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A discrete-time analogue preserving the global stability of a continuous SIRS epidemic model

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Abstract. In this paper, we consider the backward Euler discretization derived from a continuous SIRS epidemic model, which contains a remaining problem that our discrete model has two solutions for a given initial infected population; one is positive and the other is negative. Under an additional positiveness condition on infected population, we show that the backward Euler discretization is one of simple discrete-time analogue which preserves the global asymptotic stability of equilibria of the corresponding continuous model.

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1 Introduction

In order to investigate the long term behavior of disease transmission in a population, Mena-Lorca and Hethcote [9] considered a continuous-time SIRS (Susceptible-Infected-Recovered-Susceptible) epidemic model. The SIRS epidemic model has been considered to be appropriate when describing the phenomena that susceptible individuals become infectious, then removed with immunity after recovery from infection and then susceptible again when the temporary immunity fades away. Later, various kinds of continuous SIRS epidemic models and a significant body of work have been carried out, for example, [4, 7, 10, 12, 14, 15, 16] and the references cited therein.

First, we consider the following SIRS epidemic model:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - \mu_1 S(t) - \beta S(t)I(t) + \delta R(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t) - (\mu_2 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu_3 + \delta)R(t),\ t > 0. \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infective and recovered individuals at time t, respectively. B is the recruitment rate of the population and γ is the natural recovery rate of the infective individuals. μ_1, μ_2 and μ_3 with $\mu_1 \leq \min\{\mu_2, \mu_3\}$ are the natural death rates of the susceptible, infective and recovered individuals, respectively. β is the disease transmission rate and δ is the rate at which recovered individuals lose immunity and return to the susceptible class.

By constructing elegant Lyapunov functions, Vargas-De-León [12] established a complete analysis of the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium of (1.1).

On the other hand, discrete schemes which preserve the global asymptotic stability of the equilibria of the corresponding continous-time epidemic models has been extensively investigated. For SIR epidemic models, by applying a discrete time analogue of Lyapunov functional techniques in McCluskey [8], Enatsu et al. [3] recently established the complete global stability results for difference equations with a variation of the backward Euler method. For SIRS epidemic models, based on the ideas in Izzo and Vecchio [5], Izzo et al. [6] and Mickens' nonstandard discretization, Sekiguchi [11] proved the global stability of a disease-free equilibrium and permanence for a difference equation derived

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by the corresponding continuous-time model with delays by applying techniques in Wang [13]. However, in those cases, how to choose the discrete schemes which preserve the global asymptotic stability of the endemic equilibrium of the corresponding continous models was an open problem.

In this paper, under an additional positiveness condition on infected population, we show that the backward Euler discretization is one of discrete-time analogues preserving the global asymptotic stability of equilibria of the corresponding continuous model. Let us consider the following difference equations:

$$\begin{cases} S(n+1)-S(n)=B-\mu_1 S(n+1)-\beta S(n+1)I(n+1)+\delta R(n+1), \\ I(n+1)-I(n)=\beta S(n+1)I(n+1)-(\mu_2+\gamma)I(n+1),\ I(n+1)>0, \\ R(n+1)-R(n)=\gamma I(n+1)-(\mu_3+\delta)R(n+1), \quad n=0,1,\dots \end{cases} \tag{1.2}$$

with the initial conditions

$$S(0)=\phi_1(0)>0,\ I(0)=\phi_2(0)>0,\ R(0)=\phi_3(0)>0. \tag{1.3}$$

Remark 1.1. Since the backward Euler discretization is an implicit method, $I(1)$ is given in terms of the following quadratic equation:

$$\begin{aligned} &\beta\{(1+\mu_2+\gamma)(1+\mu_3+\delta)-\delta\gamma\}I(1)^2 \\ &+[\{(1+\mu_1)(1+\mu_2+\gamma)-\beta(B+S(0)+I(0))\}(1+\mu_3+\delta)-\beta\delta R(0)]I(1) \\ &-(1+\mu_1)(1+\mu_3+\delta)I(0)=0. \end{aligned} \tag{1.4}$$

If the condition $I(1)>0$ is not assumed in (1.2), then the equation (1.4) has one positive and one negative root (see also Lemma 2.1 below). This is why the condition $I(n+1)>0$ for $n=0,1,\dots$ is needed. Recently, the similar results for the difference equations which are discrete analogues of a continuous-time SIS epidemic model are also obtained in Enatsu et al. [1].

We define a threshold parameter for the system (1.2) as follows.

$$R_0=\frac{\beta B}{\mu_1(\mu_2+\gamma)}. \tag{1.5}$$

The system (1.2) always has a disease-free equilibrium $E^0=(S^0,0,0)$, $S^0=\frac{B}{\mu_1}$. If $R_0>1$, then the system (1.2) admits a unique endemic equilibrium $E^*=(S^*,I^*,R^*)\in \mathrm{Int}(\mathbb{R}^3_{+0})$, where

$$S^*=\frac{\mu_2+\gamma}{\beta},\ I^*=\frac{S^0(\mu_3+\delta)}{\mu_3+\delta+\gamma}\left(1-\frac{1}{R_0}\right),\ R^*=\frac{S^0\gamma}{\mu_3+\delta+\gamma}\left(1-\frac{1}{R_0}\right). \tag{1.6}$$

By constructing a discrete-time analogue of Lyapunov functions, we obtain the following result:

Theorem 1.1. *If $R_0>1$, then the endemic equilibrium E^* of the system (1.2) is globally asymptotically stable. If $R_0\le 1$, then the disease-free equilibrium E^0 of the system (1.2) is globally asymptotically stable.*

The organization of this paper is as follows. In Section 2, we offer some basic results. In Section 3, we obtain the permanence of the system (1.2) for $R_0>1$. In Section 4, we prove Theorem 1.1. Finally, a conclusion is offered in Section 5.

2 Basic results

In this section, we introduce basic results for the system (1.2).

Lemma 2.1. *Let $(S(n),I(n),R(n))$ be the solution of the system (1.2) with the initial conditions (1.3). Then $S(n)>0$, $I(n)>0$, $R(n)>0$ hold for all $n>0$, and (1.2) is equivalent to the following iteration system:*

$$\begin{aligned} I(n+1)=&\frac{-\tilde{B}_n+\sqrt{\tilde{B}_n^2+4\tilde{A}\tilde{C}_n}}{2\tilde{A}}=\frac{2\tilde{C}_n}{\tilde{B}_n+\sqrt{\tilde{B}_n^2+4\tilde{A}\tilde{C}_n}}, \\ R(n+1)=&\frac{\gamma I(n+1)+R(n)}{1+\mu_3+\delta}, \\ S(n+1)=&\frac{\delta R(n+1)+B+S(n)}{\beta I(n+1)+1+\mu_1}=\frac{\delta\gamma I(n+1)+(1+\mu_3+\delta)(B+S(n))+\delta R(n)}{(1+\mu_3+\delta)(\beta I(n+1)+1+\mu_1)},\ n=0,1,\dots, \end{aligned} \tag{2.1}$$

where

$$\begin{cases} \tilde{A} = \beta\{(1+\mu_2+\gamma)(1+\mu_3+\delta)-\delta\gamma\}, \\ \tilde{B}_n = \{(1+\mu_1)(1+\mu_2+\gamma)-\beta(B+S(n)+I(n))\}(1+\mu_3+\delta)-\beta\delta R(n), \\ \tilde{C}_n = (1+\mu_1)(1+\mu_3+\delta)I(n). \end{cases}$$

Proof. It is evident that the first equation of (1.2) is equivalent to the third equation of (2.1) and the third equation of (1.2) is equivalent to the second equation of (2.1). The second equation of (1.2) is equivalent to

$$\begin{aligned}(1+\mu_2+\gamma)I(n+1)-I(n) =&\beta S(n+1)I(n+1)\\ =&\beta\frac{\{\delta\gamma I(n+1)+(1+\mu_3+\delta)(B+S(n))+\delta R(n)\}I(n+1)}{(1+\mu_3+\delta)(\beta I(n+1)+1+\mu_1)},\end{aligned}$$

which is equivalent to a quadratic equation $P(x)=0$ with $x=I(n+1)$, where

$$\begin{aligned}P(x) =&(1+\mu_2+\gamma)(1+\mu_3+\delta)(\beta x+1+\mu_1)x-\beta\{\delta\gamma x+(1+\mu_3+\delta)(B+S(n))+\delta R(n)\}x-I(n)(1+\mu_3+\delta)(\beta x+1+\mu_1)\\ =&\beta\{(1+\mu_2+\gamma)(1+\mu_3+\delta)-\delta\gamma\}x^2+[\{(1+\mu_1)(1+\mu_2+\gamma)-\beta(B+S(n)+I(n))\}(1+\mu_3+\delta)-\beta\delta R(n)]x\\ &-(1+\mu_1)(1+\mu_3+\delta)I(n)\\ =&Ax^2+\tilde{B}_n x-\tilde{C}_n.\end{aligned}$$

For $S(n)>0$, $I(n)>0$ and $R(n)>0$, it is evident that $I(n+1)$ defined by the first equation of (2.1) is a unique positive solution of $P(x)=0$.

Assume that there exists a nonnegative integer n_0 such that $S(n)>0$, $I(n)>0$ and $R(n)>0$, $n=0,1,\dots,n_0$. By the first equation of (2.1), we have $I(n_0+1)>0$. By the second and the third equations of (2.1), we have $R(n_0+1)>0$ and $S(n_0+1)>0$. Hence, by induction, the solution $(S(n),I(n),R(n))$ is unique satisfying $S(n)>0$, $I(n)>0$ and $R(n)>0$ for all $n>0$. Thus, (1.2) is equivalent to (2.1). This completes the proof of this lemma. □

Lemma 2.2. *For the system* (1.2) *with the initial conditions* (1.3), *it holds that*

$$\limsup_{n\to+\infty} N(n) \le \frac{B}{\mu_1}, \tag{2.2}$$

where $N(n)=S(n)+I(n)+R(n)$.

Proof. From the system (1.2), it follows that

$$\begin{aligned}N(n+1)-N(n) =&B-\mu_1 S(n+1)-\mu_2 I(n+1)-\mu_3 R(n+1)\\ =&B-\mu_1 N(n+1)-(\mu_2-\mu_1)I(n+1)-(\mu_3-\mu_1)R(n+1)\\ \le&B-\mu_1 N(n+1).\end{aligned} \tag{2.3}$$

Suppose that $N(n+1)$ is eventually monotone decreasing for $n\ge0$. Then, there exists $\lim_{n\to+\infty} N(n+1)=N^*>0$ and by letting $n\to+\infty$, we have $0\le B-\mu_1 N^*$, from which we obtain (2.2).

Next, we suppose that $N(n+1)$ is not eventually monotone decreasing for $n\ge0$. Then, there exists a sequence $\{n_l\}_{l=1}^{+\infty}$ such that

$$N(n_l+1)\ge N(n_l), \quad \lim_{l\to+\infty} N(n_l+1)=\limsup_{n\to+\infty} N(n+1).$$

From (2.3) at $n=n_l$, we have

$$0\le N(n_l+1)-N(n_l)\le B-\mu_1 N(n_l+1).$$

By letting $l\to+\infty$, we obtain $0\le B-\mu_1\limsup_{n\to+\infty} N(n+1)$, that is, $\limsup_{n\to+\infty} N(n+1)\le B/\mu_1$. This completes the proof. □

3 Permanence for $R_0>1$

Throughout this section, we assume $R_0>1$. Now we prove the permanence of the system (1.2) for $R_0>1$. Applying techniques in Enatsu et al. [3, Lemma 4.1], we prepare the following basic lemmas:

Lemma 3.1. *If* $I(n+1)<I(n)$, *then* $S(n+1)<S^*$. *Inversely, if* $S(n+1)\ge S^*$, *then* $I(n+1)\ge I(n)$.

Proof. Assume that $I(n+1) < I(n)$. By the second equation of (1.2), we have

$$I(n+1) - I(n) = \beta S(n+1)I(n+1) - (\mu_2+\gamma)I(n+1) = \beta(S(n+1) - S^*)I(n+1). \tag{3.1}$$

Since $I(n+1) > 0$ holds for all $n \geq 0$, we obtain $S(n+1) < S^*$. Inversely, assume that $S(n+1) \geq S^*$. Then, it is evident that $I(n+1) \geq I(n)$ holds true. □

We now offer a simplified proof for permanence of the system (1.2) than that of Wang [13] (see also Xu and Ma [14]).

Lemma 3.2. *For any solution of the system* (1.2) *with the initial conditions* (1.3)*, it holds that*

$$\begin{cases} \liminf_{n\to+\infty} S(n) \geq v_1 := \dfrac{B}{\mu_1 + \beta(B/\mu_1)} > 0, \\ \liminf_{n\to+\infty} I(n) \geq v_2 := \left(\dfrac{1}{1+\mu_2+\gamma}\right)^{l_0(q)} qI^* > 0, \\ \liminf_{n\to+\infty} R(n) \geq v_3 := \dfrac{\gamma}{\mu_3+\delta} v_2 > 0, \end{cases} \tag{3.2}$$

where $0 < q < \frac{B\beta I^* - \mu_1\delta R^*}{\beta I^*(B+\delta R^*)}$ *and* $l_0(q) > 0$ *is a sufficiently large integer such that*

$$S^* < S^{\triangle} := \frac{B}{r_q}\left\{1 - \left(\frac{1}{1+r_q}\right)^{l_0(q)}\right\}, \quad r_q = \mu_1 + \beta qI^*. \tag{3.3}$$

Proof. Let $(S(t), I(t), R(t))$ be a solution of the system (1.2) with the initial conditions (1.3). By Lemma 2.1, we have $\limsup_{n\to+\infty} I(n) \leq \frac{B}{\mu_1}$ and $S^* < \frac{B}{\mu_1}$. For $\varepsilon > 0$ sufficiently small, there exists a $N_1 = N_1(\varepsilon) > 0$ such that $I(n) < \frac{B}{\mu_1} + \varepsilon$ for $n > N_1$. Then, by the first equation of (1.2), we derive

$$S(n+1) - S(n) > B - \left\{\mu_1 + \beta\left(\frac{B}{\mu_1} + \varepsilon\right)\right\}S(n+1),$$

which yields

$$\liminf_{n\to+\infty} S(n) > \frac{B}{\mu_1 + \beta(B/\mu_1 + \varepsilon)}. \tag{3.4}$$

Since (3.4) holds for arbitrary $\varepsilon > 0$, we obtain $\liminf_{n\to+\infty} S(n) \geq v_1$. By the following relation:

$$\begin{aligned} B\beta I^* - \mu_1\delta R^* &= B\beta I^* - \mu_1\delta\frac{\gamma}{\mu_3+\delta}I^* \\ &= \mu_1(\mu_2+\gamma)\left(R_0 - \frac{\gamma}{\mu_2+\gamma}\frac{\delta}{\mu_3+\delta}\right)I^* \\ &> \mu_1(\mu_2+\gamma)(R_0 - 1)I^* > 0, \end{aligned}$$

we have

$$S^* = \frac{B + \delta R^*}{\mu_1 + \beta I^*} = \frac{B}{\frac{B(\mu_1+\beta I^*)}{B+\delta R^*}} = \frac{B}{\mu_1 + \frac{B\beta I^* - \mu_1\delta R^*}{B+\delta R^*}} < \frac{B}{\mu_1 + \beta qI^*}$$

for any $0 < q < \frac{B\beta I^* - \mu_1\delta R^*}{\beta I^*(B+\delta R^*)}$. Thus, there exists a positive integer $l_0(q)$ such that (3.3) holds.

We first prove the claim that it is not possible that for any solution of (1.2), there exists a nonnegative constant n_0 such that $I(n+1) \leq qI^*$ for all $n \geq n_0$. Suppose on the contrary that there exist a solution of (1.2) and a nonnegative integer n_0 such that $I(n+1) \leq qI^*$ for all $n \geq n_0$. We then obtain

$$S(n+1) - S(n) \geq B - (\mu_1 + \beta qI^*)S(n+1) = B - r_q S(n+1), \quad \text{for } n \geq n_0,$$

which yields

$$\begin{aligned} S(n+1) \quad &\geq \quad \left(\frac{1}{1+r_q}\right)^{n+1-(n_0+1)} S(n_0+1) + \frac{B}{1+r_q}\sum_{l=0}^{n-n_0}\left(\frac{1}{1+r_q}\right)^l \\ &\geq \quad \frac{B}{r_q}\left\{1 - \left(\frac{1}{1+r_q}\right)^{n+1-(n_0+1)}\right\}, \quad \text{for any } n \geq n_0. \end{aligned}$$

Therefore, we have

$$S(n+1) \geq \frac{B}{r_q}\left\{1 - \left(\frac{1}{1+r_q}\right)^{l_0(q)}\right\} = S^{\triangle} > S^*, \quad \text{for any } n \geq n_0 + l_0(q). \tag{3.5}$$

By the second part of Lemma 3.1, we obtain $I(n+1)-I(n)\geq 0$ for any $n\geq n_0+l_0(q)$. This yields

$$I(n+1)\geq I(n_0+l_0(q)) \quad \text{for any } n\geq n_0+l_0(q). \tag{3.6}$$

We then have

$$\begin{aligned} I(n+1)-I(n) &= \beta S(n+1)I(n+1)-(\mu_2+\gamma)I(n+1)\\ &= \{\beta S(n+1)-(\mu_2+\gamma)\}I(n+1)\\ &> \{\beta S^{\triangle}-(\mu_2+\gamma)\}I(n+1)\\ &> \beta(S^{\triangle}-S^*)I(n_0+l_0(q))>0, \text{ for any } n\geq n_0+l_0(q), \end{aligned}$$

which implies that $\lim_{n\to+\infty} I(n)=+\infty$. However, by Lemma 2.1, it holds that $\limsup_{n\to+\infty} I(n)\leq \frac{B}{\mu_1}$, which leads to a contradiction. Hence, the claim is proved.

By the claim, we are left to consider the following two possibilities;

$$\begin{cases} \text{(i) } I(n)\geq qI^* \text{ for all } n \text{ sufficiently large,}\\ \text{(ii) } I(n) \text{ oscillates about } qI^* \text{ for all } n \text{ sufficiently large.} \end{cases}$$

If the first case holds, then we get the conclusion of the proof. We now show that if the second case holds, then $I(n)\geq v_2$ for all n sufficiently large. Let $n_3<n_4$ be sufficiently large such that

$$I(n_3-1),\ I(n_4+1)>qI^*, \quad \text{and} \quad I(n)\leq qI^* \quad \text{for any } n_3\leq n\leq n_4.$$

By the second equation of (1.2), we have $I(n+1)-I(n)\geq -(\mu_2+\gamma)I(n+1)$, that is, $I(n+1)\geq \frac{1}{1+\mu_2+\gamma}I(n)$ for any $n\geq n_3$. It follows that

$$I(n+1)\geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{n+1-n_3} I(n_3)\geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{n+1-n_3} qI^*$$

for any $n\geq n_3$. Therefore, we obtain

$$I(n+1)\geq \left(\frac{1}{1+\mu_2+\gamma}\right)^{l_0(q)} qI^*=v_2, \text{ for any } n_3\leq n\leq n_3+l_0(q). \tag{3.7}$$

If $n_4\geq n_3+l_0(q)$, then by applying the similar discussion to (3.5) and (3.6) in place of n_0 by n_3, we obtain $I(n)\geq v_2$ for $n_3+l_0(q)\leq n\leq n_4$. Hence, $I(n)\geq v_2$ holds for $n_3\leq n\leq n_4$. Since the interval $n_4\leq n\leq n_5$ can be arbitrarily chosen, $I(n)\geq v_2$ holds for all n sufficiently large. Thus, we have $\liminf_{t\to+\infty} I(t)\geq v_2$, which yields $\liminf_{t\to+\infty} R(t)\geq v_3$. The proof is complete. □

By Lemmas 2.1 and 3.2, we obtain the permanence of system (1.2) for $R_0>1$.

4 Global stability

In this section, we prove Theorem 1.1.

4.1 Global stability of the endemic equilibrium E^* for $R_0>1$

In this subsection, we prove the first part of Theorem 1.1. Put $N^*=S^*+I^*+R^*$ and

$$x_n=\frac{S(n)}{S^*},\ y_n=\frac{I(n)}{I^*},\ z_n=\frac{R(n)}{R^*},\ w_n=\frac{N(n)}{N^*}. \tag{4.1}$$

Proof of the first part of Theorem 1.1. We consider the following sequence:

$$U_\delta^{E^*}(n)=\begin{cases} \frac{(S(n)-S^*)^2}{2S^*}+I^*g\left(\frac{I(n)}{I^*}\right)+\frac{\delta}{\gamma S^*}\frac{(R(n)-R^*)^2}{2}\\ +\frac{\delta\gamma}{\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^*}\frac{\{(N(n)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n)-R^*)\}^2}{2},\\ \text{if either} \quad \mu_1<\mu_2 \quad \text{or} \quad \mu_1<\mu_3,\\ \frac{(S(n)-S^*)^2}{2S^*}+I^*g\left(\frac{I(n)}{I^*}\right)+\frac{\delta}{\gamma S^*}\frac{(R(n)-R^*)^2}{2}+\frac{\delta}{4\mu_1 S^*}\frac{(N(n)-N^*)^2}{2},\\ \text{if} \quad \mu_1=\mu_2=\mu_3, \end{cases} \tag{4.2}$$

where $\alpha_{21} = \mu_2 - \mu_1$, $\alpha_{31} = \mu_3 - \mu_1$ and $g(x) = x - 1 - \ln x \geq g(1) = 0$. First, we calculate $\frac{(S(n+1)-S^*)^2}{2S^*} - \frac{(S(n)-S^*)^2}{2S^*}$.

$$\begin{aligned}
&\frac{(S(n+1)-S^*)^2}{2S^*} - \frac{(S(n)-S^*)^2}{2S^*}\\
=&\frac{1}{2S^*}(S(n+1)+S(n)-2S^*)(S(n+1)-S(n))\\
=&\left(\frac{S(n+1)}{S^*}-1\right)(S(n+1)-S(n)) - \frac{1}{2S^*}(S(n+1)-S(n))^2\\
\leq&\left(\frac{S(n+1)}{S^*}-1\right)(S(n+1)-S(n))\\
=&\left(\frac{S(n+1)}{S^*}-1\right)\{B-\mu_1 S(n+1) - \beta S(n+1)I(n+1) + \delta R(n+1)\}\\
=&\left(\frac{S(n+1)}{S^*}-1\right)\{-\mu_1(S(n+1)-S^*) + \beta(S^*I^* - S(n+1)I(n+1)) + \delta(R(n+1)-R^*)\}\\
=&(x_{n+1}-1)\{-\mu_1 S^*(x_{n+1}-1) + \beta S^* I^*(1-x_{n+1}y_{n+1}) + \delta R^*(z_{n+1}-1)\}\\
=&-\mu_1 S^*(x_{n+1}-1)^2 + \beta S^* I^*(x_{n+1}-1)(1-x_{n+1}y_{n+1}) + \delta R^*(x_{n+1}-1)(z_{n+1}-1).
\end{aligned} \tag{4.3}$$

Second, we calculate $I^*\{g(\frac{I(n+1)}{I^*}) - g(\frac{I(n)}{I^*})\}$. By applying the following inequality (cf. Enatsu *et al.* [3]):

$$-\ln\frac{I(n+1)}{I(n)} = \ln\left\{1-\left(1-\frac{I(n)}{I(n+1)}\right)\right\} \leq -\left(1-\frac{I(n)}{I(n+1)}\right) = -\frac{I(n+1)-I(n)}{I(n+1)},$$

we obtain

$$\begin{aligned}
I^*\left\{g\left(\frac{I(n+1)}{I^*}\right) - g\left(\frac{I(n)}{I^*}\right)\right\} =&I^*\left(\frac{I(n+1)-I(n)}{I^*} - \ln\frac{I(n+1)}{I(n)}\right)\\
\leq&I^*\left(\frac{I(n+1)-I(n)}{I^*} - \frac{I(n+1)-I(n)}{I(n+1)}\right)\\
=&\left(1-\frac{I^*}{I(n+1)}\right)(I(n+1)-I(n))\\
=&\left(1-\frac{I^*}{I(n+1)}\right)(\beta S(n+1)I(n+1) - (\mu_2+\gamma)I(n+1))\\
=&\left(1-\frac{I^*}{I(n+1)}\right)(\beta S(n+1)I(n+1) - \beta S^* I(n+1))\\
=&\beta S^* I^*\left(1-\frac{1}{y_{n+1}}\right)(x_{n+1}y_{n+1} - y_{n+1})\\
=&\beta S^* I^*(y_{n+1}-1)(x_{n+1}-1).
\end{aligned} \tag{4.4}$$

Second, we have

$$\begin{aligned}
\frac{(R(n+1)-R^*)^2}{2} - \frac{(R(n)-R^*)^2}{2} =&\frac{1}{2}(R(n+1)+R(n)-2R^*)(R(n+1)-R(n))\\
=&(R(n+1)-R^*)(R(n+1)-R(n)) - \frac{1}{2}(R(n+1)-R(n))^2\\
\leq&(R(n+1)-R^*)(R(n+1)-R(n)).
\end{aligned}$$

By the third equation of system (1.2), we get

$$\frac{(R(n+1)-R^*)^2}{2} - \frac{(R(n)-R^*)^2}{2} \leq (R(n+1)-R^*)\{\gamma I(n+1) - (\mu_3+\delta)R(n+1)\}.$$

Moreover, by $I(n+1) = N(n+1) - S(n+1) - R(n+1)$, we obtain

$$\begin{aligned}
&\frac{(R(n+1)-R^*)^2}{2} - \frac{(R(n)-R^*)^2}{2}\\
\leq&(R(n+1)-R^*)\{\gamma I(n+1) - (\mu_3+\delta)R(n+1)\}\\
=&(R(n+1)-R^*)\{\gamma(N(n+1)-S(n+1)-R(n+1)) - (\mu_3+\delta)R(n+1)\}\\
=&(R(n+1)-R^*)\{\gamma(N(n+1)-N^*) - \gamma(S(n+1)-S^*) - (\mu_3+\gamma+\delta)(R(n+1)-R^*)\}\\
=&\gamma R^* N^*(z_{n+1}-1)(w_{n+1}-1) - \gamma R^* S^*(z_{n+1}-1)(x_{n+1}-1) - (\mu_3+\gamma+\delta)(R^*)^2(z_{n+1}-1)^2.
\end{aligned} \tag{4.5}$$

For the first case either $\mu_1 < \mu_2$ or $\mu_1 < \mu_3$, we have

$$\begin{aligned}
&\frac{\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\}^2}{2}-\frac{\{(N(n)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n)-R^*)\}^2}{2}\\
=\ &\frac{1}{2}\left\{(N(n+1)+N(n)-2N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)+R(n)-2R^*)\right\}\\
&\times\left\{(N(n+1)-N(n))+\frac{\alpha_{21}}{\gamma}(R(n+1)-R(n))\right\}.
\end{aligned}$$

By $N(n+1)+N(n)-2N^* = 2(N(n+1)-N^*)-(N(n+1)-N(n))$ and $R(n+1)+R(n)-2R^* = 2(R(n+1)-R^*)-(R(n+1)-R(n))$, we have

$$\begin{aligned}
&\frac{1}{2}\left\{(N(n+1)+N(n)-2N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)+R(n)-2R^*)\right\}\left\{(N(n+1)-N(n))+\frac{\alpha_{21}}{\gamma}(R(n+1)-R(n))\right\}\\
=&\left\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\right\}\left\{(N(n+1)-N(n))+\frac{\alpha_{21}}{\gamma}(R(n+1)-R(n))\right\}\\
&-\frac{1}{2}\left\{(N(n+1)-N(n))+\frac{\alpha_{21}}{\gamma}(R(n+1)-R(n))\right\}^2\\
\le&\left\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\right\}\left\{(N(n+1)-N(n))+\frac{\alpha_{21}}{\gamma}(R(n+1)-R(n))\right\}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\frac{\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\}^2}{2}-\frac{\{(N(n)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n)-R^*)\}^2}{2}\\
\le&\left\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\right\}\\
&\times\left\{B-\mu_1N(n+1)-\alpha_{21}I(n+1)-\alpha_{31}R(n+1)+\alpha_{21}I(n+1)-\frac{\alpha_{21}(\mu_3+\delta)}{\gamma}R(n+1)\right\}\\
=&\left\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\right\}\left[B-\mu_1N(n+1)-\left\{\alpha_{31}+\frac{\alpha_{21}(\mu_3+\delta)}{\gamma}\right\}R(n+1)\right]\\
=&\left\{(N(n+1)-N^*)+\frac{\alpha_{21}}{\gamma}(R(n+1)-R^*)\right\}\left[-\mu_1(N(n+1)-N^*)-\left\{\alpha_{31}+\frac{\alpha_{21}(\mu_3+\delta)}{\gamma}\right\}(R(n+1)-R^*)\right]\\
=&-\mu_1(N^*)^2(w_{n+1}-1)^2-\left\{\alpha_{31}+\frac{\alpha_{21}(\mu_1+\mu_3+\delta)}{\gamma}\right\}N^*R^*(w_{n+1}-1)(z_{n+1}-1)\\
&-\frac{\alpha_{21}}{\gamma}\left\{\alpha_{31}+\frac{\alpha_{21}(\mu_3+\delta)}{\gamma}\right\}(R^*)^2(z_{n+1}-1)^2. \qquad (4.6)
\end{aligned}$$

We note that the following relation holds.

$$\begin{aligned}
(x_{n+1}-1)(1-x_{n+1}y_{n+1})+(y_{n+1}-1)(x_{n+1}-1) =&-(x_{n+1}-1)(x_{n+1}y_{n+1}-y_{n+1})\\
=&-y_{n+1}(x_{n+1}-1)^2\\
\le&0, \qquad (4.7)
\end{aligned}$$

Combining (4.3), (4.4), (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned}
U_\delta^{E^*}(n+1)-U_\delta^{E^*}(n)\le&-\mu_1S^*(x_{n+1}-1)^2-\beta S^*I^*y_{n+1}(x_{n+1}-1)^2+\gamma S^*R^*(x_{n+1}-1)(z_{n+1}-1)\\
&+\frac{\delta R^*N^*}{S^*}(z_{n+1}-1)(w_{n+1}-1)-\gamma R^*S^*(z_{n+1}-1)(x_{n+1}-1)\\
&-\frac{\delta(\mu_3+\gamma+\delta)(R^*)^2}{\gamma S^*}(z_{n+1}-1)^2-\frac{\mu_1\delta\gamma(N^*)^2}{\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^*}(w_{n+1}-1)^2\\
&-\frac{\delta N^*R^*}{S^*}(w_{n+1}-1)(z_{n+1}-1)-\frac{\delta\alpha_{21}\{\gamma\alpha_{31}+\alpha_{21}(\mu_3+\delta)\}(R^*)^2}{\gamma\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^*}(z_{n+1}-1)^2\\
=&-S^*(\mu_1+\beta I^*y_{n+1})(x_{n+1}-1)^2-\frac{\mu_1\delta\gamma(N^*)^2}{\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^*}(w_{n+1}-1)^2\\
&-\left[\frac{\delta\alpha_{21}\{\gamma\alpha_{31}+\alpha_{21}(\mu_3+\delta)\}(R^*)^2}{\gamma\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^*}+\frac{\delta(\mu_3+\gamma+\delta)(R^*)^2}{\gamma S^*}\right](z_{n+1}-1)^2. \qquad (4.8)
\end{aligned}$$

For the second case $\mu_1 = \mu_2 = \mu_3$, by $N^* = B/\mu_1$, we obtain

$$\begin{aligned}\frac{\delta(N(n+1)-N^*)^2}{4\mu_1 S^*} - \frac{\delta(N(n)-N^*)^2}{4\mu_1 S^*} \leq& \frac{\delta}{4\mu_1 S^*}(N(n+1)-N^*)(B-\mu_1 N(n+1)) \\ =& -\frac{\delta}{4S^*}(N(n+1)-N^*)^2 \\ =& -\frac{\delta(N^*)^2}{4S^*}(w_{n+1}-1)^2. \end{aligned} \tag{4.9}$$

Combining (4.3), (4.4), (4.5), (4.7) and (4.9), we have

$$\begin{aligned} U_\delta^{E^*}(n+1) - U_\delta^{E^*}(n) \leq& -\mu_1 S^* (x_{n+1}-1)^2 - \beta S^* I^* y_{n+1}(x_{n+1}-1)^2 \\ &+ \gamma S^* R^* (x_{n+1}-1)(z_{n+1}-1) + \frac{\delta R^* N^*}{S^*}(z_{n+1}-1)(w_{n+1}-1) \\ &- \gamma R^* S^* (z_{n+1}-1)(x_{n+1}-1) - \frac{\delta(\mu_3+\gamma+\delta)(R^*)^2}{\gamma S^*}(z_{n+1}-1)^2 \\ &- \frac{\delta(N^*)^2}{4S^*}(w_{n+1}-1)^2 \\ =& -\mu_1 S^* (x_{n+1}-1)^2 - \beta S^* I^* y_{n+1}(x_{n+1}-1)^2 \\ &- \frac{\delta(R^*)^2}{S^*}(z_{n+1}-1)^2 + \frac{\delta R^* N^*}{S^*}(z_{n+1}-1)(w_{n+1}-1) - \frac{\delta(N^*)^2}{4S^*}(w_{n+1}-1)^2 \\ &- \frac{\delta(R^*)^2}{\gamma S^*}(\mu_3+\delta)(z_{n+1}-1)^2 \\ =& -(\mu_1 S^* + \beta S^* I^* y_{n+1})(x_{n+1}-1)^2 - \frac{\delta}{S^*}\left\{R^*(z_{n+1}-1) - \frac{N^*}{2}(w_{n+1}-1)\right\}^2 \\ &- \frac{\delta(R^*)^2}{\gamma S^*}(\mu_3+\delta)(z_{n+1}-1)^2. \end{aligned} \tag{4.10}$$

From (4.8) and (4.10), for the both cases, we obtain $U_\delta^{E^*}(n+1) - U_\delta^{E^*}(n) \leq 0$ for all $n > 0$. Therefore, $\lim_{n\to+\infty}(U_\delta^{E^*}(n+1) - U_\delta^{E^*}(n)) = 0$, from which we obtain $\lim_{n\to+\infty} S(n+1) = S^*$, $\lim_{n\to+\infty} R(n+1) = R^*$ and $\lim_{n\to+\infty} N(n+1) = N^*$. This yields $\lim_{n\to+\infty} I(n+1) = I^*$. Since $U_\delta^{E^*}(n) \geq \frac{(S(n)-S^*)^2}{2S^*} + I^* g(\frac{I(n)}{I^*}) + \frac{\delta}{\gamma S^*}\frac{(R(n)-R^*)^2}{2}$ for all $n \geq 0$, E^* is uniformly stable. Hence, E^* is globally asymptotically stable. The proof is complete. □

4.2 Global stability of the disease-free equilibrium E^0 for $R_0 \leq 1$

In this subsection, we prove the second part of Theorem 1.1.

Proof of the second part of Theorem 1.1. We consider the following sequence:

$$U_\delta^{E^0}(n) = \begin{cases} \frac{(S(n)-S^0)^2}{2S^0} + I(n) + \frac{\delta}{\gamma S^0}\frac{R(n)^2}{2} + \frac{\delta\gamma}{\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S^0}\frac{\{(N(n)-N^0)+\frac{\alpha_{21}}{\gamma}R(n)\}^2}{2}, \\ \text{if either} \quad \mu_1 < \mu_2 \quad \text{or} \quad \mu_1 < \mu_3, \\ \frac{(S(n)-S^0)^2}{2S^0} + I(n) + \frac{\delta}{\gamma S^0}\frac{R(n)^2}{2} + \frac{\delta}{4\mu_1 S^0}\frac{\left(N(n)-N^0\right)^2}{2}, \\ \text{if} \quad \mu_1 = \mu_2 = \mu_3, \end{cases}$$

where $N^0 = S^0 = B/\mu_1$. Similar to the proof of the first part of Theorem 1.1, we first calculate $\frac{(S(n+1)-S^0)^2}{2S^0} - \frac{(S(n)-S^0)^2}{2S^0}$.

$$\begin{aligned} \frac{(S(n+1)-S^0)^2}{2S^0} - \frac{(S(n)-S^0)^2}{2S^0} \leq& \left(\frac{S(n+1)}{S^0} - 1\right)(S(n+1) - S(n)) \\ =& \left(\frac{S(n+1)}{S^0} - 1\right)\{B - \mu_1 S(n+1) - \beta S(n+1)I(n+1) + \delta R(n+1)\} \\ =& \left(\frac{S(n+1)}{S^0} - 1\right)\{-\mu_1(S(n+1)-S^0) - \beta S(n+1)I(n+1) + \delta R(n+1)\} \\ =& -\mu_1\frac{(S(n+1)-S^0)^2}{S^0} - \beta S(n+1)I(n+1)\left(\frac{S(n+1)}{S^0} - 1\right) \\ &+ \delta R(n+1)\left(\frac{S(n+1)}{S^0} - 1\right). \end{aligned} \tag{4.11}$$

Second, by $I(n+1) = N(n+1) - S(n+1) - R(n+1)$, we have

$$\begin{aligned}
\frac{R(n+1)^2}{2} - \frac{R(n)^2}{2} \leq & R(n+1)(R(n+1) - R(n)) \\
= & R(n+1)\left\{\gamma I(n+1) - (\mu_3 + \delta)R(n+1)\right\} \\
= & R(n+1)\{\gamma(N(n+1) - S(n+1) - R(n+1)) - (\mu_3 + \delta)R(n+1)\} \\
= & R(n+1)\{\gamma(N(n+1) - N^0) - \gamma(S(n+1) - S^0) - (\mu_3 + \gamma + \delta)R(n+1)\} \\
= & \gamma R(n+1)(N(n+1) - N^0) - \gamma R(n+1)(S(n+1) - S^0) - (\mu_3 + \gamma + \delta)R(n+1)^2. \qquad (4.12)
\end{aligned}$$

For the first case either $\mu_1 < \mu_2$ or $\mu_1 < \mu_3$, by $N(n+1) + N(n) - 2N^0 = 2(N(n+1) - N^0) - (N(n+1) - N(n))$ and $R(n+1) + R(n) = 2R(n+1) - (R(n+1) - R(n))$, we have

$$\begin{aligned}
& \frac{\{(N(n+1) - N^0) + \frac{\alpha_{21}}{\gamma}R(n+1)\}^2}{2} - \frac{\{(N(n) - N^0) + \frac{\alpha_{21}}{\gamma}R(n)\}^2}{2} \\
\leq & \left\{(N(n+1) - N^0) + \frac{\alpha_{21}}{\gamma}R(n+1)\right\} \\
& \times\left\{B - \mu_1 N(n+1) - \alpha_{21}I(n+1) - \alpha_{31}R(n+1) + \alpha_{21}I(n+1) - \frac{\alpha_{21}(\mu_3 + \delta)}{\gamma}R(n+1)\right\} \\
= & \left\{(N(n+1) - N^0) + \frac{\alpha_{21}}{\gamma}R(n+1)\right\}\left[-\mu_1(N(n+1) - N^0) - \left\{\alpha_{31} + \frac{\alpha_{21}(\mu_3 + \delta)}{\gamma}\right\}R(n+1)\right] \\
= & -\mu_1(N(n+1) - N^0)^2 - \left\{\alpha_{31} + \frac{\alpha_{21}(\mu_1 + \mu_3 + \delta)}{\gamma}\right\}(N(n+1) - N^0)R(n+1) \\
& - \frac{\alpha_{21}}{\gamma}\left\{\alpha_{31} + \frac{\alpha_{21}(\mu_3 + \delta)}{\gamma}\right\}R(n+1)^2. \qquad (4.13)
\end{aligned}$$

Combining (4.11), (4.12) and (4.13), we obtain

$$\begin{aligned}
U_\delta^{E^0}(n+1) - U_\delta^{E^0}(n) \leq & -\mu_1\frac{(S(n+1) - S^0)^2}{S^0} - \beta I(n+1)\frac{(S(n+1) - S^0)^2}{S^0} \\
& - \frac{\mu_1\delta\gamma}{\{\gamma\alpha_{31} + \alpha_{21}(\mu_1 + \mu_3 + \delta)\}S^0}(N(n+1) - N^0)^2 \\
& - \left[\frac{\delta\alpha_{21}\{\gamma\alpha_{31} + \alpha_{21}(\mu_3 + \delta)\}}{\gamma\{\gamma\alpha_{31} + \alpha_{21}(\mu_1 + \mu_3 + \delta)\}S^0} + \frac{\delta(\mu_3 + \gamma + \delta)}{\gamma S^0}\right]R(n+1)^2 \\
= & -(\mu_1 + \beta I(n+1))\frac{(S(n+1) - S^0)^2}{S^0} - \frac{\mu_1\delta\gamma}{\{\gamma\alpha_{31} + \alpha_{21}(\mu_1 + \mu_3 + \delta)\}S^0}(N(n+1) - N^0)^2 \\
& - \left[\frac{\delta\alpha_{21}\{\gamma\alpha_{31} + \alpha_{21}(\mu_3 + \delta)\}}{\gamma\{\gamma\alpha_{31} + \alpha_{21}(\mu_1 + \mu_3 + \delta)\}S^0} + \frac{\delta(\mu_3 + \gamma + \delta)}{\gamma S^0}\right]R(n+1)^2. \qquad (4.14)
\end{aligned}$$

For the second case $\mu_1 = \mu_2 = \mu_3$, by $N^0 = B/\mu_1$, we obtain

$$\begin{aligned}
\frac{\delta(N(n+1) - N^0)^2}{4\mu_1 S^0} - \frac{\delta(N(n) - N^0)^2}{4\mu_1 S^0} \leq & \frac{\delta}{4\mu_1 S^0}(N(n+1) - N^0)(B - \mu_1 N(n+1)) \\
= & -\frac{\delta}{4S^0}(N(n+1) - N^0)^2. \qquad (4.15)
\end{aligned}$$

Combining (4.11), (4.12) and (4.15), we have

$$\begin{aligned}
U_\delta^{E^0}(n+1) - U_\delta^{E^0}(n) \leq & -\mu_1\frac{(S(n+1) - S^0)^2}{S^0} - \beta I(n+1)\frac{(S(n+1) - S^0)^2}{S^0} \\
& - \frac{\delta}{S^0}R(n+1)^2 + \frac{\delta}{S^0}R(n+1)(N(n+1) - N^0) - \frac{\delta}{4S^0}(N(n+1) - N^0)^2 \\
& - \frac{\delta}{\gamma S^0}(\mu_3 + \delta)R(n+1)^2 \\
= & -(\mu_1 + \beta I(n+1))\frac{(S(n+1) - S^0)^2}{S^0} - \frac{\delta}{S^0}\left(R(n+1) - \frac{N(n+1) - N^0}{2}\right)^2 \\
& - \frac{\delta}{\gamma S^0}(\mu_3 + \delta)R(n+1)^2. \qquad (4.16)
\end{aligned}$$

From (4.14) and (4.16), for the both cases, we obtain $U_\delta^{E^0}(n+1) - U_\delta^{E^0}(n) \leq 0$ for all $n > 0$. Therefore, we have $\lim_{n\to+\infty}(U_\delta^{E^0}(n+1) - U_\delta^{E^0}(n)) = 0$. This yields $\lim_{n\to+\infty} S(n+1) = S^0$, $\lim_{n\to+\infty} R(n+1) = 0$ and $\lim_{n\to+\infty} N(n+1) = N^0$, from which we obtain $\lim_{n\to+\infty} I(n+1) = 0$. Since $U_\delta^{E^0}(n) \geq \frac{(S(n)-S^0)^2}{2S^0} + I(n) + \frac{\delta}{\gamma S^0}\frac{R(n)^2}{2}$ for all $n \geq 0$, E^0 is uniformly stable. Hence, E^0 is globally asymptotically stable. The proof is complete. □

5 Conclusions

In this paper, for the first-order difference equations (1.2), we show that the disease-free equilibrium E^0 is globally asymptotically stable if $R_0 \leq 1$ and the endemic equilibrium E^* is globally asymptotically stable if $R_0 > 1$. Theorem 1.1 implies that the backward Euler discretization preserves the global stability of the equilibria of the corresponding continuous SIRS epidemic model.

In order to apply key properties of Lyapunov's direct method to the stability analysis for the model, we use the backward Euler method with the positiveness condition $I(n+1) > 0$ for $n = 0, 1, \ldots$. In particular, for $E = (S, I, R)$ and $N = S + I + R$, we define

$$U_\delta^E(n) = \begin{cases} \frac{(S(n)-S)^2}{2S} + Ig\left(\frac{I(n)}{I}\right) + \frac{\delta}{\gamma S}\frac{(R(n)-R)^2}{2} \\ + \frac{\delta\gamma}{\{\gamma\alpha_{31}+\alpha_{21}(\mu_1+\mu_3+\delta)\}S}\frac{\{(N(n)-N)+\frac{\alpha_{21}}{\gamma}(R(n)-R)\}^2}{2}, \\ \text{if either} \quad \mu_1 < \mu_2 \quad \text{or} \quad \mu_1 < \mu_3, \\ \frac{(S(n)-S)^2}{2S} + Ig\left(\frac{I(n)}{I}\right) + \frac{\delta}{\gamma S}\frac{(R(n)-R)^2}{2} + \frac{\delta}{4\mu_1 S}\frac{(N(n)-N)^2}{2}, \\ \text{if} \quad \mu_1 = \mu_2 = \mu_3, \end{cases} \tag{5.1}$$

and present a unified construction of a discrete-time analogue of Lyapunov functions in the proof of the global stability of the disease-free equilibrium E^0 for $R_0 \leq 1$ and the endemic equilibrium E^* for $R_0 > 1$ of system (1.2), respectively as follows;

$$U_\delta^{E^0}(n) := \lim_{E\to E^0} U_\delta^E(n), \text{ and } U_\delta^{E^*}(n) := \lim_{E\to E^*} U_\delta^E(n).$$

Based on the idea in Enatsu et al. [4, Section 1] that $\lim_{x\to+0} xg\left(\frac{y}{x}\right) = y$ holds for each $y > 0$, we construct suitable Lyapunov sequences and the term $I^*g(\frac{I(n)}{I^*})$ in $U_\delta^{E^*}(n)$ corresponds to the term $I(n)$ in $U_\delta^{E^0}(n)$. Making use of the relation, the global stability of two equilibria of (1.2) is fully determined by a threshold parameter R_0.

In fact, by applying variation of the backward Euler method, we consider the following difference equations:

$$\begin{cases} S(n+1) - S(n) = B - \mu_1 S(n+1) - \beta S(n+1)\sum_{j=0}^{m} I(n-j) + \delta R(n+1), \\ I(n+1) - I(n) = \beta S(n+1)\sum_{j=0}^{m} I(n-j) - (\mu_2+\gamma)I(n+1), \\ R(n+1) - R(n) = \gamma I(n+1) - (\mu_3+\delta)R(n+1), \quad n = 0, 1, \ldots \end{cases} \tag{5.2}$$

with the initial conditions $S(0) = \phi_1(0) > 0$, $I(j) = \phi_2(j) \geq 0$ for $j = -m, \ldots, 0$ with $\phi_2(0) > 0$ and $R(0) = \phi_3(0) > 0$. The equations (5.2) have a unique positive solution $(S(n), I(n), R(n))$.

For the case $\delta = 0$, by applying a discrete-time analogue of Lyapunov functional techniques in McCluskey [8], Enatsu et al. [3] has obtained the complete global stability results that the disease-free equilibrium E^0 of (5.2) is globally asymptotically stable if and only if $R_0 \leq 1$ and the endemic equilibrium E^* of (5.2) is globally asymptotically stable if and only if $R_0 > 1$. Furthermore, their result is applied to the global stability analysis of difference equations for SIR epidemic models with a class of nonlinear incidence rates in Enatsu et al. [2].

In addition, by applying Lyapunov functional techniques on the corresponding continuous-time SIRS epidemic model in Enatsu et al. [4], we can obtain the similar result and we still need the additional condition $\mu_1 S^* - \delta R^* \geq 0$ on eusuring the global stability of E^* for $R_0 > 1$. In contrast, we show that the backward Euler method preserves the complete global stability of the equilibria E^0 and E^* under the positiveness conditions on I.

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