_i(0) = \phi_3^i, \quad i = 1, 2, \\ (\phi_1^1, \phi_2^1, \phi_3^1, \phi_1^2, \phi_2^2, \phi_3^2) \in \mathbb{R}_{+0}^6, \end{cases} \tag{6.2}$$

where b is the recruitment rate of the population, μ is the natural death rate of the population, β is the proportionality constant, γ is the natural recovery rate of the infective individuals. Susceptible, infected and recovered individuals of every i group leave for j group ($i \neq j, i, j = 1, 2$) at a per capita rate α. We assume that two groups are connected each other by the direct communication, etc. When the infective individuals αI_j in j group travel into i group, disease is transmitted to the susceptible individuals S_i in i group with the incidence rate $\kappa\alpha S_i I_j$ with a transmission rate $\kappa\alpha$.

The basic reproduction number of system (6.1) is

$$R_0 = \frac{b(\beta + \kappa\alpha)}{\mu(\mu + \gamma)}. \tag{6.3}$$

System (6.1) has a unique positive solution $(S_1(t), I_1(t), R_1(t), S_2(t), I_2(t), R_2(t))$ satisfying the initial condition (6.2) and always has a disease-free equilibrium $E^0 = (b/\mu, 0, 0, b/\mu, 0, 0)$, and an endemic equilibrium $E^* = (S^*, I^*, R^*, S^*, I^*, R^*)$ if $R_0 > 1$, where

$$S^* = \frac{\mu+\gamma}{\beta+\kappa\alpha},\ I^* = \frac{b}{\mu+\gamma} - \frac{\mu}{\beta+\kappa\alpha},\ R^* = \frac{\gamma}{\mu}\left(\frac{b}{\mu+\gamma} - \frac{\mu}{\beta+\kappa\alpha}\right). \tag{6.4}$$

Then, condition (i) in (5.6) becomes $\frac{\alpha}{(\mu+\alpha)+\beta I^*} < a_2 = a_4 = 1 < \frac{(\mu+\alpha)+\beta I^*}{\alpha}$ which is satisfied. Hence, by Theorem 5.1, we establish that the global dynamics of system (6.1) is fully determined by a threshold parameter R_0; the global stability of the disease-free equilibrium E^0 and the endemic equilibrium E^* of (6.1) is completely determined by R_0.

Example 6.2 (See Li *et al.* [16, Section 4]).

$$\begin{cases} \frac{dS}{dt} = q_1\mu A - (\mu+p)S - \beta SI + \varepsilon V, \\ \frac{dV}{dt} = q_2\mu A + pS - (\mu+\varepsilon)V, \\ \frac{dE}{dt} = \beta SI - (\mu+\gamma)E, \\ \frac{dI}{dt} = \gamma E - (\mu+\alpha+\delta)I. \end{cases} \tag{6.5}$$

(6.5) is equivalent to the following system

$$\begin{cases} \frac{dS_1}{dt} = b_1 - (\mu+p)S_1 - \beta S_1 I_2 + \varepsilon S_2, \\ \frac{dS_2}{dt} = b_2 + pS_1 - (\mu+\varepsilon)S_2, \\ \frac{dI_1}{dt} = \beta S_1 I_2 - (\mu+\gamma)I_1, \\ \frac{dI_2}{dt} = \gamma I_1 - (\mu+\gamma_2)I_2, \end{cases} \tag{6.6}$$

which is the case that

$$\begin{cases} b_1 = q_1\mu A,\ b_2 = q_2\mu A,\ q_2 = 1-q_1,\ \beta_{11} = 0,\ \beta_{12} = \beta,\ \beta_{21} = \beta_{22} = 0, \\ l_{11} = l_{21} = p,\ l_{22} = l_{12} = \varepsilon,\ m_{11} = m_{21} = \gamma,\ m_{22} = m_{12} = 0, \\ \gamma_1 = 0,\ \gamma_2 = \alpha+\delta, \end{cases} \tag{6.7}$$

and the basic reproduction number of system (6.5) is

$$R_0 = \frac{\beta\gamma A(\varepsilon + q_1\mu)}{(\mu+\gamma)(\mu+\alpha+\delta)(p+\mu+\varepsilon)}. \tag{6.8}$$

Assume that $R_0 > 1$. Then, for $a_2 = 1$ and $a_4 = \frac{\mu+\gamma}{\gamma}$, the condition (ii) in (5.6) is satisfied. Thus, we conclude that $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$.

Example 6.3 (See Li *et al.* [16, Section 2]).

$$\begin{cases} \frac{dS}{dt} = \Lambda - d_0 S - S(\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3), \\ \frac{dI_1}{dt} = S(\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3) - (d_1 + \delta_{21})I_1 + \delta_{12} I_2 + \delta_{13} I_3, \\ \frac{dI_2}{dt} = \delta_{21} I_1 - (d_2 + \delta_{12} + \delta_{32})I_2 + \delta_{23} I_3 + \delta_{23} I_3, \\ \frac{dI_3}{dt} = \delta_{32} I_2 - (d_3 + \delta_{13} + \delta_{23} + \delta_{43})I_3, \end{cases} \tag{6.9}$$

and $\frac{dT}{dt} = \delta_{43} I_3 - d_T T$. Then, (6.9) is equivalent to the following system

$$\begin{cases} \frac{dS_1}{dt} = \Lambda - d_0 S_1 - S_1(\beta_{11} I_1 + \beta_{12} I_2 + \beta_{13} I_3), \\ \frac{dS_2}{dt} = \Lambda - d_0 S_2, \\ \frac{dS_3}{dt} = \Lambda - d_0 S_3, \\ \frac{dI_1}{dt} = S_1(\beta_{11} I_1 + \beta_{12} I_2 + \beta_{13} I_3) - (d_1 + m_{21})I_1 + m_{12} I_2 + m_{13} I_3, \\ \frac{dI_2}{dt} = m_{21} I_1 - (d_2 + m_{12} + m_{32})I_2 + m_{23} I_3 + m_{23} I_3, \\ \frac{dI_3}{dt} = m_{32} I_2 - (d_3 + m_{13} + m_{23} + \gamma_3)I_3, \end{cases} \tag{6.10}$$

which is the case that

$$\begin{cases} b_1 = b_2 = b_3 = \Lambda,\ \beta_{1j} = \beta_j,\ j = 1,2,3,\ \beta_{ij} = 0,\ i = 2,3,\ j = 1,2,3, \\ m_{ij} = \delta_{ij},\ i \neq j,\ j = 1,2,3,\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = \delta_{43}. \end{cases} \tag{6.11}$$

Then, we have that

$$\tilde{\mathbf{V}} = \begin{bmatrix} d_1 + \delta_{21} & 0 & 0 \\ 0 & d_2 + \delta_{12} + \delta_{32} & 0 \\ 0 & 0 & d_3 + \delta_{13} + \delta_{23} + \delta_{43} \end{bmatrix}, \tag{6.12}$$

and for $\mathbf{S} = (S_1, S_2, \ldots, S_n)^T$, put

$$\tilde{\mathbf{M}}(\mathbf{S}) = \tilde{\mathbf{V}}^{-1}\tilde{\mathbf{F}}(\mathbf{S}), \quad \tilde{\mathbf{F}}(\mathbf{S}) = \begin{bmatrix} S_1\beta_1 & S_1\beta_2 + \delta_{12} & S_1\beta_3 + \delta_{13} \\ \delta_{21} & 0 & \delta_{23} \\ 0 & \delta_{32} & 0 \end{bmatrix}, \tag{6.13}$$

and consider the following threshold parameter

$$\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)). \tag{6.14}$$

Since $n = 3$ and $\tilde{\mathbf{F}}$ is irreducible (which implies that $\tilde{\mathbf{M}}$ is irreducible) and (1.20) holds, for $R_0 > 1$, by Lemma 4.2, there exists a positive 3 column vector $\mathbf{v} = (v_1, v_2, v_3)^T$ such that (1.18) holds. Hence, by Corollary 1.1, we conclude that $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$.

Finally, we conclude this section by noting that we can also apply Theorem 1.1 to the stage-progression models for HIV/AIDS with amelioration in Guo *et al.* [20, 21].

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