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Global stability of SIR epidemic models with a wide class of nonlinear incidence rates and distributed delays

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Abstract. In this paper, we establish the global asymptotic stability of equilibria for an SIR model of infectious diseases with distributed time delays governed by a wide class of nonlinear incidence rates. We obtain the global properties of the model by proving the permanence and constructing suitable Lyapunov functionals. Under some suitable assumptions on the nonlinear term in the incidence rate, the global dynamics of the model is completely determined by the basic reproduction number R_0 and the distributed delays do not influence the global dynamics of the model.

Keywords: SIR epidemic models, nonlinear incidence rate, global asymptotic stability, permanence, distributed delays, Lyapunov functional.
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1 Introduction

Mathematical models which describe the dynamics of infectious diseases have played a crucial role in the disease control in epidemiological aspect. In order to understand the mechanism of disease transmission, many authors have proposed various kinds of epidemic models (see also [1–22] and the references therein).

One of the basic SIR epidemic models is given as follows (see Hethcote [7]).

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu N(t) - \dfrac{\beta S(t)I(t)}{N(t)} - \mu S(t), \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t)}{N(t)} - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t), \end{cases} \tag{1.1}$$

where $N(t) = S(t) + I(t) + R(t)$. The initial conditions of (1.1) is $S(0) \geq 0$, $I(0) \geq 0$ and $R(0) \geq 0$ with $N(0) = S(0) + I(0) + R(0) \equiv N_0 > 0$. For system (1.1), since $N'(t) = 0$ holds for all $t > 0$, we have $N(t) \equiv N_0$ for all $t \geq 0$.

$S(t)$, $I(t)$ and $R(t)$ denote the proportions of the population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection, respectively. Hence, N(t) denotes the total population size. $\mu > 0$ represents the birth rate of the population and the death rates of susceptibles, infected and recovered individuals. We assume that all newborns are susceptibles. $\sigma > 0$ represents the recovery rate of infectives, and $\beta > 0$ represents the product of the average number of contacts of an individual per unit time. For system (1.1), individuals leave the susceptible class at a rate $\beta S(t)I(t)/N(t)$, which is called standard incidence rate. By defining

$$\tilde{S}(t) = \frac{S(t)}{N_0},\ \tilde{I}(t) = \frac{I(t)}{N_0},\ \tilde{R}(t) = \frac{R(t)}{N_0}, \tag{1.2}$$

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and dividing the equations in (1.1) by the constant total population size N_0 yields the following form ("˜" is dropped for convenience of readers).

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \beta S(t)I(t) - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t). \end{cases} \tag{1.3}$$

On the other hand, many authors have suggested that the bilinear incidence rate should be modified into a nonlinear incidence rate because the effect concerning the nonlinearity of incidence rates has been observed for some disease transmissions. For example, Capasso and Serio [4] studied the cholera epidemic spread in Bari in 1973 and introduced an incidence rate which takes a form $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$, and Brown and Hasibuan [3] studied infection model of the two-spotted spider mites, *Tetranychus urticae* and introduced an incidence rate which takes a form $(S(t)I(t))^b$. In order to study the impact of those nonlinearity, Korobeinikov and Maini [10] considered a variety of models with the incidence rate of the form $F(S(t))G(I(t))$. Later, Korobeinikov [11,12] obtained the global properties of the following basic SIR epidemic model with more general framework of the incidence rate.

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - f(S(t),I(t)) - \mu S(t), \\ \dfrac{dI(t)}{dt} = f(S(t),I(t)) - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t). \end{cases} \tag{1.4}$$

However, it is advocated in [2, 5] that more realistic models should incorporate time delays, which enable us to investigate the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.) after an incubation time denoting the time during which the infectious agents develop in the vector. This is called the phenomena of time delay effect which now has important biological meanings in epidemic models.

In this paper, we establish the global asymptotic stability of equilibria for a SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays by modifying Lyapunov functional techniques in Huang *et al.* [9], Korobeinikov [11,12] and McCluskey [17]. Our results indicate that the global dynamics is fully determined by a single threshold number R_0 independently of time delay effects under some biologically feasible conditions on the nonlinearity of the incidence rate.

The organization of this paper is as follows. In Section 2, for an SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays, we establish our main results. In Section 3, we offer a basic result. In Section 4, we show the global stability of the disease-free equilibrium of the system. In Section 5, we show the permanence of the system and establish the global asymptotic stability of the positive equilibrium for the system using a key lemma (see Lemma 5.2). Finally, we offer a discussion in Section 6.

2 Main results

In the present paper, we consider the following SIR epidemic model with a wide class of nonlinear incidence rates and distributed delays:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \displaystyle\int_0^h p(\tau) f(S(t),I(t-\tau))d\tau - \mu S(t), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^h p(\tau) f(S(t),I(t-\tau))d\tau - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - \mu R(t), \end{cases} \tag{2.1}$$

with the initial conditions

$$S(\theta) = \varphi_1(\theta),\ \ I(\theta) = \varphi_2(\theta),\ \ R(\theta) = \varphi_3(\theta),\ \ -h \le \theta \le 0,\ \ h > 0, \tag{2.2}$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C$ such that $\varphi_i(\theta) = \varphi_i(0) \ge 0$ $(-h \le \theta \le 0,\ i = 1,3)$, $\varphi_2(\theta) \ge 0$ $(-h \le \theta \le 0)$. C denotes the Banach space $C([-h,0], \mathbb{R}^3_{+0})$ of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}^3_{+0}$ with the supremum norm, where $\mathbb{R}_{+0} = \{x \in \mathbb{R} | x \ge 0\}$. From a biological meaning, we assume that $\varphi_i(0) > 0$ for $i = 1,2,3$.

h is a maximum time taken to become infectious and the transmission of the infection is governed by an incidence rate $\int_0^h p(\tau) f(S(t),I(t-\tau))d\tau$. Here, $p(\tau)$ denotes the fraction of vector population in which the time taken to become

infectious is τ [20]. We assume that $p(\tau)$ is continuous on $[0,h]$ satisfying $\int_0^h p(\tau)d\tau = 1$, and $f : \mathbb{R}^2_{+0} \to \mathbb{R}_{+0}$ is continuously differentiable on $\mathbb{R}^2_{+0}$ satisfying $f(0,I) = f(S,0) = 0$ for $S, I \geq 0$ and the following conditions:

$$\begin{cases} \text{(H1) } f(S,I) \text{ is a strictly monotone increasing function of } S \geq 0, \text{ for any fixed } I > 0, \\ \qquad \text{and a monotone increasing function of } I \geq 0, \text{ for any fixed } S \geq 0, \\ \text{(H2) } \phi(S,I) = \frac{f(S,I)}{I} \text{ is a bounded and monotone decreasing function of } I > 0, \text{ for any fixed } S \geq 0, \\ \qquad \text{and } K(S) \equiv \lim_{I \to +0} \phi(S,I) \text{ is continuous on } S \geq 0 \text{ and a monotone increasing function of } S \geq 0. \end{cases}$$

We note that $K(S) > 0$ holds for any $S > 0$. The basic reproduction number of system (2.1) becomes

$$R_0 = \frac{K(S_0)}{\mu + \sigma}, \ S_0 = 1. \tag{2.3}$$

R_0 denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population. If $R_0 \leq 1$, then under the conditions (H1) and (H2), system (2.1) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$. On the other hand, if $R_0 > 1$, then system (2.1) also admits a unique positive equilibrium $E_* = (S^*, I^*, R^*)$, where $S^*, I^*, R^* > 0$.

Our main theorems are as follows.

Theorem 2.1. *The disease-free equilibrium E_0 of system (2.1) is the only equilibrium and globally asymptotically stable, if and only if $R_0 \leq 1$.*

Theorem 2.2. *The positive equilibrium E_* of system (2.1) is globally asymptotically stable, if and only if $R_0 > 1$.*

Under the conditions (H1) and (H2), for a class of delayed epidemic models, $f(S,I)$ includes various special incidence rates. If $f(S,I) = \beta SI$, then the incidence rate becomes a bilinear form, which is proposed in [15–17, 20]. If $f(S,I) = \frac{\beta SI}{1+\alpha I}$, then the incidence rate describes saturated effects of the prevalence of infectious diseases, which is proposed in [4, 18, 22]. In addition, $f(S,I) = F(S)G(I)$, then the incidence rate is of the form proposed in Huang *et al.* [9].

3 Preliminary

In this section, we prove the following basic result, which guarantees the existence and uniqueness of the solution $(S(t), I(t), R(t))$ for system (2.1) satisfying initial conditions (2.2).

Lemma 3.1. *The solution $(S(t), I(t), R(t))$ of system (2.1) with initial conditions (2.2) uniquely exists and is positive for all $t \geq 0$. Furthermore, it holds that*

$$\lim_{t \to +\infty} (S(t) + I(t) + R(t)) = 1. \tag{3.1}$$

Proof. We notice that the right hand side of system (2.1) is completely continuous and locally Lipschitzian on C. Then, it follows that the solution of system (2.1) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to prove that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from that $\frac{dS(t)}{dt} = \mu > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us now show that $I(t) > 0$ for all $t \in [0, \alpha)$. Suppose on the contrary that there exists some $t_1 \in (0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. Integrating the second equation of system (2.1) from 0 to t_1, we see that

$$I(t_1) = I(0)\mathrm{e}^{-(\mu+\sigma)t_1} + \int_0^{t_1} \int_0^h p(\tau) f(S(u), I(u-\tau))\mathrm{e}^{-(\mu+\sigma)(t_1-u)} d\tau du > 0.$$

This contradicts $I(t_1) = 0$. From the third equation of system (2.1), we also have that $R(t) > 0$ for all $t \in [0, \alpha)$. Furthermore, for $t \in [0, \alpha)$, we obtain

$$\frac{dN(t)}{dt} = \mu - \mu(S(t) + I(t) + R(t)) = \mu(1 - N(t)), \tag{3.2}$$

which implies that $(S(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows that $(S(t), I(t), R(t))$ exists and is unique and positive for all $t \geq 0$. From (3.2), we immediately have (3.1), which completes the proof. □

Since the variable R does not appear in the first and the second equations of system (2.1), we omit the third equation of system (2.1). Thus, we consider the following 2-dimensional system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \mu - \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \sigma) I(t). \end{cases} \tag{3.3}$$

4 Global stability of the disease-free equilibrium for $R_0 \leq 1$

In this section, we give a proof of the global asymptotic stability of the disease-free equilibrium $E_0 = (S_0, 0, 0)$ of system (2.1) for $R_0 \leq 1$. The following theorem indicates that the disease can be eradicated in the host population if $R_0 \leq 1$.

Theorem 4.1. *The disease-free equilibrium $Q_0 \equiv (S_0, 0)$ of system (3.3) is the only equilibrium and globally asymptotically stable, if and only if $R_0 \leq 1$.*

Proof. From the conditions (H1) and (H2), the disease-free equilibrium is the only equilibrium for system (3.3). We now consider the following Lyapunov functional:

$$U^0(t) = U_1^0(t) + I(t) + U_+^0(t),$$

where

$$U_1^0(t) = \int_{S_0}^{S(t)} \left(1 - \frac{K(S_0)}{K(s)}\right) ds,\ U_+^0(t) = \int_0^h p(\tau) \int_{t-\tau}^t f(S(u+\tau), I(u)) \frac{K(S_0)}{K(S(u+\tau))} du d\tau.$$

We show that $\frac{dU^0(t)}{dt} \leq 0$ for all $t \geq 0$. First, we calculate $\frac{dU_1^0(t)}{dt}$. By using $\mu = \mu S_0$,

$$\begin{aligned}
\frac{dU_1^0(t)}{dt} &= \left(1 - \frac{K(S_0)}{K(S(t))}\right) \left(\mu - \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t)\right) \\
&= -\mu(S(t) - S_0) \left(1 - \frac{K(S_0)}{K(S(t))}\right) - \left(1 - \frac{K(S_0)}{K(S(t))}\right) \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau.
\end{aligned}$$

Second, calculating $\frac{dU_+^0(t)}{dt}$, we get that

$$\frac{dU_+^0(t)}{dt} = \int_0^h p(\tau) \left\{ f(S(t+\tau), I(t)) \frac{K(S_0)}{K(S(t+\tau))} - f(S(t), I(t-\tau)) \frac{K(S_0)}{K(S(t))} \right\} d\tau.$$

Therefore, it follows that

$$\begin{aligned}
\frac{dU^0(t)}{dt} &= -\mu(S(t) - S_0) \left(1 - \frac{K(S_0)}{K(S(t))}\right) - \left(1 - \frac{K(S_0)}{K(S(t))}\right) \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau \\
&\quad + \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \sigma) I(t) \\
&\quad + \int_0^h p(\tau) \left\{ f(S(t+\tau), I(t)) \frac{K(S_0)}{K(S(t+\tau))} - f(S(t), I(t-\tau)) \frac{K(S_0)}{K(S(t))} \right\} d\tau \\
&= -\mu(S(t) - S_0) \left(1 - \frac{K(S_0)}{K(S(t))}\right) + \int_0^h p(\tau) \left\{ \frac{\phi(S(t+\tau), I(t))}{\mu + \sigma} \cdot \frac{K(S_0)}{K(S(t+\tau))} - 1 \right\} (\mu + \sigma) I(t) d\tau.
\end{aligned}$$

By the condition (H1), we obtain that

$$-\mu(S(t) - S_0) \left(1 - \frac{K(S_0)}{K(S(t))}\right) \leq 0,$$

with equality if and only if $S(t) = S_0$. It follows from the condition (H2) that

$$\frac{\phi(S(t+\tau), I(t))}{\mu + \sigma} \cdot \frac{K(S_0)}{K(S(t+\tau))} \leq \frac{K(S(t+\tau))}{\mu + \sigma} \cdot \frac{K(S_0)}{K(S(t+\tau))} = \frac{K(S_0)}{\mu + \sigma} = R_0.$$

Therefore, $R_0 \leq 1$ ensures that $\frac{dU^0(t)}{dt} \leq 0$ for all $t > 0$, where $\frac{dU^0(t)}{dt} = 0$ holds if $S(t) = S_0$. Hence, it immediately follows from system (3.3) that Q_0 is the largest invariant set in $\{(S(t), I(t)) \in \mathbb{R}_{+0}^2 | \frac{dU^0(t)}{dt} = 0\}$. From the Lyapunov-LaSalle asymptotic stability theorem [14], we obtain that Q_0 is the only equilibrium of system (3.3) and globally asymptotically stable. This completes the proof. □

Proof of Theorem 2.1. By Theorem 4.1, we immediately obtain the conclusion of this theorem. □

Remark 4.1. To establish the global asymptotic stability of the disease-free equilibrium E_0 for $R_0 \leq 1$, the condition of the monotonicity of $f(S, I)$ of $I \geq 0$ for any fixed $S \geq 0$ in (H1) is not necessary for our analysis.

5 Permanence and global stability of the positive equilibrium for $R_0 > 1$

In this section, we show the permanence and the global asymptotic stability of the positive equilibrium $E_* = (S^*, I^*, R^*)$ for system (2.1) for $R_0 > 1$.

Corollary 5.1. *If $R_0 > 1$, then system (2.1) has a unique positive equilibrium E_* satisfying the following equations:*

$$\mu - \mu S^* - f(S^*, I^*) = 0,\ f(S^*, I^*) - (\mu + \sigma)I^* = 0,\ \sigma I^* - \mu R^* = 0.$$

Proof. At a fixed point (S, I, R) of system (1.1), the following equations hold.

$$\begin{cases} \mu - \mu S - (\mu + \sigma)I = 0, \\ f(S, I) - (\mu + \sigma)I = 0, \\ \sigma I - \mu R = 0. \end{cases} \tag{5.1}$$

Substituting the second equation of (5.1) into the first equation of (5.1), we consider the following equation:

$$H(I) := \frac{f(1 - \frac{\mu+\sigma}{\mu}I, I)}{I} - (\mu + \sigma) = 0.$$

By the hypothesis (H2), H is strictly monotone decreasing on $(0, +\infty)$ satisfying

$$\lim_{I \to +0} H(I) = K(S_0) - (\mu + \sigma) = (\mu + \sigma)(R_0 - 1) > 0,$$

and $H(\frac{\mu}{\mu+\sigma}) = -(\mu+\sigma) < 0$ holds. Hence, there exists a unique $0 < I^* < \frac{\mu}{\mu+\sigma}$ such that $H(I^*) = 0$. By (5.1), we obtain $S^* = 1 - \frac{\mu+\sigma}{\mu}I^* > 0$ and $R^* = \frac{\sigma I^*}{\mu} > 0$. This implies that (2.1) has a unique positive equilibrium $E_* = (S^*, I^*, R^*)$. □

5.1 Permanence

In this subsection, we show the permanence of system (2.1) by using techniques in Song *et al.* [19] and Wang [21]. From (3.1), let us put sufficiently small $\varepsilon_S > 0$ and sufficiently large $T_S > 0$ satisfying $K(S(t)) \le K(S_0) + \varepsilon_S$ holds for any $t \ge T_S$. The following theorem indicates that the disease eventually persists in the host population if $R_0 > 1$.

Theorem 5.1. *If $R_0 > 1$, then for any solution of system (2.1), it holds that*

$$\liminf_{t \to +\infty} S(t) \ge v_1,\ \liminf_{t \to +\infty} I(t) \ge v_2 := qI^* \mathrm{e}^{-(\mu+\sigma)\rho h},\ \liminf_{t \to +\infty} R(t) \ge v_3 := \frac{\sigma v_2}{\mu},$$

where $v_1 > 0$ satisfies $\mu - K(v_1) - \mu v_1 = 0$, and $q > 0$ and $\rho \ge 1$ satisfy

$$S^* < \frac{\mu - (K(S_0) + \varepsilon_S)qI^*}{\mu}\big(1 - \mathrm{e}^{-\mu\rho h}\big),\ 0 < q < \frac{\mu}{(K(S_0) + \varepsilon_S)I^*}. \tag{5.2}$$

Proof. Let $(S(t), I(t), R(t))$ be a solution of system (2.1) with initial condition (2.2). By Lemma 3.1, it follows that $\limsup_{t \to +\infty} I(t) \le 1$, which implies from the condition (H2) that, for any $\varepsilon_I > 0$, there is an integer $T_I \ge 0$ such that

$$\begin{aligned} \frac{dS(t)}{dt} &= \mu - \int_0^h p(\tau)\frac{f(S(t), I(t-\tau))}{I(t-\tau)}I(t-\tau)d\tau - \mu S(t) \\ &\ge \mu - K(S(t))\int_0^h p(\tau)I(t-\tau)d\tau - \mu S(t) \\ &= \mu - K(S(t))(1 + \varepsilon_I) - \mu S(t), \end{aligned}$$

for $t \ge T_I + h$. Let us now consider the auxiliary equation

$$\frac{dS(t)}{dt} = \mu - K(S(t)) - \mu S(t).$$

Then, one can immediately obtain that $\lim_{t \to +\infty} S(t) = v_1 > 0$. Since (5.3) holds for arbitrary $\varepsilon_I > 0$ sufficiently small, it follows that $\liminf_{t \to +\infty} S(t) \ge v_1 > 0$.

We now prove that it is impossible that $I(t) \le qI^*$ for all sufficiently large t. Suppose on the contrary that there exists a sufficiently large $t_1 \ge T_S$ such that $I(t) \le qI^*$ holds for all $t \ge t_1$. Then, similar to the above discussion, we have that for any $t \ge t_1 + h$,

$$\frac{dS(t)}{dt} = \mu - \int_0^h p(\tau)\phi(S(t), I(t-\tau))I(t-\tau)d\tau - \mu S(t) \ge \mu - (K(S_0) + \varepsilon_S)qI^* - \mu S(t),$$

which yields for $t \geq t_1 + h$,

$$\begin{aligned} S(t) &\geq S(t_1+h)\mathrm{e}^{-\mu(t-t_1-h)} + \mathrm{e}^{-\mu t}\int_{t_1+h}^{t} \mathrm{e}^{\mu s}(\mu - (K(S_0)+\varepsilon_S)qI^*)ds \\ &= S(t_1+h)\mathrm{e}^{-\mu(t-t_1-h)} + \frac{\mu-(K(S_0)+\varepsilon_S)qI^*}{\mu}\big(1-\mathrm{e}^{-\mu(t-t_1-h)}\big). \end{aligned} \tag{5.3}$$

Hence, it follows from (5.3) that for $t \geq t_1 + h + \rho h$,

$$S(t) > \frac{\mu-(K(S_0)+\varepsilon_S)qI^*}{\mu}\big(1-\mathrm{e}^{-\mu\rho h}\big) = S^{\triangle} > S^*. \tag{5.4}$$

Now, we define the following functional.

$$V(t) = I(t) + \int_0^h p(\tau)\int_t^{t+\tau} f(S(u), I(u-\tau))dud\tau. \tag{5.5}$$

Calculating the derivative of $V(t)$ along solutions of system (2.1) gives as follows.

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^h p(\tau)f(S(t), I(t-\tau))d\tau - (\mu+\sigma)I(t) + \int_0^h p(\tau)\left\{f(S(t+\tau), I(t)) - f(S(t), I(t-\tau))\right\}d\tau \\ &= \int_0^h p(\tau)f(S(t+\tau), I(t))d\tau - (\mu+\sigma)I(t). \end{aligned}$$

For $t \geq t_1 + h + \rho h$, it follows from (5.4) and the relation $\mu + \sigma = \phi(S^*, I^*)$ that

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^h p(\tau)\left\{\phi(S(t+\tau), I(t)) - (\mu+\sigma)\right\}I(t)d\tau \\ &> \int_0^h p(\tau)\left\{\phi(S(t+\tau), I^*) - \phi(S^*, I^*) + \phi(S^*, I^*) - (\mu+\sigma)\right\}I(t)d\tau \\ &= \int_0^h p(\tau)\{\phi(S(t+\tau), I^*) - \phi(S^*, I^*)\}I(t)d\tau \\ &\geq \{\phi(S^{\triangle}, I^*) - \phi(S^*, I^*)\}I(t). \end{aligned} \tag{5.6}$$

Setting

$$\underline{i} = \min_{\theta\in[-h,0]} I(\theta + t_1 + \rho h + 2h),$$

we claim that $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. Otherwise, if there is a $T \geq 0$ such that $I(t) \geq \underline{i}$ for $t_1 + h + \rho h \leq t \leq t_1 + 2h + \rho h + T$, $I(t_1 + 2h + \rho h + T) = \underline{i}$ and $\frac{d}{dt}I(t)|_{t=t_1+2h+\rho h+T} \leq 0$, it follows from the second equation of system (2.1), the conditions (H1) and (H2) that for $t_2 = t_1 + 2h + \rho h + T$,

$$\begin{aligned} \frac{dI(t)}{dt}\Big|_{t=t_2} &= \int_0^h p(\tau)f(S(t_2), I(t_2-\tau))d\tau - (\mu+\sigma)I(t_2) \\ &= \int_0^h p(\tau)\phi(S(t_2), I(t_2-\tau))I(t_2-\tau)d\tau - (\mu+\sigma)I(t_2) \\ &> \int_0^h p(\tau)\phi(S(t_2), I^*)I(t_2-\tau)d\tau - (\mu+\sigma)I(t_2) \\ &\geq \left\{\phi(S(t_2), I^*) - (\mu+\sigma)\right\}I(t_2) \\ &\geq \left\{\phi(S^{\triangle}, I^*) - (\mu+\sigma)\right\}\underline{i} \\ &> \left\{\phi(S^*, I^*) - (\mu+\sigma)\right\}\underline{i} = 0. \end{aligned}$$

This is a contradiction. Therefore $I(t) \geq \underline{i}$ for all $t \geq t_1 + h + \rho h$. It follows from (5.6) that

$$\frac{dV(t)}{dt} > \{\phi(S^{\triangle}, I^*) - \phi(S^*, I^*)\}\underline{i} > 0, \text{ for } t \geq t_1 + 2h + \rho h,$$

which implies that $\lim_{t\to+\infty} V(t) = +\infty$. However, it holds from (3.1) and (5.5) that $\limsup_{t\to+\infty} V(t) < +\infty$. Hence the claim holds.

Thus, we proved that it is impossible that $I(t) \le qI^*$ for all sufficiently large t. This implies that we are left to consider the following two possibilities.

$$\begin{cases} \text{(i) } I(t) \ge qI^* \text{ for all } t \text{ sufficiently large,} \\ \text{(ii) } I(t) \text{ oscillates about } qI^* \text{ for all } t \text{ sufficiently large.} \end{cases}$$

If the first case holds, then we immediately get the conclusion of the proof. If the second case holds, we show that $I(t) \ge qI^* \exp\left(-(\mu+\sigma)\rho h\right)$ for all t sufficiently large. Let $t_3 < t_4$ be sufficiently large such that

$$I(t_3) = I(t_4) = qI^*,\ I(t) < qI^*,\ t_3 < t < t_4.$$

If $t_4 - t_3 \le \rho h$, then from the second equation of system (3.3), we have $\frac{dI(t)}{dt} > -(\mu+\sigma)I(t)$, that is,

$$I(t) > I(t_3)\exp\left(-(\mu+\sigma)(t-t_3)\right) \ge qI^*\exp\left(-(\mu+\sigma)\rho h\right) = v_2.$$

If $t_4 - t_3 > \rho h$, we obtain $I(t) \ge v_2$ for $t_3 \le t \le t_3 + \rho h$. We now claim that $I(t) \ge v_2$ for all $t_3 + \rho h \le t \le t_4$. Otherwise, there is a $T^* > 0$ such that $I(t) \ge v_2$ for $t_3 \le t \le t_3 + \rho h + T^* < t_4$, $I(t_3 + \rho h + T^*) = v_2$ and $\frac{dI(t)}{dt}|_{t=t_3+\rho h+T^*} \le 0$. On the other hand, for $t_0 = t_3 + \rho h + T^*$, it follows from the relation $\phi(S(t_0), I(t_0)) > \phi(S(t_0), I^*) \ge \phi(S^\triangle, I^*) > \phi(S^*, I^*)$ that

$$\begin{aligned} \frac{dI(t)}{dt}\Big|_{t=t_0} &= \int_0^h p(\tau) f(S(t_0), I(t_0-\tau))d\tau - (\mu+\sigma)I(t_0) \\ &= \int_0^h p(\tau)\phi(S(t_0), I(t_0-\tau))I(t_0-\tau)d\tau - (\mu+\sigma)I(t_0) \\ &> \left\{\phi(S(t_0), I^*) - (\mu+\sigma)\right\} I(t_0) \\ &\ge \left\{\phi(S^\triangle, I^*) - (\mu+\sigma)\right\} I(t_0) \\ &> \left\{\phi(S^*, I^*) - (\mu+\sigma)\right\} I(t_0) = 0, \end{aligned}$$

which is a contradiction. Hence $I(t) \ge qI^*\exp\left(-(\mu+\sigma)\rho h\right) = v_2$ for $t_3 \le t \le t_4$. Since the interval $[t_3, t_4]$ is arbitrarily chosen, we conclude that $I(t) \ge v_2$ holds for all t sufficiently large. Thus, we obtain $\liminf_{t\to+\infty} I(t) \ge v_2$, from which we have $\liminf_{t\to+\infty} R(t) \ge v_3$. Hence, this completes the proof. □

5.2 Global stability of the positive equilibrium

In this subsection, we give a proof of the global asymptotic stability of the positive equilibrium E_* for $R_0 > 1$.

For a fixed $0 \le \tau \le h$, we put

$$y_t = \frac{I(t)}{I^*},\ \tilde{y}_{t,\tau} = \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}. \tag{5.7}$$

The following lemma plays a key role to obtain Theorems 2.2 and 5.2.

Lemma 5.1. *Assume that system (2.1) has a positive equilibrium E_*. Under the conditions (H1) and (H2), it holds that*

$$g(y_t) - g(\tilde{y}_{t,\tau}) \ge 0, \tag{5.8}$$

for all $t \ge 0$ and $0 \le \tau \le h$, where $g(x) = x - 1 - \ln x \ge 0$, for $x > 0$.

Proof. By the definitions of y_t and $\tilde{y}_{t,\tau}$, we have that $\tilde{y}_{t,\tau} - 1 = \frac{f(S(t+\tau),I(t)) - f(S(t+\tau),I^*)}{f(S(t+\tau),I^*)}$ and

$$y_t - \tilde{y}_{t,\tau} = \frac{I(t)}{I^*} - \frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)} = \frac{I(t)}{f(S(t+\tau), I^*)}\left\{\phi(S(t+\tau), I^*) - \phi(S(t+\tau), I(t))\right\}.$$

Then, it follows from the conditions (H1) and (H2) that

$$\begin{aligned} g(y_t) - g(\tilde{y}_{t,\tau}) &= y_t - \tilde{y}_{t,\tau} - \ln\frac{y_t}{\tilde{y}_{t,\tau}} = y_t - \tilde{y}_{t,\tau} - \frac{y_t}{\tilde{y}_{t,\tau}} + 1 + \frac{y_t}{\tilde{y}_{t,\tau}} - 1 - \ln\frac{y_t}{\tilde{y}_{t,\tau}} \\ &= \frac{1}{\tilde{y}_{t,\tau}}(\tilde{y}_{t,\tau} - 1)(y_t - \tilde{y}_{t,\tau}) + g\left(\frac{y_t}{\tilde{y}_{t,\tau}}\right) \ge 0. \end{aligned}$$

Hence, this completes the proof. □

Now, we are in a position to prove the global asymptotic stability of the positive equilibrium E_* for $R_0 > 1$, by applying the technique established by Huang *et al.* [9], Korobeinikov [11, 12] and McCluskey [17].

Theorem 5.2. *The positive equilibrium $Q_* \equiv (S^*, I^*)$ of the reduced system (3.3) is globally asymptotically stable, if and only if $R_0 > 1$.*

Proof. We now define the following functional:

$$U^*(t) = U_1^*(t) + U_+^*(t), \tag{5.9}$$

where

$$U_1^*(t) = \int_{S^*}^{S(t)} \left(1 - \frac{\phi(S^*, I^*)}{\phi(s, I^*)}\right) ds + g\left(\frac{I(t)}{I^*}\right),\ U_+^*(t) = f(S^*, I^*) \int_0^h p(\tau) \int_{t-\tau}^t g\left(\frac{f(S(u+\tau), I(u))}{f(S(u+\tau), I^*)}\right) du d\tau. \tag{5.10}$$

We note that $U_1^*(t)$ satisfies $\frac{\partial U_1^*}{\partial S} = 1 - \frac{\phi(S^*, I^*)}{\phi(S, I^*)}$ and $\frac{\partial U_1^*}{\partial I} = 1 - \frac{I^*}{I}$, which implies that the point $(S(t), I(t)) = (S^*, I^*)$ is a stational point of the function $U_1^*(t)$ and it is the unique stational point and the global minimum of the function U_1^*.

Using the relation $\mu = \mu S^* + f(S^*, I^*)$ and $\mu + \sigma = \phi(S^*, I^*)$, the time derivative of the function $U_1^*(t)$ along the positive solution of system (3.3) becomes

$$\begin{aligned}
\frac{dU_1^*(t)}{dt} &= \left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right)\left\{\mu - \int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \mu S(t)\right\} \\
&\quad + \left(1 - \frac{I^*}{I(t)}\right)\left(\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - (\mu + \sigma) I(t)\right) \\
&= \left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right)\left\{\int_0^h p(\tau)\{f(S^*, I^*) - f(S(t), I(t-\tau))\} d\tau - \mu(S(t) - S^*)\right\} \\
&\quad + \left(1 - \frac{I^*}{I(t)}\right)\left(\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau - \phi(S^*, I^*) I(t)\right) \\
&= \mu S^* \left(1 - \frac{S(t)}{S^*}\right)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) \int_0^h p(\tau)\left\{1 - \frac{f(S(t), I(t-\tau))}{f(S^*, I^*)}\right\} d\tau \\
&\quad + f(S^*, I^*)\left(1 - \frac{I^*}{I(t)}\right) \int_0^h p(\tau)\left\{\frac{f(S(t), I(t-\tau))}{f(S^*, I^*)} - \frac{I(t)}{I^*}\right\} d\tau,
\end{aligned} \tag{5.11}$$

and the time derivative of the function $U_+^*(t)$ becomes

$$\frac{dU_+^*(t)}{dt} = f(S^*, I^*) \int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}\right) - g\left(\frac{f(S(t), I(t-\tau))}{f(S(t), I^*)}\right)\right\} d\tau. \tag{5.12}$$

From (5.11) and (5.12), we obtain that

$$\begin{aligned}
\frac{dU^*(t)}{dt} &= \mu S^* \left(1 - \frac{S(t)}{S^*}\right)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \int_0^h p(\tau)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)} + \frac{f(S(t), I(t-\tau))}{f(S(t), I^*)}\right) d\tau \\
&\quad + f(S^*, I^*) \int_0^h p(\tau)\left(1 - \frac{I(t)}{I^*} - \frac{I^*}{I(t)} \frac{f(S(t), I(t-\tau))}{f(S^*, I^*)}\right) d\tau \\
&\quad + f(S^*, I^*) \int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}\right) - g\left(\frac{f(S(t), I(t-\tau))}{f(S(t), I^*)}\right)\right\} d\tau \\
&= \mu S^* \left(1 - \frac{S(t)}{S^*}\right)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) \\
&\quad + f(S^*, I^*) \int_0^h p(\tau)\left\{g\left(\frac{f(S(t+\tau), I(t))}{f(S(t+\tau), I^*)}\right) - g\left(\frac{I(t)}{I^*}\right)\right\} d\tau \\
&\quad - f(S^*, I^*) \int_0^h p(\tau)\left\{g\left(\frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) + g\left(\frac{I^*}{I(t)} \frac{f(S(t), I(t-\tau))}{f(S^*, I^*)}\right)\right\} d\tau.
\end{aligned} \tag{5.13}$$

From the condition (H1), we obtain

$$\left(1 - \frac{S(t)}{S^*}\right)\left(1 - \frac{\phi(S^*, I^*)}{\phi(S(t), I^*)}\right) \le 0,$$

with equality holds if and only if $S(t) = S^*$, and using Lemma 5.1, we have $g(\frac{f(S(t+\tau),I(t))}{f(S(t+\tau),I^*)}) - g(\frac{I(t)}{I^*}) \leq 0$ for all $0 \leq \tau \leq h$. This implies that $\frac{dU^*(t)}{dt} \leq 0$ holds for all $t \geq 0$ since S^* and $f(S^*, I^*)$ are nonnegative. Therefore, it follows from (5.13) that $\frac{dU^*(t)}{dt} = 0$ holds if $S(t) = S^*$ and $f(S^*, I(t-\tau)) = \frac{f(S^*,I^*)}{I^*} I(t)$ for almost all $\tau \in [0, h]$. By Hale and Lunel [6, Theorem 5.3.1], solutions of system (3.3) limit to M, the largest invariant subset of $\{\frac{dU^*(t)}{dt} = 0\}$. We now show that M consists of only the positive equilibrium Q_*. For each element of M, we have $S(t) = S^*$ and, since M is invariant, $\frac{dS(t)}{dt} = 0$. Using the first equation of system (3.3) and the relation $\mu = \mu S^* + f(S^*, I^*)$, we obtain that

$$\begin{aligned} 0 = \frac{dS(t)}{dt} &= \mu - \int_0^h p(\tau) f(S^*, I(t-\tau)) d\tau - \mu S^* = \mu - \frac{f(S^*, I^*)}{I^*} I(t) - \mu S^* \\ &= \mu S^* + f(S^*, I^*) - \frac{f(S^*, I^*)}{I^*} I(t) - \mu S^* \\ &= f(S^*, I^*) \left(1 - \frac{I(t)}{I^*}\right). \end{aligned}$$

Thus, each element of M satisfies $S(t) = S^*$ and $I(t) = I^*$. Since the permanence result (see Lemma 3.1 and Theorem 5.1) for system (3.3) is already known, by the LaSalle invariance principle [14], Q^* is the only equilibrium of system (3.3) on the line and globally asymptotically stable. Hence, the proof is complete. □

Proof of Theorem 2.2. By Lemma 3.1, Theorems 5.1 and 5.2, we obtain the conclusion of this theorem. □

6 Discussion

In this paper, we establish the global asymptotic stability of the disease-free equilibrium for $R_0 \leq 1$, and the positive equilibrium for $R_0 > 1$ by modifying Lyapunov functional techniques in Huang *et al.* [9], Korobeinikov [11, 12] and McCluskey [17]. From a biological motivation, we do not only extend the nondelayed model (1.4) in Korobeinikov [11,12] to the delayed model (2.1) but also obtain the permanence result and the global properties for (2.1) with distributed time delays governed by a wide class of nonlinear incidence rate $\int_0^h p(\tau) f(S(t), I(t-\tau)) d\tau$. It is noteworthy that the global dynamics is completely determined by the basic reproduction number R_0 independently of the length of an incubation period of the diseases as long as the infection rate has a suitable monotone property characterized by (H1) and (H2).

It has been generally considered reasonable to expect that a biologically feasible functional response is associated with monotonicity with respect to the proportion of susceptible and infected individuals, and is concave, or at least nonconvex with respect to the proportion of infective individuals (see, e.g., [4, 10–12]). Noting that $\phi(S, I) = \frac{f(S,I)}{I}$ denotes the infection force per unit proportion of infective individuals, the conditions that $f(S, I)$ is monotone increasing of I and $\phi(S, I)$ is monotone decreasing of I in (H1) and (H2) describe the crowding (saturation) effects. Thus, one can see that the conditions (H1) and (H2) are natural assumptions which have a biological meaning. Our result further indicates that the disease dynamics is fully determined when the saturation effects appear (see, e.g., [4, 18, 22]).

Finally, we have to stress that Lemma 5.1 plays a vital role to establish the global asymptotic stability of the positive equilibrium E_* of system (2.1) for $R_0 > 1$. These techniques are also applicable to various kinds of epidemic models (e.g. SIRS models, SEIR models, etc.). These will be our future consideration.

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