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Global stability of SIRS epidemic models with a class of nonlinear incidence rates and distributed delays

Yoichi Enatsu*
Department of Pure and Applied Mathematics, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan
E-mail: yo1.gc-rw.docomo@akane.waseda.jp

Yukihiko Nakata
Basque Center for Applied Mathematics
Mazarredo, 14 E-48009 Bilbao, Spain
E-mail: nakata@bcamath.org

Yoshiaki Muroya
Department of Mathematics, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan
E-mail: ymuroya@waseda.jp

Abstract. In this article, we establish the global asymptotic stability of a disease-free equilibrium and an endemic equilibrium of an SIRS epidemic model with a class of nonlinear incidence rates and distributed delays. By using strict monotonicity of the incidence function and constructing a Lyapunov functional, we obtain sufficient conditions under which the endemic equilibrium is globally asymptotically stable. When the nonlinear incidence rate is a saturated incidence rate, our result provides a new global stability condition for a small rate of immunity loss.

Keywords: SIRS epidemic model, nonlinear incidence rate, global asymptotic stability, distributed delays, Lyapunov functional.
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1 Introduction

To investigate the global behavior of the prevalence of infectious diseases, stability analysis of equilibria for epidemic models have been carried out (see [1–19] and the references therein).

Mena-Lorca and Hethcote [15] considered several SIRS epidemic models with a bilinear incidence rate of the form $\beta S(t)I(t)$ and a standard incidence rate of the form $\beta S(t)I(t)/N(t)$, where $N(t) = S(t) + I(t) + R(t)$. A threshold parameter of the models was also found in Mena-Lorca and Hethcote [15] to determine whether the disease dies out or approaches to an endemic equilibrium. Later, to investigate the effect of an immunity loss of diseases, a significant body of work concerning the stability analysis for the SIRS epidemic models has been carried out (see, for example, [7–12,16–19] and the references therein).

In modeling of those communicable diseases, an incidence rate has played a vital role in ensuring that the model can give a reasonable qualitative description for the disease dynamics. A bilinear incidence rate and a standard incidence rate were frequently used in the literature of mathematical modeling. In contrast, many authors suggested that transmission of the infection would have a nonlinear incidence rate. For example, Capasso and Serio [2] studied the cholera epidemic spread in Bari in 1973 and gave an assumption that the incidence rate takes the nonlinear form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$, which was interpreted as a saturated incidence rate. The saturation effect was originally introduced for the Holling functional response of the predator in a prey-predator system. This incidence rate includes crowding effect of the infective individuals. Korobeinikov and Maini [9] thereafter formulated a variety of models with an incidence rate of the form $F(S(t))G(I(t))$ and Korobeinikov [10, 11] obtained the global properties of basic SIR and SIRS epidemic models with a more general framework of the incidence rate $F(S(t), I(t))$.

Recently, Xu and Ma [18] investigated the spread of vector-borne diseases and formulated an SIRS epidemic model with the saturated incidence rate $\beta S(t)\frac{I(t-h)}{1+\alpha I(t-h)}$. Some authors have now proposed several reasons for the nonlinearity of the incidence rates and introduced various nonlinear incidence functions with delays (see also [7, 19]).

*Corresponding author.

In this article, to study the impact of nonlinearity of those incidence rates and time delay effects, we consider the global dynamics of the following SIRS epidemic model with a class of nonlinear incidence rates and distributed delays:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu S(t) - \beta S(t) \displaystyle\int_0^h f(\tau) G(I(t-\tau)) d\tau + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t) \displaystyle\int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu+\gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu+\delta) R(t). \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective and recovered individuals at time t, respectively. $B > 0$ is the recruitment rate of the population, $\mu > 0$ is the natural death rate of the population, $\beta > 0$ is the proportionality constant, $\gamma > 0$ is the natural recovery rate of the infective individuals, and $\delta \geq 0$ is the rate at which recovered individuals lose immunity and return to the susceptible class. $h > 0$ is a maximum time taken to become infectious, and the transmission of the infection is governed by an incidence rate $\beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau$. $f(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ. Here, $f : [0, h] \to [0, +\infty)$ is continuous on $[0, h]$ satisfying $\int_0^h f(\tau) d\tau = 1$. The initial conditions of system (1.1) take the form

$$S(\theta) = \varphi_1(\theta), \quad I(\theta) = \varphi_2(\theta), \quad R(\theta) = \varphi_3(\theta), \quad -h \leq \theta \leq 0, \tag{1.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i(\theta) = \varphi_i(0) \geq 0$ $(-h \leq \theta \leq 0,\ i = 1, 3)$, $\varphi_2(\theta) \geq 0$ $(-h \leq \theta \leq 0)$. C denotes the Banach space $C([-h, 0], \mathbb{R}^3_{+0})$ of continuous functions mapping the interval $[-h, 0]$ into $\mathbb{R}^3_{+0}$ with the supremum norm, where $\mathbb{R}_{+0} = \{x \in \mathbb{R} | x \geq 0\}$. From a biological meaning, we assume $\varphi_i(0) > 0$ for $i = 1, 2, 3$.

Throughout this article, we further assume that

(H1) $G(I)$ is continuous and monotone increasing on $[0, +\infty)$ with $G(0) = 0$,

(H2) $I/G(I)$ is monotone increasing on $(0, +\infty)$ with $\lim_{I \to +0} (I/G(I)) = 1$.

Under the hypotheses (H1) and (H2), G is Lipschitz continuous on $[0, +\infty)$ and $0 < G(I) \leq I$ holds for all $I > 0$. The basic reproduction number of the system (1.1) becomes

$$R_0 = \frac{B\beta}{\mu(\mu+\gamma)}. \tag{1.3}$$

R_0 denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population.

By the fundamental theory of functional differential equations, system (1.1) has a unique solution $(S(t), I(t), R(t))$, and $S(t) > 0$, $I(t) > 0$ and $R(t) > 0$ hold for all $t \geq 0$. It is clear that system (1.1) always has a disease-free equilibrium $E_0 = (B/\mu, 0, 0)$, and if $R_0 > 1$, then system (1.1) allows a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$, $S^* > 0$, $I^* > 0$, and $R^* > 0$ (see Lemma 4.1).

In this article, applying Lyapunov functional techniques for a delayed SIR epidemic model in McCluskey [13] and the property that the total population of system (1.1) converges to a positive constant B/μ, we obtain sufficient conditions which ensure the global asymptotic stability of the endemic equilibrium E_* for $R_0 > 1$ (see Lemma 5.1). The main results are as follows.

Theorem 1.1. *If $R_0 < 1$, then the disease-free equilibrium E_0 of system (1.1) is globally asymptotically stable.*

Theorem 1.2. *Let us assume that $R_0 > 1$ holds. Then system (1.1) is permanent. Moreover, we assume that the following conditions hold:*

(I) *There exist positive constants C_1 and C_2 such that*

$$\inf_{0 \leq I \leq B/\mu} \frac{G(I) - G(I^*)}{I - I^*} \geq C_1 > 0 \text{ and } \inf_{0 < I \leq B/\mu} \frac{\frac{I}{G(I)} - \frac{I^*}{G(I^*)}}{I - I^*} \geq C_2 > 0,$$

(II) $\delta^2 - 4C_1C_2(\mu+\gamma)(\mu+\delta)\frac{B}{\mu+\beta G(B/\mu)} < 0$.

Then the endemic equilibrium E_* of system (1.1) is globally asymptotically stable.

We show the global asymptotic stability of the endemic equilibrium for a small rate of immunity loss δ as long as the infection rate has suitable properties concerning the concavity of function G characterized by the hypotheses (H1) and (H2).

The organization of this article is as follows. In Section 2, we offer a basic result for system (1.1). In Section 3, we prove Theorem 1.1. In Section 4, we establish the permanence and global asymptotic stability of the endemic equilibrium E_* of system (1.1) for $R_0 > 1$. In Section 5, we prove Theorem 1.2 by means of a Lyapunov functional on a reduced system which is derived from system (1.1) with a key lemma (see Lemma 5.1). To show the feasibility of our global stability conditions of the endemic equilibrium for $R_0 > 1$, we present numerical examples in Section 6. Finally, a discussion is offered in Section 7.

2 Global stability of the endemic equilibrium E^* for $R_0 > 1$

In this section, we introduce a basic result on boundedness of the solution of system (1.1).

Lemma 2.1. *For any solution of system* (1.1) *with the initial conditions* (1.2)*, it holds that*

$$\lim_{t\to+\infty}(S(t)+I(t)+R(t)) = \frac{B}{\mu}. \tag{2.1}$$

Proof. Let $N(t) = S(t) + I(t) + R(t)$. Then, it follows from system (1.1) that

$$\frac{dN(t)}{dt} = B - \mu S(t) - \mu I(t) - \mu R(t) = B - \mu N(t).$$

Hence, we obtain $\lim_{t\to+\infty} N(t) = B/\mu$. □

3 Stability of the Disease-Free Equilibrium E_0 for $R_0 < 1$

3.1 Local stability of the disease-free equilibrium E_0

In this subsection, we investigate the local asymptotic stability of the disease-free equilibrium E_0 of system (1.1).

Lemma 3.1. *If $R_0 < 1$, then the disease-free equilibrium E_0 of system* (1.1) *is locally asymptotically stable. If $R_0 > 1$, then the disease-free equilibrium E_0 of system* (1.1) *is unstable.*

Proof. The characteristic equation of system (1.1) at E_0 is of the form

$$(\lambda+\mu)\left\{\lambda+(\mu+\gamma)\left(1-R_0\int_0^h f(\tau)\mathrm{e}^{-\lambda\tau}d\tau\right)\right\}(\lambda+\mu+\delta) = 0. \tag{3.1}$$

It is clear that both $\lambda = -\mu$ and $\lambda = -(\mu+\delta)$ are roots of (3.1). All other roots of (3.1) are determined by the following equation:

$$\lambda+(\mu+\gamma)\left(1-R_0\int_0^h f(\tau)\mathrm{e}^{-\lambda\tau}d\tau\right) = 0. \tag{3.2}$$

For the case $R_0 < 1$, we suppose on the contrary that E_0 is not locally asymptotically stable. Then, there exists a root $\lambda = \tilde{\lambda}$ such that $\mathrm{Re}\tilde{\lambda} \geq 0$. However, from (3.2), we obtain

$$\mathrm{Re}\tilde{\lambda} = (\mu+\gamma)\left\{R_0\mathrm{e}^{-\mathrm{Re}\tilde{\lambda}\tau}\int_0^h f(\tau)\cos(\mathrm{Im}\tilde{\lambda}\tau)d\tau - 1\right\} \leq (\mu+\gamma)(R_0-1) < 0,$$

which is a contradiction. Hence, if $R_0 < 1$, then the disease-free equilibrium E_0 of system (1.1) is locally asymptotically stable. Now, we put

$$Q(\lambda) := \lambda+(\mu+\gamma)\left(1-R_0\int_0^h f(\tau)\mathrm{e}^{-\lambda\tau}d\tau\right). \tag{3.3}$$

For the case $R_0 > 1$, it is directly seen that $Q(0) = (\mu+\gamma)(1-R_0) < 0$ and $\lim_{\lambda\to+\infty} Q(\lambda) = +\infty$ holds for $\lambda \in \mathbb{R}$. Therefore, (3.1) has at least one positive real root. Hence, if $R_0 > 1$, then the disease-free equilibrium E_0 is unstable. □

3.2 Global stability of the disease-free equilibrium E_0

In this subsection, by constructing a Lyapunov functional, we prove Theorem 1.1.

Proof of Theorem 1.1. By $R_0 < 1$, we choose $\varepsilon_s > 0$ sufficiently small such that

$$\beta\left(\frac{B}{\mu} + \varepsilon_s\right) < \mu + \gamma. \tag{3.4}$$

We consider the following Lyapunov functional:

$$W(t) = I(t) + (\mu+\gamma)\int_0^h f(\tau)\int_{t-\tau}^t I(u)du d\tau.$$

From Lemma 3.1, there exists a $T = T(\varepsilon_S) > 0$ such that $S(t) < \frac{B}{\mu} + \varepsilon_S$ for all $t > T$. We then obtain

$$\begin{aligned}
\frac{dW(t)}{dt} &= \beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t) \\
&\quad +(\mu+\gamma)\int_0^h f(\tau)\left(I(t) - I(t-\tau)\right)d\tau \\
&\le \beta\left(\frac{B}{\mu} + \varepsilon_s\right)\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)\int_0^h f(\tau)I(t-\tau)d\tau \\
&\le \beta\left(\frac{B}{\mu} + \varepsilon_s\right)\int_0^h f(\tau)I(t-\tau)d\tau - (\mu+\gamma)\int_0^h f(\tau)I(t-\tau)d\tau \\
&= \int_0^h f(\tau)\left\{\beta\left(\frac{B}{\mu} + \varepsilon_s\right) - (\mu+\gamma)\right\}I(t-\tau)d\tau
\end{aligned}$$

for $t > T + h$. From (3.4) and the arbitrarity of $\varepsilon_s > 0$, we verify that $\frac{dW(t)}{dt} \le 0$ holds for $t > T + h$. Thus, it holds that $\lim_{t\to+\infty} W(t) = 0$, which implies that $\lim_{t\to+\infty} I(t) = 0$. It follows that $\lim_{t\to+\infty} R(t) = 0$ and $\lim_{t\to+\infty} S(t) = B/\mu$ hold. By Lemma 3.1 and Lyapunov-LaSalle asymptotic stability theorem, the disease-free equilibrium E_0 is globally asymptotically stable. □

Remark 3.1. *To establish the global asymptotic stability of the disease-free equilibrium E_0 of system ((1.1)) for $R_0 < 1$, the hypothesis of the monotonicity of $G(I)$ in (H1) is not necessary.*

4 Permanence for $R_0 > 1$

4.1 Existence and uniqueness of the endemic equilibrium E_*

In this subsection, by the hypothesis (H2), we give a lemma of the unique existence of E_* for $R_0 > 1$:

Lemma 4.1. *If $R_0 > 1$, then system (1.1) has a unique endemic equilibrium E_* satisfying*

$$\begin{cases} B - \mu S^* - \beta S^* G(I^*) + \delta R^* = 0, \\ \beta S^* G(I^*) - (\mu+\gamma)I^* = 0, \\ \gamma I^* - (\mu+\delta)R^* = 0. \end{cases} \tag{4.1}$$

Proof. At a fixed point (S, I, R) of system (1.1), the following equations hold.

$$\begin{cases} B - \mu S - \left(\mu + \gamma - \dfrac{\gamma\delta}{\mu+\delta}\right)I = 0, \\ \beta S G(I) - (\mu+\gamma)I = 0, \\ \gamma I - (\mu+\delta)R = 0. \end{cases} \tag{4.2}$$

Substituting the second equation of (4.2) into the first equation of (4.2), we consider the following equation:

$$H(I) := B - \frac{\mu(\mu+\gamma)I}{\beta G(I)} - \frac{\mu(\mu+\gamma+\delta)}{\mu+\delta}I = 0.$$

By the hypothesis (H2), H is strictly monotone decreasing on $(0, +\infty)$ satisfying

$$\lim_{I\to+0} H(I) = B - \frac{\mu(\mu+\gamma)}{\beta} = B\left(1 - \frac{1}{R_0}\right) > 0$$

and $H(I) < 0$ for all $I > \frac{B(\mu+\delta)}{\mu(\mu+\gamma+\delta)}$. Thus, there exists a unique $I^* > 0$ such that $H(I^*) = 0$. By (4.2), we obtain $S^* = \frac{(\mu+\gamma)I^*}{\beta G(I^*)} > 0$ and $R^* = \frac{\gamma I^*}{\mu+\delta} > 0$. Hence, system (1.1) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$. □

4.2 Permanence for $R_0 > 1$

In this subsection, we show the permanence of system (1.1). The following lemma indicates that the disease eventually persists in the host population if $R_0 > 1$.

Lemma 4.2. *If $R_0 > 1$, then for any solution of system (1.1) with the initial conditions (1.2), it holds that*

$$\liminf_{t\to+\infty} S(t) \geq v_1 := \frac{B}{\mu + \beta G(B/\mu)},\ \liminf_{t\to+\infty} I(t) \geq v_2 := qI^*\mathrm{e}^{-(\mu+\gamma)(h+\rho h)},\ \liminf_{t\to+\infty} R(t) \geq v_3 := \frac{\gamma v_2}{\mu+\delta},$$

where $0 < q < \frac{B\frac{G(I^*)}{I^*} - \frac{\mu\delta\gamma}{\beta(\mu+\delta)}}{B + \frac{\delta\gamma}{\mu+\delta}I^*} < 1$ and $\rho > 0$ satisfy $S^* < S^\triangle := \frac{B}{k}(1 - \mathrm{e}^{-k\rho h})$, $k = \mu + \beta G(qI^*)$.

Proof. By Lemma 2.1, it holds that $\limsup_{t\to+\infty} I(t) \leq \frac{B}{\mu}$. This yields that, for any $\varepsilon_I > 0$ sufficiently small, there exists a $T_1 = T_1(\varepsilon_I) > 0$ such that $I(t) < \frac{B}{\mu} + \varepsilon_I$ for all $t > T_1$. From the hypothesis (H1), for $t > T_1 + h$, we derive

$$\frac{dS(t)}{dt} \geq B - \left\{\mu + \beta G\left(\frac{B}{\mu} + \varepsilon_I\right)\right\}S(t),$$

which implies that

$$\liminf_{t\to+\infty} S(t) \geq \frac{B}{\mu + \beta G(B/\mu + \varepsilon_I)}.$$

As the above inequality holds for arbitrary $\varepsilon_I > 0$, it follows that $\liminf_{t\to+\infty} S(t) \geq v_1$.

We now show that $\liminf_{t\to+\infty} I(t) \geq v_2$. First, we prove that it is impossible that $I(t) \leq qI^*$ for all $t \geq \rho h$. Suppose on the contrary that $I(t) \leq qI^*$ for all $t \geq \rho h$. Since we have $\frac{G(I^*)}{I^*} = \frac{\mu+\gamma}{\beta S^*}$ and $S^* < S^* + I^* + R^* = B/\mu$, it holds that

$$\begin{aligned}
\beta BG(I^*) - \frac{\mu\delta\gamma}{\mu+\delta}I^* &= \left(\beta B\frac{G(I^*)}{I^*} - \frac{\mu\delta\gamma}{\mu+\delta}\right)I^* \\
&= \left(\beta B\frac{\mu+\gamma}{\beta S^*} - \frac{\mu\delta\gamma}{\mu+\delta}\right)I^* \\
&> \left\{\beta B\frac{\mu+\gamma}{\beta S^*} - \mu(\mu+\gamma)\right\}I^* \\
&= \frac{(\mu+\gamma)(B-\mu S^*)I^*}{S^*} > 0,
\end{aligned}$$

which yields

$$S^* = \frac{B + \frac{\delta\gamma}{\mu+\delta}I^*}{\mu + \beta G(I^*)} = \frac{B}{\frac{B(\mu+\beta G(I^*))}{B+\frac{\delta\gamma}{\mu+\delta}I^*}} = \frac{B}{\mu + \frac{\beta BG(I^*) - \frac{\mu\delta\gamma}{\mu+\delta}I^*}{B+\frac{\delta\gamma}{\mu+\delta}I^*}} < \frac{B}{\mu+\beta qI^*} \leq \frac{B}{\mu+\beta G(qI^*)},$$

for any $0 < q < \frac{B\frac{G(I^*)}{I^*} - \frac{\mu\delta\gamma}{\beta(\mu+\delta)}}{B+\frac{\delta\gamma}{\mu+\delta}I^*}$. From the first equation of system (1.1), one can obtain

$$\frac{dS(t)}{dt} \geq B - (\mu + \beta G(qI^*))S(t),\ \text{for } t \geq \rho h + h,$$

which yields

$$S(t) \geq \mathrm{e}^{-k(t-\rho h-h)}\left\{S(\rho h+h) + B\int_{\rho h+h}^{t} \mathrm{e}^{k(\theta-\rho h-h)}d\theta\right\} > \frac{B}{k}(1 - \mathrm{e}^{-k(t-\rho h-h)}) \tag{4.3}$$

for $t \geq \rho h + h$. Hence, it follows from (4.3) that

$$S(t) > \frac{B}{k}(1 - \mathrm{e}^{-k\rho h}) = S^\triangle > S^*,\ \text{for } t \geq 2\rho h + h. \tag{4.4}$$

For $t \geq 0$, we define

$$V(t) = I(t) + \beta S^* \int_0^h f(\tau)\int_{t-\tau}^{t} G(I(u))du d\tau. \tag{4.5}$$

By the hypothesis (H2), calculating the derivative of $V(t)$ along the solution of system (1.1) gives as

$$\begin{aligned}
\frac{dV(t)}{dt} &= \beta\int_0^h f(\tau)G(I(t-\tau))(S(t)-S^*)d\tau + \beta S^*G(I(t)) - (\mu+\gamma)I(t)\\
&= \beta\int_0^h f(\tau)G(I(t-\tau))(S(t)-S^*)d\tau + \left\{\beta S^*\frac{G(I(t))}{I(t)} - (\mu+\gamma)\right\}I(t)\\
&\geq \beta\int_0^h f(\tau)G(I(t-\tau))(S(t)-S^*)d\tau + \left\{\beta S^*\frac{G(I^*)}{I^*} - (\mu+\gamma)\right\}I(t)\\
&= \beta\int_0^h f(\tau)G(I(t-\tau))(S(t)-S^*)d\tau\\
&> \beta\int_0^h f(\tau)G(I(t-\tau))(S^\triangle-S^*)d\tau, \text{ for } t\geq 2\rho h+h. \qquad (4.6)
\end{aligned}$$

Setting $\underline{i} = \min_{\theta\in[-h,0]} I(\theta+2\rho h+2h)$, we claim that $I(t)\geq \underline{i}$ for all $t\geq 2\rho h+h$. Otherwise, if there is a $T\geq 0$ such that $I(t)\geq\underline{i}$ for $2\rho h+h\leq t\leq 2\rho h+2h+T$, $I(2\rho h+2h+T)=\underline{i}$ and $\frac{dI(t)}{dt}|_{t=2\rho h+2h+T}\leq 0$, then it follows from (4.4) that

$$\begin{aligned}
\frac{dI(t)}{dt}\Big|_{t=2\rho h+2h+T} &= \beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t)\\
&\geq \beta S(t)G(I(t)) - (\mu+\gamma)I(t)\\
&\geq \left\{\beta S(t)\frac{G(I^*)}{I^*} - (\mu+\gamma)\right\}\underline{i}\\
&> \left\{\beta S^\triangle\frac{G(I^*)}{I^*} - (\mu+\gamma)\right\}\underline{i}\\
&> \left\{\beta S^*\frac{G(I^*)}{I^*} - (\mu+\gamma)\right\}\underline{i} = 0.
\end{aligned}$$

This is a contradiction. Therefore, $I(t)\geq\underline{i}$ for all $t\geq 2\rho h+h$. By the hypothesis (H1), it follows from (4.6) that

$$\frac{dV(t)}{dt} > \beta G(\underline{i})(S^\triangle - S^*) > 0, \text{ for } t\geq 2\rho h+2h,$$

which implies that $\lim\limits_{t\to+\infty} V(t) = +\infty$. However, from Lemma 2.1, it holds that $\limsup\limits_{t\to+\infty} V(t)\leq \frac{B}{\mu}+\beta S^*G(\frac{B}{\mu}) < +\infty$. This leads to a contradiction. Hence the claim is proved.

As the above claim holds, we are left to consider two possibilities:

$$\begin{cases} \text{(i)} & I(t)\geq qI^* \text{ for all } t \text{ sufficiently large,}\\ \text{(ii)} & I(t) \text{ oscillates about } qI^* \text{ for all } t \text{ sufficiently large.}\end{cases}$$

If the first case holds, then we immediately get the conclusion. If the second case holds, then we show that $I(t)\geq v_2$ for all t sufficiently large. Let $t_1<t_2$ be sufficiently large such that

$$I(t_1)=I(t_2)=qI^*,\ I(t)<qI^*,\ t_1<t<t_2.$$

If $t_2-t_1\leq h+\rho h$, then it follows from the second equation of system (1.1) that

$$\frac{dI(t)}{dt}\geq -(\mu+\gamma)I(t),$$

that is,

$$I(t)\geq I(t_1)\mathrm{e}^{-(\mu+\gamma)(t-t_1)} = qI^*\mathrm{e}^{-(\mu+\gamma)(h+\rho h)} = v_2$$

holds for all $t\geq t_1$. If $t_2-t_1\leq h+\rho h$, then we similarly verify that $I(t)\geq v_2$ holds for $t_1\leq t\leq t_1+h+\rho h$. We now claim that $I(t)\geq v_2$ for all $t_1+h+\rho h\leq t\leq t_2$. Otherwise, there is a $T^*>0$, such that $I(t)\geq v_2$ for $t_1\leq t\leq t_1+h+\rho h+T^*$, $I(t_1+h+\rho h+T^*)=v_2$ and $\frac{dI(t)}{dt}|_{t=t_1+h+\rho h+T^*}\leq 0$. Then, from (4.4), we get

$$\begin{aligned}
\frac{dI(t)}{dt}\Big|_{t=t_1+h+\rho h+T^*} &= \beta S(t)\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t)\\
&\geq \beta S^\triangle G(I(t)) - (\mu+\gamma)I(t)\\
&\geq \left\{\beta S^\triangle\frac{G(v_2)}{v_2} - (\mu+\gamma)\right\}v_2.
\end{aligned}$$

However, by the hypothesis (H2), it holds that

$$\frac{dI(t)}{dt}\Big|_{t=t_1+h+\rho h+T^*} \geq \left\{\beta S^{\triangle}\frac{G(I^*)}{I^*} - (\mu+\gamma)\right\}v_2 > 0,$$

which is a contradiction. Hence, $I(t) \geq v_2$ for $t_1 \leq t \leq t_2$. As the interval $[t_1, t_2]$ is arbitrarily chosen, $I(t) \geq v_2$ holds for all t sufficiently large. Thus, we obtain $\liminf_{t\to+\infty} I(t) \geq v_2$, from which we have $\liminf_{t\to+\infty} R(t) \geq v_3$.

5 Global Stability of the Endemic Equilibrium E_* for $R_0 > 1$

From Lemma 2.1, we see that the limit set of system (1.1) in the first octant of $\mathbb{R}^3$ locates on the plane $S+I+R = B/\mu$. Hence, the dynamics of system (1.1) in the first octant of $\mathbb{R}^3$ is equivalent to the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \dfrac{B(\mu+\delta)}{\mu} - (\mu+\delta)S(t) - \beta S(t)\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau - \delta I(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)\displaystyle\int_0^h f(\tau)G(I(t-\tau))d\tau - (\mu+\gamma)I(t). \end{cases} \tag{5.7}$$

We now discuss the global asymptotic stability of the endemic equilibrium $\tilde{E}_*$ of system (5.1) for $R_0 > 1$. By Lemma 4.1, system (5.1) has a unique endemic equilibrium $\tilde{E}_* := (S^*, I^*)$ if $R_0 > 1$. For simplicity, we put

$$x_t = \frac{S(t)}{S^*},\ y_t = \frac{I(t)}{I^*},\ \tilde{y}_t = \frac{G(I(t))}{G(I^*)}. \tag{5.8}$$

The following lemma plays an important role to obtain Theorem 1.2.

Lemma 5.1. *For all $t \geq 0$, under the condition (I), it holds that*

$$g(y_t) - g(\tilde{y}_t) \geq C_1C_2I^*(y_t-1)^2, \tag{5.9}$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$ defined for all $x > 0$.

Proof. First, we obtain

$$\tilde{y}_t - 1 = \frac{G(I(t)) - G(I^*)}{G(I^*)}$$

and

$$y_t - \tilde{y}_t = \frac{I(t)}{I^*} - \frac{G(I(t))}{G(I^*)} = \frac{G(I(t))}{I^*}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right).$$

Then, it follows from the condition (I) that

$$\begin{aligned}(\tilde{y}_t - 1)(y_t - \tilde{y}_t) &= \frac{G(I(t))}{I^*G(I^*)}(G(I(t)) - G(I^*))\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right) \\ &\geq \frac{C_1C_2G(I(t))}{I^*G(I^*)}(I(t)-I^*)^2 = \frac{C_1C_2I^*G(I(t))}{G(I^*)}(y_t-1)^2 \\ &= C_1C_2I^*\tilde{y}_t(y_t-1)^2. \end{aligned} \tag{5.10}$$

By the hypotheses (H1) and (H2), we have

$$g(y_t) - g(\tilde{y}_t) = y_t - \tilde{y}_t - \ln\frac{y_t}{\tilde{y}_t} = y_t - \tilde{y}_t - \frac{y_t}{\tilde{y}_t} + 1 + \frac{y_t}{\tilde{y}_t} - 1 - \ln\frac{y_t}{\tilde{y}_t} = \frac{1}{\tilde{y}_t}(\tilde{y}_t-1)(y_t-\tilde{y}_t) + g\left(\frac{y_t}{\tilde{y}_t}\right) \geq C_1C_2I^*(y_t-1)^2.$$

Hence, we get the conclusion of this lemma. □

Now, we are in a position to prove the global asymptotic stability of the endemic equilibrium $\tilde{E}_*$ for $R_0 > 1$ by applying techniques in McCluskey [13, Proof of Theorem 4.1].

Theorem 5.1. *If $R_0 > 1$ and the conditions (I) and (II) hold, then the endemic equilibrium $\tilde{E}_*$ of system (5.1) is globally asymptotically stable.*

Proof. Let us consider the following Lyapunov functional:

$$U(t) = \frac{1}{\beta G(I^*)} U_S(t) + \frac{I^*}{\beta S^* G(I^*)} U_I(t) + U_+(t), \tag{5.11}$$

where

$$U_S(t) = g\left(\frac{S(t)}{S^*}\right),\ U_I(t) = g\left(\frac{I(t)}{I^*}\right),\ U_+(t) = \int_0^h f(\tau) \int_{t-\tau}^t g\left(\frac{G(I(s))}{G(I^*)}\right) ds d\tau. \tag{5.12}$$

We now show that $\frac{dU(t)}{dt} \leq 0$. First, we calculate $\frac{dU_S(t)}{dt}$:

$$\frac{dU_S(t)}{dt} = \frac{S(t) - S^*}{S^* S(t)} \left\{ \frac{B(\mu+\delta)}{\mu} - (\mu+\delta)S(t) - \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - \delta I(t) \right\}.$$

By the relation $\frac{B(\mu+\delta)}{\mu} = (\mu+\delta)S^* + \beta S^* G(I^*) + \delta I^*$, we have

$$\begin{aligned}
\frac{dU_S(t)}{dt} =& \frac{S(t) - S^*}{S^* S(t)} \left\{ (\mu+\delta)S^* + \beta S^* G(I^*) + \delta I^* - (\mu+\delta)S(t) - \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - \delta I(t) \right\} \\
=& \frac{S(t) - S^*}{S^* S(t)} \left[-(\mu+\delta)(S(t) - S^*) + \beta \int_0^h f(\tau) \left\{ S^* G(I^*) - S(t) G(I(t-\tau)) \right\} d\tau - \delta (I(t) - I^*) \right] \\
=& -\frac{(\mu+\delta)(S(t) - S^*)^2}{S^* S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{S^*}{S(t)}\right) \left(1 - \frac{S(t)}{S^*} \frac{G(I(t-\tau))}{G(I^*)}\right) d\tau - \frac{\delta (S(t) - S^*)(I(t) - I^*)}{S^* S(t)} \\
=& -\frac{S^*(\mu+\delta)(x_t - 1)^2}{S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{1}{x_t}\right) (1 - x_t \tilde{y}_{t-\tau}) d\tau - \frac{\delta I^*}{S(t)} (x_t - 1)(y_t - 1) \\
=& -\frac{S^*(\mu+\delta)(x_t - 1)^2}{S(t)} + \beta G(I^*) \int_0^h f(\tau) \left(1 - \frac{1}{x_t} - x_t \tilde{y}_{t-\tau} + \tilde{y}_{t-\tau}\right) d\tau - \frac{\delta I^*}{S(t)} (x_t - 1)(y_t - 1).
\end{aligned} \tag{5.13}$$

Secondly, we calculate $\frac{dU_I(t)}{dt}$:

$$\frac{dU_I(t)}{dt} = \frac{I(t) - I^*}{I^* I(t)} \left\{ \beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - (\mu+\gamma) I(t) \right\}.$$

By the relation $(\mu+\gamma)I^* = \beta S^* G(I^*)$, we have

$$\begin{aligned}
\frac{dU_I(t)}{dt} =& \frac{I(t) - I^*}{I^* I(t)} \left(\beta S(t) \int_0^h f(\tau) G(I(t-\tau)) d\tau - \beta S^* \frac{G(I^*)}{I^*} I(t) \right) \\
=& \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left(1 - \frac{I^*}{I(t)}\right) \left(\frac{S(t)}{S^*} \frac{G(I(t-\tau))}{G(I^*)} - \frac{I(t)}{I^*} \right) d\tau \\
=& \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left(1 - \frac{1}{y_t}\right) (x_t \tilde{y}_{t-\tau} - y_t) d\tau \\
=& \beta S^* \frac{G(I^*)}{I^*} \int_0^h f(\tau) \left(x_t \tilde{y}_{t-\tau} - \frac{x_t \tilde{y}_{t-\tau}}{y_t} - y_t + 1 \right) d\tau.
\end{aligned} \tag{5.14}$$

Finally, calculating $\frac{dU_+(t)}{dt}$ gives as follows:

$$\begin{aligned}
\frac{dU_+(t)}{dt} &= \int_0^h f(\tau) \left(g\left(\frac{G(I(t))}{G(I^*)}\right) - g\left(\frac{G(I(t-\tau))}{G(I^*)}\right) \right) d\tau \\
&= \int_0^h f(\tau) \left(g(\tilde{y}_t) - g(\tilde{y}_{t-\tau}) \right) d\tau \\
&= \int_0^h f(\tau) (\tilde{y}_t - \ln \tilde{y}_t - \tilde{y}_{t-\tau} + \ln \tilde{y}_{t-\tau}) d\tau.
\end{aligned} \tag{5.15}$$

Combining (5.3), (5.7), (5.8), and (5.9), it follows from Lemma 5.1 that

$$\begin{aligned}\frac{dU(t)}{dt} &= \frac{1}{\beta G(I^*)}\left\{-\frac{S^*(\mu+\delta)(x_t-1)^2}{S(t)} + \beta G(I^*)\int_0^h f(\tau)\left(1-\frac{1}{x_t}-x_t\tilde{y}_{t-\tau}+\tilde{y}_{t-\tau}\right)d\tau - \frac{\delta I^*}{S(t)}(x_t-1)(y_t-1)\right\}\\ &\quad + \frac{I^*}{\beta S^* G(I^*)}\left\{\beta S^*\frac{G(I^*)}{I^*}\int_0^h f(\tau)\left(x_t\tilde{y}_{t-\tau}-\frac{x_t\tilde{y}_{t-\tau}}{y_t}-y_t+1\right)d\tau\right\} + \int_0^h f(\tau)(\tilde{y}_t-\ln\tilde{y}_t-\tilde{y}_{t-\tau}+\ln\tilde{y}_{t-\tau})d\tau\\ &= -\frac{S^*(\mu+\delta)(x_t-1)^2}{\beta G(I^*)S(t)} - \frac{\delta I^*}{\beta G(I^*)S(t)}(x_t-1)(y_t-1) - (g(y_t)-g(\tilde{y}_t)) - \int_0^h f(\tau)\left\{g\left(\frac{1}{x_t}\right)+g\left(\frac{x_t\tilde{y}_{t-\tau}}{y_t}\right)\right\}d\tau\\ &\le -\frac{S^*(\mu+\delta)(x_t-1)^2+\delta I^*(x_t-1)(y_t-1)+\beta G(I^*)S(t)C_1C_2I^*(y_t-1)^2}{\beta G(I^*)S(t)}.\end{aligned}$$

By Lemma 4.2, for any $0<\varepsilon<v_1$, there exists a $T_\varepsilon>0$ such that $S(t)>v_1-\varepsilon$ for all $t>T_\varepsilon$. From the condition (II), we may restrict this $\varepsilon>0$ sufficiently small such that

$$\delta^2 - 4C_1C_2(\mu+\gamma)(\mu+\delta)(v_1-\varepsilon) < 0.$$

Then, for all $t>T_\varepsilon$, it follows

$$\begin{aligned}(\delta I^*)^2 - 4C_1C_2\beta S^*G(I^*)(\mu+\delta)I^*S(t) =& (I^*)^2\left\{\delta^2 - 4C_1C_2(\mu+\gamma)(\mu+\delta)S(t)\right\}\\ <& (I^*)^2\left\{\delta^2 - 4C_1C_2(\mu+\gamma)(\mu+\delta)(v_1-\varepsilon)\right\}\\ <& 0,\end{aligned}$$

from which we obtain $\frac{dU(t)}{dt}\le 0$ for all $t>T_\varepsilon$. We recall that $\frac{dU(t)}{dt}=0$ if $x_t=1$ and $y_t=1$, or equivalently, if $S(t)=S^*$ and $I(t)=I^*$ for all $t>T_\varepsilon$. It follows from Lemmas 2.1, 4.2 and LaSalle's invariance principle that $\tilde{E}_*$ of system (5.1) is globally asymptotically stable. □

Proof of Theorem 1.2. Summarizing results of Lemmas 2.1, 4.1, 4.2, and Theorem 5.1, we obtain the conclusion of this theorem. □

6 Applications

In this section, we illustrate some examples in order to validate the feasibility of our analytical results for $R_0>1$ for the following model proposed in Xu and Ma [18]:

$$\begin{cases}\dfrac{dS(t)}{dt} = B-\mu S(t)-\beta S(t)G(I(t-h))+\delta R(t),\\ \dfrac{dI(t)}{dt} = \beta S(t)G(I(t-h))-(\mu+\gamma)I(t),\\ \dfrac{dR(t)}{dt} = \gamma I(t)-(\mu+\delta)R(t)\end{cases} \tag{6.1}$$

with $G(I)=\frac{I}{1+\alpha I}$, $\alpha>0$.

If $R_0<1$, then the disease-free equilibrium E_0 of system (6.1) is globally asymptotically stable. If $R_0>1$, then system (6.1) is permanent. Concerning the global stability of a unique endemic equilibrium E_*, Muroya et al [17, Corollary 4.1] established the following theorem:

Theorem A. If $R_0>1$ and

$$\delta > \overline{\delta}_1(\alpha) := \frac{\beta\gamma}{\alpha(\mu+\gamma)+\beta} - \mu, \tag{6.2}$$

then the endemic equilibrium E_* of system (6.1) is globally asymptotically stable.

Theorem A improves the global stability condition of the endemic equilibrium in Xu and Ma [18, Theorem 3.1]. In contrast, similar to Theorem 1.2, we establish the following result:

Corollary 6.1. *If $R_0>1$ and*

$$0\le\delta<\underline{\delta}_2(\alpha), \tag{6.3}$$

where

$$\underline{\delta}_2(\alpha) := \frac{2C_1C_2(\mu+\gamma)B}{\mu+\beta G(B/\mu)} + \sqrt{\left\{\frac{2C_1C_2(\mu+\gamma)B}{\mu+\beta G(B/\mu)}\right\}^2 + \frac{4\mu C_1C_2(\mu+\gamma)B}{\mu+\beta G(B/\mu)}},$$

then the endemic equilibrium E_* of system (6.1) is globally asymptotically stable.

Parameter	Description	Value	Reference
β	Transmission rate	0.05 per day per inidividual	Assumed
B	Recruitment rate	30 individuals per day	Assumed
μ	Natural death rate	0.02 per day	[4]
γ	Recovery rate of infectives	0.077 per day	[4]
h	Latency period	0.1 days	Assumed

Table 1: Parameters of system (6.1) and their values in Figure 1. For the above parameter values, we have $R_0 = 257.732\cdots > 1$.

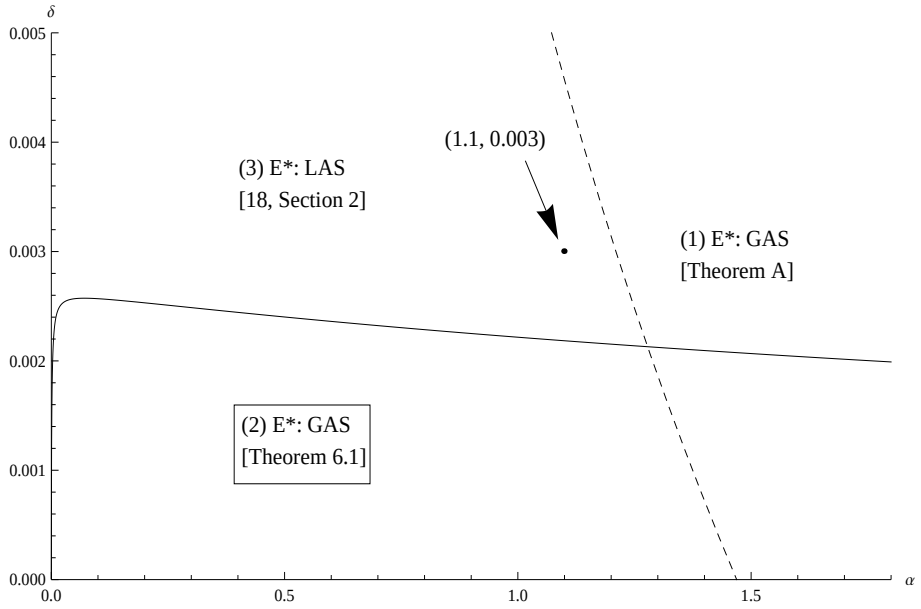

Figure 1: Curves of $\overline{\delta}_1(\alpha)$ (dotted line) and $\underline{\delta}_2(\alpha)$ (dashed line) for the parameter values in Table 1 [(1): $\delta > \overline{\delta}_1(\alpha)$, (2): $0 \le \delta \le \overline{\delta}_1(\alpha)$ and $0 \le \delta < \underline{\delta}_2(\alpha)$, (3): $0 \le \delta \le \overline{\delta}_1(\alpha)$ and $\delta \ge \underline{\delta}_2(\alpha)$]. Here, GAS and LAS denote globally asymptotically stable and locally asymptotically stable, respectively.

Next, for system (6.1), using parameter values given in Table 1, we carry out some computational experiments to investigate the feasibility of our global stability condition (6.3) with respect to the rate of immunity loss $\delta \ge 0$. For the parameter values, we obtain $R_0 = 257.732\cdots > 1$ and the endemic equilibrium E_* exists. In Figure 1, we show regions of global and local stability of the endemic equilibrium E_* which were obtained in Theorem A, Corollary 6.1 and Xu and Ma [18, Section 2] in the parameter space (α, δ).

First, we consider the case $\alpha = 1.4$. Then, we obtain $\overline{\delta}_1(\alpha) = 0.0007\cdots$ by (6.2) and $\underline{\delta}_2(\alpha) = 0.0021\cdots$ by (6.3). Hence, the endemic equilibrium E_* of system (6.1) is globally asymptotically stable for any $\delta \ge 0$. From a biological point of view, the prevalence of the disease settles to an endemic steady state independent of initial conditions concerning the fractions of a host population for any rate of immunity loss.

Secondly, we consider the case $\alpha = 1.1$. Then, we obtain $\overline{\delta}_1(\alpha) = 0.0045\cdots$ by (6.2) and $\underline{\delta}_2(\alpha) = 0.0021\cdots$ by (6.3). Therefore, the endemic equilibrium E_* of system (6.1) is globally asymptotically stable for $0 \le \delta < 0.0021\cdots$ or $\delta > 0.0045\cdots$. Thus, the global stability of the endemic equilibrium E_* is guaranteed for a small rate of immunity loss δ even if the condition (6.2) in Theorem A does not hold.

In contrast, Figure 2 indicates that the endemic equilibrium E_* of system (6.1) is also globally asymptotically stable for the case $\alpha = 1.1$ and $\delta = 0.003 \in [\underline{\delta}_2(\alpha), \overline{\delta}_1(\alpha)]$ with the parameter values in Table 1. There is still an open problem to determine the global asymptotic stability of E_* of system (6.1) when both (6.2) and (6.3) fail.

7 Discussions

In this article, we consider delayed SIRS epidemic models with a class of nonlinear incidence rates. For the incidence function G, we put the hypotheses (H1) and (H2), which describe crowding effects of infective individuals. For $R_0 < 1$, we establish the global asymptotic stability of the disease-free equilibrium in Theorem 1.1 and for $R_0 > 1$, we obtain sufficient conditions of the global asymptotic stability of the endemic equilibrium in Theorem 1.2.

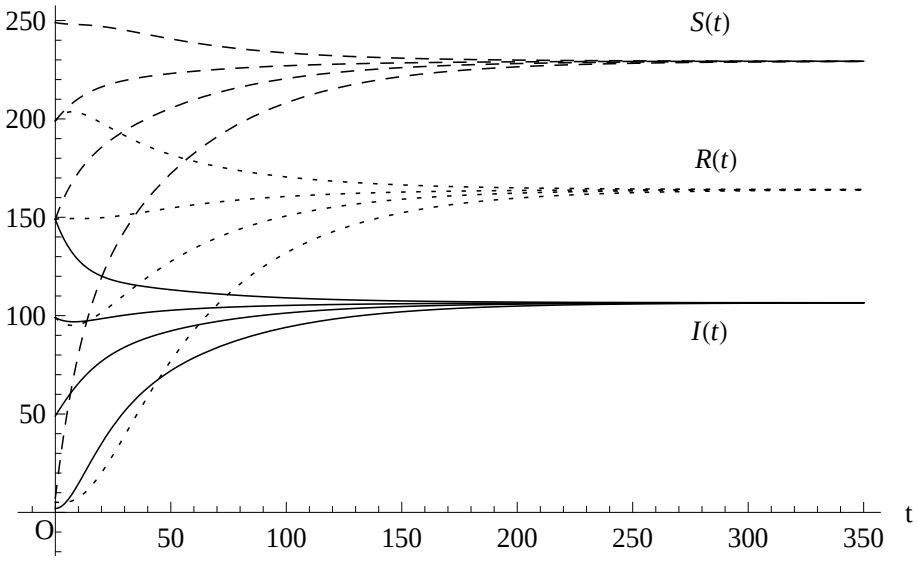

Figure 2: The graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of system (6.1). For the parameter values in Table 1 with $\alpha = 1.1$ and $\delta = 0.003$, we have $R_0 = 257.732\cdots > 1$ and $E^* = (229.338\cdots, 106.56\cdots, 164.102\cdots)$.

In particular, for $R_0 > 1$, by using strict monotonicity of the functions $G(I)$ and $I/G(I)$ on a neighborhood of I^*, we establish Lemma 5.1, which plays a key role to construct a Lyapunov functional for the reduced limit system (5.1).

For the special case $\delta = 0$ (a delayed SIR epidemic model), Beretta and Takeuchi [1] obtained the global stability of a disease-free equilibrium and the local stability of an endemic equilibrium of the model with a bilinear incidence rate (i.e. $G(I) = I$). However, in their global stability analysis of the endemic equilibrium, they required a condition that the size of time delay h should be small enough. The global stability of the endemic equilibrium for a sufficiently large h remained unsolved for a long time. Later, by applying techniques of equation deformation on a calculation of the time derivative of a Lyapunov functional, McCluskey [13] solved the problem and established that the endemic equilibrium of the model is globally asymptotically stable whenever it exists. The similar global stability results for delayed SIR epidemic models with a wide class of incidence rates were obtained in [3, 6, 14].

In contrast, for the case $\delta > 0$, there are few global stability results concerning an endemic equilibrium of delayed SIRS epidemic models. For a delayed SIRS epidemic model with the saturated incidence rate, by applying new monotone iterative techniques in [16], Muroya et al [17] recently obtained sufficient conditions, which ensure the global asymptotic stability of an endemic equilibrium of the model with an incidence rate $\beta S(t)\frac{I(t-h)}{1+\alpha I^p(t-h)}$ with $p > 0$ for large δ. Their result for $p = 1$ improved the global stability condition of the endemic equilibrium in Xu and Ma [18] for (6.1) (see Theorem A). For system (6.1), we derive Corollary 6.1 from Theorem 1.2 and find the new global stability region of the endemic equilibrium in the parameter space (α, δ), compared with that of Theorem A (see Figure 1). This illustrates that the global stability of the endemic equilibrium still holds for small δ as well as the case $\delta = 0$.

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