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A note on the global stability of an SEIR epidemic model with constant latency time and infectious period

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Abstract. In this note, under the condition for the permanence used by [Beretta and Breda, An SEIR epidemic model with constant latency time and infectious period, *Math. Biosci. Eng.* **8** (2011) 931-952], applying modified monotone sequences, we establish the global asymptotic stability of the endemic equilibrium of this SEIR epidemic model, without any other additional conditions on the global stability.

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1 Introduction

Motivated by the interesting contribution by Xu and Du [4] and Xu and Ma [6] to the stability analysis by means of iterative schemes and comparison principles, Beretta and Breda [1] recently investigated a two delayed SEIR epidemic model with a saturation incidence rate. One delay is a time taken by the infected individuals to become infectious, and the other delay is a time taken by an infectious individual to be removed from the infection. The model is as follows.

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - g(I(t))S(t), \\ \dfrac{dE(t)}{dt} = g(I(t))S(t) - g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} - \mu_2 E(t), \\ \dfrac{dI(t)}{dt} = g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} - g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_2 I(t), \\ \dfrac{dR(t)}{dt} = g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_3 R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible individuals, exposed individuals, infected individuals and recovered individuals, respectively. The exposed individuals who have been infected, take a time $\tau_1 \geq 0$ to become infectious and the infected individuals who have become infected individual take a time $\tau_2 \geq 0$ to be removed from the infection. In addition, we assume that the removed individuals cannot return to the susceptible class, because they have been quarantined and/or acquired permanent immunity to infectious diseases. $\Lambda > 0$ is the constant recruitment rate of the population, and $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_3 > 0$ are the constant death rates of the susceptible individuals, both exposed individuals and infectious individuals, and the recovered individuals, respectively. We assume that $g(I)$ is a saturated incidence rate of the form

$$g(I) = \frac{\beta I}{1 + \alpha I}, \tag{1.2}$$

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where βI is a measure of the force of infection and $\frac{1}{1+\alpha I}$ accounts for the inhibition effect on the rate of infection when I becomes large. We assume that

$$\mu_1 = \min_{i=1,2,3}\{\mu_i\}, \quad \mu_2 = \max_{i=1,2,3}\{\mu_i\}. \tag{1.3}$$

By taking into account that the rate of infection at time t is $g(I(t))S(t)$ and that the exposed individuals that become infection I at time t are those infected at the previous time $t-\tau_1$, multiplied for the fraction equation for the exposed survived in the time interval $[t-\tau_1, t]$, we also get the evolution equations for $I(t)$ and $R(t)$, while the one for $S(t)$ is standard.

Since in (1.1), the evolution equations for $S(t)$ and $I(t)$ do not contain the variables $E(t)$ and $R(t)$, we may consider the following reduced system:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \mu_1 S(t) - g(I(t))S(t), \\ \dfrac{dI(t)}{dt} = g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} - g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_2 I(t). \end{cases} \tag{1.4}$$

We here note that the second equation of (1.4) can be rewritten by formally integrating the delay differential equations for $I(t)$ as follows:

$$I(t) = \int_{\tau_1}^{\tau_1+\tau_2} g(I(t-\theta))S(t-\theta)e^{-\mu_2\theta}d\theta. \tag{1.5}$$

The initial conditions of (1.4) are given by biological reasons as

$$S(\theta) = \varphi_1(\theta) \text{ and } I(\theta) = \varphi_2(\theta),\ \theta \in [-(\tau_1+\tau_2), 0] \text{ with } S(0) > 0,\ I(0) > 0, \tag{1.6}$$

where φ_1 and φ_2 are nonnegative continuous functions on a closed interval $[-(\tau_1+\tau_2), 0]$ and satisfy

$$I(0) = \int_{-(\tau_1+\tau_2)}^{-\tau_1} g(\varphi_2(\theta))\varphi_1(\theta)e^{\mu_2\theta}d\theta. \tag{1.7}$$

The basic reproduction number of system (1.4) becomes

$$R_0 = \frac{\beta\Lambda}{\mu_1\mu_2}e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2}), \tag{1.8}$$

which depends on the delays τ_1 and τ_2.

One can see immediately that system (1.4) always has a disease-free equilibrium $\mathbf{E}_0 = (\Lambda/\mu, 0)$. Apart from the above equilibrium, if $R_0 > 1$, then system (1.4) allows a unique endemic equilibrium $\mathbf{E}_+ = (S_+, I_+)$ satisfying the following equations.

$$\begin{cases} \Lambda - \mu_1 S_+ - g(I_+)S_+ = 0, \\ g(I_+)S_+e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2}) - \mu_2 I_+ = 0, \end{cases} \tag{1.9}$$

and hence, we have that

$$S_+ = \frac{\Lambda}{\mu_1}\frac{\beta+\alpha\mu_1 R_0}{R_0(\beta+\alpha\mu_1)},\ I_+ = \frac{\mu_1(R_0-1)}{\beta+\alpha\mu_1}. \tag{1.10}$$

By applying iterative schemes as used for a delayed SIR epidemic model in Xu and Du [4] and the comparison principles, Beretta and Breda [1] established the following result:

Theorem A (See Beretta and Breda [1]). *If $R_0 \le 1$, then the disease-free equilibrium of system* (1.1) *is globally asymptotically stable. If $R_0 > 1$, then there exists an endemic equilibrium of system* (1.1) *which uniquely exists and is locally asymptotically stable. Moreover, if*

$$R_0 > \frac{\beta}{\alpha\mu_1}, \tag{1.11}$$

then system (1.1) *is permanent, and in particular, if*

$$\frac{\beta}{\alpha\mu_1} < 1, \tag{1.12}$$

then the endemic equilibrium of system (1.1) *is globally asymptotically stable.*

Because Beretta and Breda [1] needed more restricted condition $R_0 > 1 > \frac{\beta}{\alpha\mu_1}$, to compare with the permanence conditions of system (1.1) such that $R_0 > 1$ and $R_0 > \frac{\beta}{\alpha\mu_1}$, there remains an open question for the case $R_0 > \frac{\beta}{\alpha\mu_1} \geq 1$ on the global asymptotic stability of the endemic equilibrium of system (1.1).

On the other hand, Muroya *et al.* [3] (see also Muroya *et al.* [2]) investigated improvement on monotone iterative techniques in Xu and Ma [5] to obtain the global stability of the endemic equilibrium of a delayed SIRS epidemic model.

Motivated by the above results, in this note, applying the techniques in Muroya *et al.* [3] to construct a strictly monotone decreasing sequence $\{\bar{S}_n\}_{n=1}^{\infty}$ of upper bound of $\limsup_{t\to+\infty} S(t)$ and a strictly monotone increasing sequence $\{\underline{S}_n\}_{n=1}^{\infty}$ of lower bound of $\liminf_{t\to+\infty} S(t)$ (see Lemma 3.2), we establish the following theorem which completely solves the above open question and improves the result in Beretta and Breda [1].

Theorem 1.1. *Assume $R_0 > 1$. Then there exists an endemic equilibrium of system* (1.1) *which is locally asymptotically stable. Moreover, if* (1.11) *holds, then the endemic equilibrium of system* (1.1) *is globally asymptotically stable.*

The organization of this paper is as follows. In Section 2, we give some known results for system (1.1). In Section 3, using monotone techniques similar to Muroya *et al.* [3], we first offer our basic Lemmas 3.1 and 3.2, from which we prove Theorem 1.1. In Section 4, we offer numerical examples in order to investigate the feasibility of the sufficient condition (1.11) ensuring the global stability of the endemic equilibrium of system (1.1).

2 Some known results in Beretta and Breda [1]

In this section, we state some known results for system (1.1) by Beretta and Breda [1]. By the comparison principle, the following result is obtained in Beretta and Breda [1, Lemma 2.1].

Lemma 2.1. *The compact set*

$$\Omega := \left\{ (S, E, I, R) \in \mathbb{R}^4_{+0} : \frac{\Lambda}{\mu_2} \leq S + E + I + R \leq \frac{\Lambda}{\mu_1} \right\} \tag{2.1}$$

is globally attractive and invariant for the solutions of (1.1)*, where*

$$\mathbb{R}^4_{+0} = \{(x_1, x_2, x_3, x_4) : x_i \geq 0,\ i = 1, 2, 3, 4\}. \tag{2.2}$$

Moreover, the following results are also obtaind in Beretta and Breda [1, Lemma 2.2 and Theorems 2.7-2.8].

Lemma 2.2. *If $R_0 > 1$, then the endemic equilibrium of system* (1.1) *uniquely exists and is locally asymptotically stable. Moreover, if the condition* (1.11) *holds, then system* (1.1) *is permanent.*

3 Monotone iterative techniques to the reduced model

In this section, we restrict our attention to obtain the improved result for the global stability of the endemic equilibrium of system (1.4) for the case $R_0 > 1$ in Theorem 1.1, because the global stability for $R_0 \leq 1$ and the permanence for $R_0 > 1$ has been already completed by Beretta and Breda [1]. By Lemma 2.1, we have

$$\bar{S} := \limsup_{t\to+\infty} S(t) \leq \frac{\Lambda}{\mu_1}, \quad \bar{I} := \limsup_{t\to+\infty} I(t) \leq \frac{\Lambda}{\mu_1}, \tag{3.1}$$

and by Lemma 2.2, under the condition (1.11), there exist some positive constants v_1 and v_2 such that

$$\underline{S} := \liminf_{t\to+\infty} S(t) \geq v_1, \quad \underline{I} := \liminf_{t\to+\infty} I(t) \geq v_2. \tag{3.2}$$

Then, similar to Muroya *et al.* [3, Lemma 4.2], from the first equation of (1.4), we have the following lemma.

Lemma 3.1.

$$\begin{cases} 0 \leq \Lambda - \mu_1\bar{S} - \bar{S}g(\underline{I}), \\ 0 \geq \Lambda - \mu_1\underline{S} - \underline{S}g(\bar{I}). \end{cases} \tag{3.3}$$

Proof. Assume that $S(t)$ is eventually monotone decreasing for $t \geq 0$. Then, by Lemma 2.2, there exists $\lim_{t\to+\infty} S(t) = \bar{S} = \underline{S} = S^* > 0$. Then, by the first equation of (1.4), we obtain that

$$0 = \Lambda - \mu_1 S^* - \{\lim_{t\to+\infty} g(I(t))\}S^*,$$

from which by the continuity and strong monotonicity of $g(I)$ with respect to I, there exists $\lim_{t\to+\infty} I(t) = \bar{I} = \underline{I} = I^* > 0$ such that

$$g(I^*)S^* = \Lambda - \mu_1 S^*,$$

and from the second equation of (1.4), we obtain that (S^*, I^*) satisfies the equation (1.9). Since the positive equilibrium $\mathbf{E}_+ = (S_+, I_+)$ is unique, we have that $S^* = S_+ = \bar{S} = \underline{S}$ and $I^* = I_+ = \bar{I} = \underline{I}$. Thus, by (1.9), the inequality (3.3) holds.

We now suppose that $S(t)$ is not eventually monotone decreasing for $t \geq 0$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} S'(t_n) \geq 0$ and $\lim_{n\to+\infty} S(t_n) = \bar{S}$. Therefore, we immediately derive the first equation of (3.3). Similarly, we obtain the second equation of (3.3). This completes the proof. $\square$

Then, we obtain that

$$\bar{S} \leq \frac{\Lambda}{\mu_1 + g(\underline{I})}, \ \underline{S} \geq \frac{\Lambda}{\mu_1 + g(\bar{I})}. \tag{3.4}$$

Now, we construct four sequences $\{\bar{S}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$, $\{\bar{I}_n\}_{n=1}^{\infty}$ and $\{\underline{I}_n\}_{n=1}^{\infty}$ for $R_0 > 1$ as follows. First, $\{\bar{S}_n\}_{n=1}^{\infty}$ of upper bound of $\limsup_{t\to+\infty} S(t)$ and a strictly monotone increasing sequence $\{\underline{S}_n\}_{n=1}^{\infty}$ of lower bound of $\liminf_{t\to+\infty} S(t)$, we use the following sequences constructed by (3.4) which are derived by similar monotone techniques in Muroya *et al.* [3].

$$\bar{S}_n = \frac{\Lambda}{\mu_1 + g(\underline{I}_{n-1})}, \ \underline{S}_n = \frac{\Lambda}{\mu_1 + g(\bar{I}_n)}. \tag{3.5}$$

However, in order to construct a strictly monotone decreasing sequence $\{\bar{I}_n\}_{n=1}^{\infty}$ of upper bound of $\limsup_{t\to+\infty} I(t)$ and a strictly monotone increasing sequence $\{\underline{I}_n\}_{n=1}^{\infty}$ of lower bound of $\liminf_{t\to+\infty} I(t)$, we use the following one on the proof of permanence in Beretta and Breda [1].

$$\bar{I}_n = \frac{1}{\alpha}\left(R_0 \frac{\bar{S}_n}{\bar{S}_1} - 1\right), \ \underline{I}_n = \frac{1}{\alpha}\left(R_0 \frac{\underline{S}_n}{\bar{S}_1} - 1\right). \tag{3.6}$$

As a result, we consider the following four sequences:

$$\begin{cases} \bar{S}_n = \dfrac{\Lambda}{\mu_1 + g(\underline{I}_{n-1})}, & \bar{I}_n = \dfrac{1}{\alpha}\left(R_0 \dfrac{\bar{S}_n}{\bar{S}_1} - 1\right), \\ \underline{S}_n = \dfrac{\Lambda}{\mu_1 + g(\bar{I}_n)}, \quad \text{and} & \underline{I}_n = \dfrac{1}{\alpha}\left(R_0 \dfrac{\underline{S}_n}{\bar{S}_1} - 1\right), \quad n = 1, 2, 3, \cdots \end{cases} \tag{3.7}$$

with

$$\underline{I}_0 = 0. \tag{3.8}$$

Lemma 3.2. *Let $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\underline{S}_n\}_{n=1}^{\infty}$ be the sequences defined by* (3.7) *with* (3.8)*. Then,*

$$\begin{cases} \bar{S}_1 = \dfrac{\Lambda}{\mu_1}, \quad \bar{I}_1 = \dfrac{1}{\alpha}(R_0 - 1), \quad \underline{S}_1 = \dfrac{\Lambda}{\mu_1 + \frac{\beta}{\alpha}(1 - \frac{1}{R_0})}, \quad \text{and} \\ \underline{I}_1 = \dfrac{1}{\alpha}\left\{\dfrac{R_0}{1 + \frac{\beta}{\alpha\mu_1}(1 - \frac{1}{R_0})} - 1\right\} = \dfrac{\mu_1(R_0 - 1)(R_0 - \frac{\beta}{\alpha\mu_1})}{\alpha\mu_1 R_0 + \beta(R_0 - 1)}. \end{cases} \tag{3.9}$$

Moreover, if $R_0 > 1$ and (1.11) *hold, if and only if,*

$$\underline{I}_1 = \frac{\mu_1(R_0 - 1)(R_0 - \frac{\beta}{\alpha\mu_1})}{\alpha\mu_1 R_0 + \beta(R_0 - 1)} > 0 = \underline{I}_0, \tag{3.10}$$

and there exist four positive constants such that the two sequences $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\bar{I}_n\}_{n=1}^{\infty}$ are strongly monotone decreasing sequences and converge to S_+ and I_+, respectively, and the two sequences $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\underline{I}_n\}_{n=1}^{\infty}$ are strongly monotone increasing sequences and converge to S_+ and I_+, respectively, as n tends to $+\infty$.

Proof. By (3.7) with (3.8), we obtain (3.9) except the last equation. Since

$$R_0 - \left\{1 + \frac{\beta}{\alpha\mu_1}\left(1 - \frac{1}{R_0}\right)\right\} = \frac{1}{R_0}(R_0 - 1)\left(R_0 - \frac{\beta}{\alpha\mu_1}\right) > 0,$$

we have that

$$\underline{I}_1 = \frac{1}{\alpha}\left\{\frac{R_0}{1 + \frac{\beta}{\alpha\mu_1}(1 - \frac{1}{R_0})} - 1\right\} = \frac{\mu_1(R_0 - 1)(R_0 - \frac{\beta}{\alpha\mu_1})}{\alpha\mu_1 R_0 + \beta(R_0 - 1)},$$

and we obtain the last equation of (3.9). Moreover, by the second equation of (3.6), $\underline{I}_1 > 0$ is equivalent to $R_0 > \bar{S}/\underline{S} \geq 1$. Thus, $R_0 > 1$ and $R_0 > \frac{\beta}{\alpha\mu_1}$, if and only if, $\underline{I}_1 > 0 = \underline{I}_0$.

Next, if $\underline{I}_1 > 0 = \underline{I}_0$, then we will prove the remaining part of this lemma. At first, we claim that for the sequences of (3.7), it holds that

$$\begin{cases} \underline{I}_0 = 0 < \underline{I}_1 < \underline{I}_2 < \cdots < \underline{I}_n < \cdots < \bar{I}_n < \cdots < \bar{I}_2 < \bar{I}_1, \quad \text{and} \\ \underline{S}_1 < \underline{S}_2 < \cdots < \underline{S}_n < \cdots < \bar{S}_n < \cdots < \bar{S}_2 < \bar{S}_1. \end{cases}$$

The proof will be given by induction. Since $g(I)$ is increasing in I, by the iteration (3.7), one can see that

$$\begin{cases} \underline{I}_0 = 0 < \underline{I}_1 < \underline{I}_2 < \bar{I}_2 < \bar{I}_1, \quad \text{and} \\ \underline{S}_1 < \underline{S}_2 < \bar{S}_2 < \bar{S}_1. \end{cases}$$

We assume the following inequalities hold for some positive integer $n \geq 2$.

$$\begin{cases} \underline{I}_0 = 0 < \underline{I}_1 < \underline{I}_2 < \cdots < \underline{I}_{n-1} < \bar{I}_{n-1} < \cdots < \bar{I}_2 < \bar{I}_1, \quad \text{and} \\ \underline{S}_1 < \underline{S}_2 < \cdots < \underline{S}_{n-1} < \bar{S}_{n-1} < \cdots < \bar{S}_2 < \bar{S}_1. \end{cases} \tag{3.11}$$

We need to prove that (3.11) implies

$$\begin{cases} \underline{I}_0 = 0 < \underline{I}_1 < \underline{I}_2 < \cdots < \underline{I}_n < \bar{I}_n < \cdots < \bar{I}_2 < \bar{I}_1, \quad \text{and} \\ \underline{S}_1 < \underline{S}_2 < \cdots < \underline{S}_n < \bar{S}_n < \cdots < \bar{S}_2 < \bar{S}_1. \end{cases} \tag{3.12}$$

In fact, since $g(I)$ is monotone increasing with respect to I, by (3.7), we know that

$$\bar{S}_n = \frac{\Lambda}{\mu_1 + g(\underline{I}_{n-1})} < \frac{\Lambda}{\mu_1 + g(\underline{I}_{n-2})} = \bar{S}_{n-1}.$$

From the second equation of (3.7), it is obvious that

$$\bar{I}_n = \frac{1}{\alpha}\left(R_0\frac{\bar{S}_n}{\bar{S}_1} - 1\right) < \frac{1}{\alpha}\left(R_0\frac{\bar{S}_{n-1}}{\bar{S}_1} - 1\right) = \bar{I}_{n-1}.$$

From the third and fourth equations of (3.7), we similarly have

$$\underline{S}_n = \frac{\Lambda}{\mu_1 + g(\bar{I}_n)} > \frac{\Lambda}{\mu_1 + g(\bar{I}_{n-1})} = \underline{S}_{n-1}$$

and

$$\underline{I}_n = \frac{1}{\alpha}\left(R_0\frac{\underline{S}_n}{\bar{S}_1} - 1\right) > \frac{1}{\alpha}\left(R_0\frac{\underline{S}_{n-1}}{\bar{S}_1} - 1\right) = \underline{I}_{n-1}.$$

Therefore, the proof of our claim is complete.

Hence, there exist four positive constants $\underline{I} \leq \bar{I}$ and $\underline{S} \leq \bar{S}$ such that

$$\underline{I} = \lim_{n\to+\infty} \underline{I}_n \leq \lim_{n\to+\infty} \bar{I}_n = \bar{I} \quad \underline{S} = \lim_{n\to+\infty} \underline{S}_n \leq \lim_{n\to+\infty} \bar{S}_n = \bar{S}$$

and

$$\begin{cases} \bar{S} = \dfrac{\Lambda}{\mu_1 + g(\underline{I})}, \quad \bar{I} = \dfrac{1}{\alpha}\left(R_0\dfrac{\bar{S}}{\bar{S}_1} - 1\right), \\ \underline{S} = \dfrac{\Lambda}{\mu_1 + g(\bar{I})}, \quad \text{and} \quad \underline{I} = \dfrac{1}{\alpha}\left(R_0\dfrac{\underline{S}}{\bar{S}_1} - 1\right), \quad n = 1, 2, 3, \cdots. \end{cases} \tag{3.13}$$

Then, we have that

$$\begin{cases} \bar{I} - \underline{I} = \dfrac{R_0}{\alpha\bar{S}_1}(\bar{S} - \underline{S}) = \dfrac{\beta}{\alpha\mu_2}e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2})(\bar{S} - \underline{S}), \\ \bar{S} - \underline{S} = \dfrac{\Lambda}{\mu_1 + g(\underline{I})} - \dfrac{\Lambda}{\mu_1 + g(\bar{I})} = \dfrac{\Lambda}{\{\mu_1 + g(\underline{I})\}\{\mu_1 + g(\bar{I})\}}\{g(\bar{I}) - g(\underline{I})\} \\ \qquad = \dfrac{\Lambda\beta}{\{\mu_1 + g(\underline{I})\}\{\mu_1 + g(\bar{I})\}(1 + \alpha\bar{I})(1 + \alpha\underline{I})}(\bar{I} - \underline{I}), \end{cases}$$

from which we obtain

$$\begin{aligned} \bar{I} - \underline{I} &= \frac{\beta^2\Lambda e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2})}{\alpha\mu_2\{\mu_1 + g(\underline{I})\}\{\mu_1 + g(\bar{I})\}(1 + \alpha\bar{I})(1 + \alpha\underline{I})}(\bar{I} - \underline{I}) \\ &= \frac{\beta}{\alpha\mu_1}\frac{R_0}{\{1 + (\alpha + \beta/\mu_1)\underline{I}\}\{1 + (\alpha + \beta/\mu_1)\bar{I}\}}(\bar{I} - \underline{I}). \end{aligned} \tag{3.14}$$

Now, we will prove that

$$\begin{cases} \dfrac{\beta^2 \Lambda e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2})}{\alpha\mu_2\{\mu_1+g(\underline{I})\}\{\mu_1+g(\bar{I})\}(1+\alpha\bar{I})(1+\alpha\underline{I})} = 1, \\ \text{that is,} \quad R_0 = \dfrac{\alpha\mu_1}{\beta}\{1+(\alpha+\beta/\mu_1)\underline{I}\}\{1+(\alpha+\beta/\mu_1)\bar{I}\} \end{cases} \tag{3.15}$$

is equivalent to

$$R_0 = \frac{\beta}{\alpha\mu_1}, \quad \underline{I} = 0, \quad \text{and} \quad \bar{I} = \frac{1}{\alpha}\left(\frac{\beta}{\alpha\mu_1} - 1\right). \tag{3.16}$$

In fact, (3.15) is equivalent to

$$\frac{\beta\mu_1}{\alpha} R_0 = [\{\mu_1+g(\bar{I})\}(1+\alpha\underline{I})][\{\mu_1+g(\underline{I})\}(1+\alpha\bar{I})]. \tag{3.17}$$

From (3.13), we have that

$$\bar{I} = \frac{1}{\alpha(R_0\frac{\mu_1}{\mu_1+g(\underline{I})} - 1)}, \quad \underline{I} = \frac{1}{\alpha}\left(R_0\frac{\mu_1}{\mu_1+g(\bar{I})} - 1\right),$$

and we obtain that

$$(1+\alpha\bar{I})\{\mu_1+g(\underline{I})\} = \mu_1 R_0 = (1+\alpha\underline{I})\{\mu_1+g(\bar{I})\}.$$

Thus, (3.17) is equivalent to

$$\frac{\beta\mu_1}{\alpha} R_0 = [\{\mu_1+g(\bar{I})\}(1+\alpha\underline{I})][\{\mu_1+g(\underline{I})\}(1+\alpha\bar{I})] = (\mu_1 R_0)^2.$$

Then,

$$R_0 = \frac{\beta}{\alpha\mu_1},$$

and by (3.9), we obtain that

$$\underline{I} = 0, \quad \text{and} \quad \bar{I} = \frac{1}{\alpha}\left(\frac{\beta}{\alpha\mu_1} - 1\right).$$

Hence, $R_0 > 1$ and (1.11) imply

$$\frac{\beta^2 \Lambda e^{-\mu_2\tau_1}(1-e^{-\mu_2\tau_2})}{\alpha\mu_2\{\mu_1+g(\underline{I})\}\{\mu_1+g(\bar{I})\}(1+\alpha\bar{I})(1+\alpha\underline{I})} \neq 1, \tag{3.18}$$

and by (3.14) and the uniqueness of the endemic equilibrium E_+ of (1.4), we obtain that

$$\underline{I} = \bar{I} = I_+ \quad \text{and} \quad \underline{S} = \bar{S} = S_+. \tag{3.19}$$

Hence, the proof is complete. □

Remark 3.1. *For the iteration* (3.7) *with* (3.8)*, if* $R_0 = \frac{\beta}{\alpha\mu_1}$*, then*

$$\underline{I}_n = 0 \quad \textit{and} \quad \bar{I}_n = \frac{1}{\alpha}\left(\frac{\beta}{\alpha\mu_1} - 1\right), \quad n = 1, 2, 3, \cdots. \tag{3.20}$$

Proof of Theorem 1.1. By Lemma 3.2, under the condition (1.11), by the convergence of the monotone iterations (3.9), one can easily see that the endemic equilibrium $\mathbf{E}_+ = (S_+, I_+)$ of (1.4) is uniformly stable and globally attractive. For system (1.1), $\lim_{t\to+\infty} S(t) = S_+$ and $\lim_{t\to+\infty} I(t) = I_+$ yield $\lim_{t\to+\infty} E(t) = (1-e^{-\mu_2\tau_1})g(I_+)S_+/\mu_2$ and $\lim_{t\to+\infty} R(t) = (1-e^{-\mu_2(\tau_1+\tau_2)})g(I_+)S_+/\mu_3$. Moreover, since $\mathbf{E}_+$ of (1.4) is uniformly stable, the endemic equilibrium of (1.1) is also uniformly stable. Hence, the global stability of the endemic equilibrium for the reduced system (1.4) implies the global stability of the endemic equilibrium for the original system (1.1). This completes the proof. □

4 Numerical examples

In this section, by using matlab, at first, for the case $R_0 > \frac{\beta}{\alpha\mu_1} \geq 1$ to illustrate the first 20 steps of the four sequences $\{\underline{S}_n\}_{n=1}^{+\infty}$ and $\{\bar{S}_n\}_{n=1}^{+\infty}$, and $\{\underline{I}_n\}_{n=1}^{+\infty}$ and $\{\bar{I}_n\}_{n=1}^{+\infty}$ of (3.7) in Lemma 3.2, we offer Example 4.1.

Figure 1: Figures of $\{\underline{S}_n\}_{n=1}^{20}$ and $\{\bar{S}_n\}_{n=1}^{20}$, and $\{\underline{I}_n\}_{n=1}^{20}$ and $\{\bar{I}_n\}_{n=1}^{20}$ for $R_0 > \frac{\beta}{\alpha\mu_1} > 1$ show that each sequence converges monotonically to the corresponding limit value of endemic equilibrium $\mathbf{E}_+ = (S_+, I_+)$, respectively.

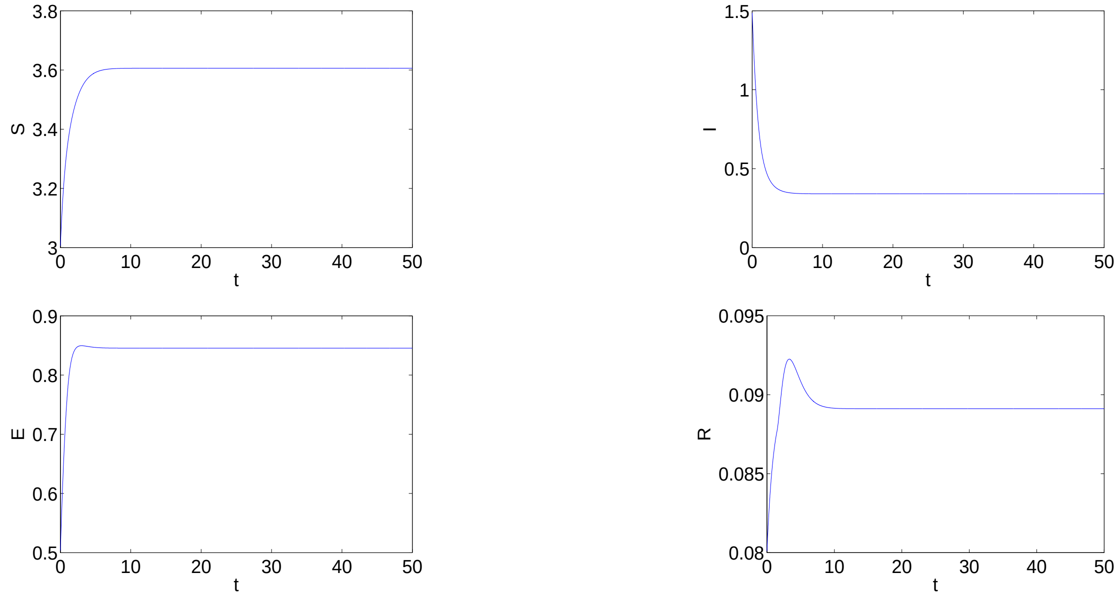

Figure 2: Figures of $S(t), I(t), E(t)$ and $R(t)$ for $R_0 > 1 > \frac{\beta}{\alpha\mu_1}$ show that the endemic equilibrium is globally asymptotically stable.

Example 4.1. *We take*

$$\lambda = 20,\ \ \beta = 10,\ \ \alpha = 1,\ \ \mu_1 = 1.2,\ \ \mu_2 = 2,\ \ \mu_3 = 1.5,\ \ \tau_1 = 1 \ \textit{and}\ \tau_2 = 1.5. \tag{4.1}$$

Then, $R_0 = 10.71644\ldots > \frac{\beta}{\alpha\mu 1} = 8.333333\ldots > 1$ and Figure *1 shows the fact that for the endemic equilibrium $\mathbf{E}_+ = (S_+, I_+)$, the two sequences $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\bar{I}_n\}_{n=1}^{\infty}$ are strongly monotone decreasing sequences and converge to S_+ and I_+, respectively, and the two sequences $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\underline{I}_n\}_{n=1}^{\infty}$ are strongly monotone increasing sequences and converge to S_+ and I_+, respectively, as n tends to $+\infty$.*

Second, we offer Examples 4.2 and 4.3 to illustrate the global stability of endemic equilibrium for the cases $R_0 > 1 > \frac{\beta}{\alpha\mu_1}$ and $R_0 > \frac{\beta}{\alpha\mu_1} \geq 1$, both of which satisfy the condition (1.11) of Theorem 1.1.

Example 4.2. *We take*

$$\lambda = 5,\ \ \beta = 5,\ \ \alpha = 10,\ \ \mu_1 = 1,\ \ \mu_2 = 1.1,\ \ \mu_3 = 1,\ \ \tau_1 = 1,\ \textit{and}\ \tau_2 = 1.5. \tag{4.2}$$

Then, $R_0 = 6.1123\cdots > 1 > \frac{\beta}{\alpha\mu_1} = 0.5$ and Figure *2 shows the fact that the endemic equilibrium of* (1.1) *is globally asymptotically stable by Theorem A.*

From the biological viewpoint, the additional condition (1.12) given by Beretta and Breda [1] seems restrictive in the meaning that the rate of infection force β should be small than $\alpha\mu_1$, where parameters α and μ_1 denote the saturation effect and the death rate of susceptible individuals, respectively. On the contrary, since R_0 is monotone increasing with respect to β, the improvement of our result is that the condition (1.11) without the condition (1.12) enables us to establish the global asymptotic stability of the endemic equilibrium of (1.1), even if the rate of infection force β is high.

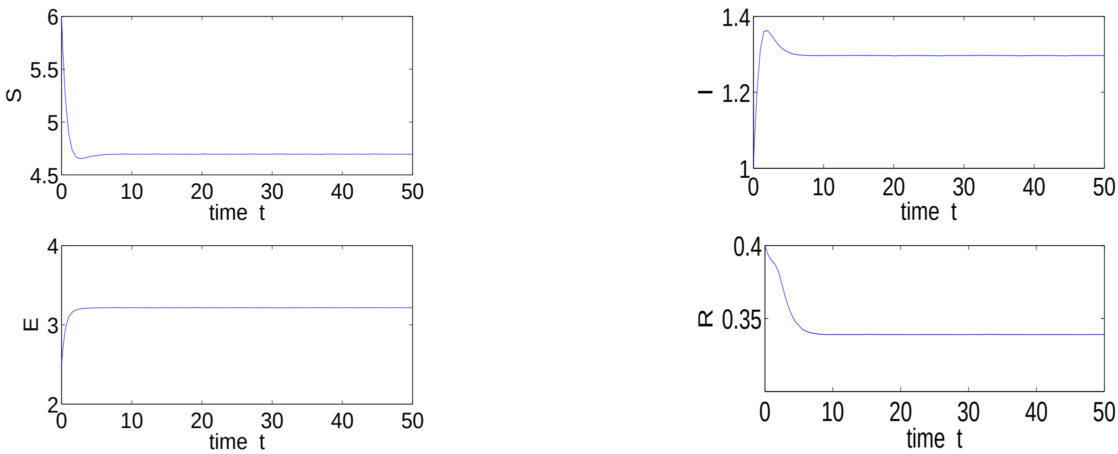

Figure 3: Figures of $S(t), I(t), E(t)$ and $R(t)$ for $R_0 > \frac{\beta}{\alpha\mu_1} > 1$ show that the endemic equilibrium is globally asymptotically stable.

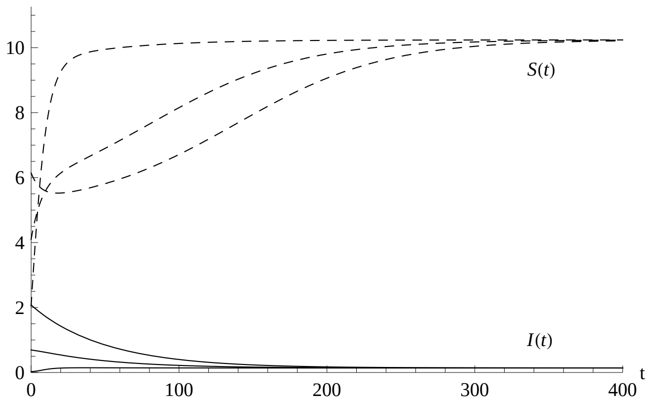

Figure 4: Graph trajectories of $S(t)$ and $I(t)$ for $1 < R_0 < \frac{\beta}{\alpha\mu_1}$ of (4.5). It seems that $\mathbf{E}_+ = (10.2397\cdots, 0.1386\cdots)$ is globally asymptotically stable.

Example 4.3. *We take*

$$\lambda = 10,\ \beta = 2,\ \alpha = 1,\ \mu_1 = 1,\ \mu_2 = 1.1,\ \mu_3 = 1,\ \tau_1 = 1\ \text{and}\ \tau_2 = 1.5. \tag{4.3}$$

Then, $R_0 = 4.889877\cdots > \frac{\beta}{\alpha\mu_1} = 2 > 1$ *and* Figure *3 shows the fact that the endemic equilibrium of (1.1) is globally asymptotically stable by Theorem 1.1.*

Note that by the definition of the basic reproduction number R_0 in (1.8), the condition (1.11) is equivalent to

$$\alpha\Lambda e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2}) > \mu_2.$$

Therefore, there exist a sufficiently small delay $\tau_1 \geq 0$ and a sufficiently large delay $\tau_2 \geq 0$ such that the condition (1.11) is satisfied, if and only if,

$$\alpha\Lambda > \mu_2, \tag{4.4}$$

which does not depend on the rate of infection force β and the death rate of susceptible individuals μ_1.

Third, we investigate numerical examples for the case $1 < R_0 \leq \frac{\beta}{\alpha\mu_1}$ which does not satisfy the condition (1.11).

Example 4.4. *We take*

$$\lambda = 0.8,\ \beta = 0.9,\ \alpha = 6,\ \mu_1 = 0.01,\ \mu_2 = 0.02,\ \tau_1 = 0.2,\ \text{and}\ \tau_2 = 0.2. \tag{4.5}$$

Then, $1 < R_0 = 14.3139\cdots < \frac{\beta}{\alpha\mu_1} = 15$*. However,* Figure *4 seems to indicate that the endemic equilibrium of* (1.1) *is globally asymptotically stable.*

Example 4.4 implies that there remains a question how to analyze the global dynamics of system (1.1) for the case

$$1 < R_0 \leq \frac{\beta}{\alpha\mu_1}. \tag{4.6}$$

Note: Recently, by using Lyapunov function approach for the corresponding age-structured model, the complete global stability of endemic equilibrium of (1.1) have been established by the paper [Huang, Beretta and Takeuchi, Global stavbility for epidemic model with constant latency and infectious periods, *Math. Biosi. Eng.* **9** (2012) 297-312.

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References

[1] E. Beretta and D. Breda, An SEIR epidemic model with constant latency time and infectious period, *Math. Biosci. Eng.* **8** (2011) 931-952.

[2] Y. Muroya, Y. Enatsu and Y. Nakata, Global stability of a delayed SIRS epidemic model with a non-monotonic incidence rate, *J. Math. Anal. Appl.* **377** (2011) 1-14.

[3] Y. Muroya, Y. Enatsu and Y. Nakata, Monotone iterative techniques to SIRS epidemic models with nonlinear incidence rates and distributed delays, *Nonlinear Analysis RWA* **12** (2011) 1897-1910.

[4] R. Xu and Y. Du, A delayed SIR epidemic model with saturation incidence and constant infectious period, *J. Appl. Math. Comput.* **35** (2010) 229-250.

[5] R. Xu and Z. Ma, Stability of a delayed SIRS epidemic model with a nonlinear incidence rate, *Chaos, Solitons and Fractals.* **41** (2009) 2319-2325.

[6] R. Xu and Z. Ma, Global stability of a delayed SEIRS epidemic model with saturation incidence, *Nonlinear Dynam.* **61** (2010) 229-239.