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Global stability of a delayed SIRS computer virus propagation model

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Abstract. We propose a delayed SIRS computer virus propagation model. Applying monotone iterative techniques and Lyapunov functional techniques, we establish sufficient conditions for the global asymptotic stability of both virus-free and virus equilibria of the model.

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1 Introduction

Due to the rapid advance of computer technologies, an increasing number of computer viruses have been recognized as a major threat to the network security. By the high similarity between computer viruses and biological viruses, the classical SIR (Susceptible-Infected-Recovered) computer virus propagation models are widely proposed and applied to pose effective measures to prevent virus infection (see for example, [1, 2, 4–8, 10, 12–14, 16–23, 25]).

One of the significant features of computer viruses is its latent characteristic property (see [18]), which means that when viruses enter a host, they hide themselves and become active after a certain period. For example, there occurs not only the case that when the susceptible computers are infected by virus, there is a time delay before becoming infected, but also the case that a recovery computer loses its anti-virus software and becomes susceptible after a time delay. Here it is assumed that all the computers connected to the network in concern are classified into the three categories.

Recently, Ren *et al.* [19] and Han and Tan [5] proposed the following delayed SIRS (Susceptible-Infected-Recovered-Susceptible) computer virus propagation model, respectively:

$$\begin{cases} \dfrac{dS(t)}{dt} = b - \beta S(t)I(t-\tau_1)\mathrm{e}^{-\mu\tau_1} + \nu R(t-\tau_2) - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau_1)\mathrm{e}^{-\mu\tau_1} - (\mu+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \nu R(t-\tau_2) - \mu R(t), \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{dS(t)}{dt} = (1-p)b - \mu S(t) - \beta S(t-\tau_1)I(t-\tau_1) + \nu R(t-\tau_2), \\ \dfrac{dI(t)}{dt} = \beta S(t-\tau_1)I(t-\tau_1) - (\mu+\gamma+\alpha)I(t), \\ \dfrac{dR(t)}{dt} = pb + \gamma I(t) - \mu R(t) - \nu R(t-\tau_2). \end{cases} \tag{1.2}$$

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For the models (1.1) and (1.2), $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infected and recovered computers at time t, respectively. Here it is assumed that a recovered computer becomes susceptible after a period of time $\tau_2 \geq 0$. However, unfortunately, by the term $-\nu R(t-\tau_2)$ in the third equation of system (1.1), there are cases that (1.1) does not generate a positive solution $(S(t), I(t), R(t))$ (see Lemma A.1 in Appendix and Example 6.1 in Section 6). The same situation occurs for (1.2) because of the term $-\nu R(t-\tau_2)$ in the third equation.

This suggests that some modification are needed in order to make a meaningful computer virus model. One of the approaches is to replace $\nu R(t-\tau_2)$ by $\nu R(t)$ in the third equation of (1.1) and (1.2), because a recovered computer does not revert to the susceptible one instantaneously. This allows us to prove the positiveness of the solutions (see Lemma 2.1). We show that the analysis on global dynamics of the revised model is similar to that of SIRS epidemic models except the delay term $\nu R(t-\tau_2)$ in the first equation of (1.1) and (1.2). To the best of our knowledge, until now there are no good analysis on the revised model. Therefore, it is meaningful to analyze the model with delay term $\nu R(t-\tau_2)$ in the first equation of system.

Motivated by the above fact, we investigate global dynamics of the following SIRS computer virus propagation model with discrete delays:

$$\begin{cases} \dfrac{dS(t)}{dt} = b - \mu_1 S(t) - \beta S(t) I(t-\tau_1) \mathrm{e}^{-\mu_1 \tau_1} + \nu R(t-\tau_2), \\ \dfrac{dI(t)}{dt} = \beta S(t) I(t-\tau_1) \mathrm{e}^{-\mu_1 \tau_1} - (\mu_2 + \gamma) I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - (\mu_3 + \nu) R(t), \end{cases} \tag{1.3}$$

with the initial conditions

$$\begin{aligned} &S(\theta) = \phi_1(\theta) \geq 0, \ I(\theta) = \phi_2(\theta) \geq 0, \ R(\theta) = \phi_3(\theta) \geq 0, \ \theta \in [-\bar{\tau}, 0], \\ &\bar{\tau} = \max(\tau_1, \tau_2), \ \text{with } \phi_1(0) > 0, \ \phi_2(0) > 0, \ \phi_3(0) > 0. \end{aligned} \tag{1.4}$$

We restrict our attention to the case

$$\nu > 0 \quad \text{and} \quad \tau_2 > 0. \tag{1.5}$$

For the model (1.3), $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible, infected and recovered computers at time t, respectively. b denotes the rate at which external computers are connected to the network, β denotes the rate at which, when having a connection to an infected computer, a susceptible computer can become infected by virus after a time delay τ_1. Hence the term $\beta I(t-\tau_1)$ is the force of infection. γ denotes the recovery rate of infected computers due to the anti-virus ability of the network and ν denotes the loss rate of recovery computers which loose its anti-virus software and return into the susceptible computers, μ_1, μ_2 and μ_3 denote the network removal rate of susceptible, infected and recovered computers, respectively.

System (1.3) always has a virus-free equilibrium $E^0 = (S^0, 0, 0)$, where

$$S^0 = \frac{b}{\mu_1}. \tag{1.6}$$

and the basic reproduction number of system (1.3) is

$$R_0 = \frac{\beta \mathrm{e}^{-\mu_1 \tau_1} b}{\mu_1 (\mu_2 + \gamma)}. \tag{1.7}$$

For $R_0 > 1$, system (1.3) admits a unique virus equilibrium $E^* = (S^*, I^*, R^*)$ such that

$$\begin{cases} b - \mu_1 S^* - \beta \mathrm{e}^{-\mu_1 \tau_1} S^* I^* + \nu R^* = 0, \\ \beta \mathrm{e}^{-\mu_1 \tau_1} S^* I^* - (\mu_2 + \gamma) I^* = 0, \\ \gamma I^* - (\mu_3 + \nu) R^* = 0. \end{cases} \tag{1.8}$$

Moreover, we assume that

$$\nu < \mu_1 \leq \min(\mu_2, \mu_3). \tag{1.9}$$

Note that for the case $\nu = 0$, system (1.3) becomes a well-known SIR epidemic model. The global stability of a unique endemic equilibrium of system (1.3) with $\nu = 0$ has been completely established by McCluskey [11]. For the case $\nu > 0$ and $\tau_2 = 0$, system (1.3) becomes a delayed SIRS epidemic model. The global dynamics is also investigated in the several literatures, for example, by Muroya *et al.* [15] with monotone iterative techniques and by Enatsu *et al.* [3] with Lyapunov functional techniques. Under the initial condition (1.4), system (1.3) has a positive solution. However, by term $\nu R(t-\tau_2)$ in the first equation of (1.1), we need some additional conditions on ν and γ for eventual boundedness of solutions and global stability of the equilibria.

In this paper, we establish the global asymptotic stability of virus-free equilibrium of (1.3) for the case $R_0 < 1$ (see Theorem 1.1). For the case $R_0 > 1$, we obtain sufficient conditions for the global asymptotic stability of a virus equilibrium by applying monotone iterative techniques by Muroya *et al.* [15] (see Theorem 1.2). Moreover, for $R_0 > 1$, by applying the similar Lyapunov functional techniques as [19, Theorem 3.3], we also obtain another sufficient condition for the global stability of the virus equilibrium (see Theorem 1.3).

Theorem 1.1. *If $R_0 < 1$, then the virus-free equilibrium $E^0 = (S^0, 0, 0)$ of system (1.3) is globally asymptotically stable for any $\tau_2 > 0$ and $0 < \nu < \mu_1$.*

Theorem 1.2. *If $R_0 > 1$, then there exists a unique virus equilibrium $E^* = (S^*, I^*, R^*)$ of system (1.3). Moreover, if*

$$2\gamma \leq 2\mu_1 - \mu_2, \tag{1.10}$$

then for any $\tau_2 > 0$ and $0 < \nu < \mu_1$, E^* is globally asymptotically stable, and if

$$\gamma < 2\mu_1 - \mu_2 < 2\gamma, \quad \text{and} \quad 0 < \nu < \frac{(2\mu_1 - \mu_2) - \gamma}{2\gamma - (2\mu_1 - \mu_2)}\mu_3, \tag{1.11}$$

then for any $\tau_2 > 0$, E^* is globally asymptotically stable.

In particular, for the case $\mu_1 = \mu_2 = \mu_3 = \mu$, if

$$\mu \geq \frac{3}{2}\gamma, \tag{1.12}$$

then for any $\tau_2 > 0$ and $0 < \nu < \mu_1$, E^* is globally asymptotically stable, and if

$$\gamma < \mu < \frac{3}{2}\gamma, \quad \text{and} \quad 0 < \nu < \frac{\mu - \gamma}{2\gamma - \mu}\mu < \mu, \tag{1.13}$$

then for any $\tau_2 > 0$, E^* is globally asymptotically stable.

Theorem 1.3. *For $R_0 > 1$, the virus equilibrium E^* is globally asymptotically stable for any $\tau_2 > 0$, if*

$$\begin{aligned} &2\mu_3 - \gamma > 0, \ \text{and} \\ &\begin{cases} 0 < \nu < \frac{2\omega(\mu_2+\gamma)-\gamma}{\omega}, & \text{for } \frac{\gamma}{2(\mu_2+\gamma)} < \omega < \frac{\gamma}{2(\mu_2+\gamma)-\mu_1}, \\ 0 < \nu < \mu_1, & \text{for } \frac{\gamma}{2(\mu_2+\gamma)-\mu_1} \leq \omega \leq 2 + \frac{2\mu_3-\gamma}{\mu_1}, \\ 0 < \nu < \frac{2\mu_3-\gamma}{\omega-2}, & \text{for } \omega > 2 + \frac{2\mu_3-\gamma}{\mu_1}, \end{cases} \end{aligned} \tag{1.14}$$

where

$$\omega = \frac{\beta I^* \mathrm{e}^{-\mu_1\tau_1}}{\mu_1 + \mu_2 + \gamma}, \quad \text{and} \quad I^* = \frac{b - \frac{\mu_1(\mu_2+\gamma)}{\beta \mathrm{e}^{\mu_1\tau_1}}}{(\mu_2 + \gamma) - \frac{\nu\gamma}{\mu_3+\nu}}.$$

In particular, for the case $\mu_1 = \mu_2 = \mu_3 = \mu > 0$, E^* is globally asymptotically stable for any $\tau_2 > 0$, if for $\omega = \frac{\beta I^* \mathrm{e}^{-\mu\tau_1}}{2\mu+\gamma}$ and $I^* = \frac{b - \frac{\mu(\mu+\gamma)}{\beta \mathrm{e}^{\mu\tau_1}}}{(\mu+\gamma) - \frac{\nu\gamma}{\mu+\nu}}$,

$$\frac{\gamma}{2} < \mu, \ \text{and} \begin{cases} 0 < \nu < \{2\omega(\mu+\gamma) - \gamma\}/\omega, & \text{for } \frac{\gamma}{2(\mu+\gamma)} < \omega < \frac{\gamma}{\mu+2\gamma}, \\ 0 < \nu < \mu, & \text{for } \frac{\gamma}{\mu+2\gamma} \leq \omega \leq 2 + \frac{2\mu-\gamma}{\mu}, \\ 0 < \nu < \frac{2\mu-\gamma}{\omega-2}, & \text{for } \omega > 2 + \frac{2\mu-\gamma}{\mu}. \end{cases} \tag{1.15}$$

The organization of this paper is as follows. In Section 2, some basic results are given. For $R_0 < 1$, we prove Theorem 1.1. In Section 3, for $R_0 > 1$, we first prove the existence of a unique virus equilibrium E^* and the permanence of system by obtaining lower positive bounds of solutions which do not depend on the initial conditions (1.4). The eventual lower bound will be used as an initial value of an iterative sequence when applying the monotone iterative techniques in Section 4. For $R_0 > 1$, by applying the monotone iterative techniques of Muroya *et al.* [15], we prove Theorem 1.2 in Section 4, and by applying the Lyapunov function techniques of Ren *et al.* [19], we prove Theorem 1.3 in Section 5. We offer some numerical examples in Section 6 and end with concluding remarks in Section 7. In Appendix, we derive concrete conditions under which system (1.1) has non-positive solutions.

2 Basic results and global stability of the virus-free equilibrium

We first offer the following lemma on positiveness and eventual boundedness of solutions $S(t)$, $I(t)$, $R(t)$ of (1.3).

Lemma 2.1. *Let* $(S(t), I(t), R(t))$ *be a solution of system* (1.3) *with the initial conditions* (1.4). *Then*

$$S(t) > 0, \ I(t) > 0 \text{ and } R(t) > 0, \text{ for all } t \geq 0. \tag{2.1}$$

Moreover, for $N(t) = S(t) + I(t) + R(t)$, *if* (1.5) *holds, then*

$$\limsup_{t\to+\infty} N(t) \leq \frac{b}{\mu_1 - \nu}. \tag{2.2}$$

Proof First, by the initial conditions (1.4), we have $S(0) > 0$, $I(0) > 0$ and $R(0) > 0$ and by continuity of the solution of (1.3), we may assume that there exists a positive t_1 such that $S(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for any $0 \leq t < t_1$. Suppose that there exists a positive t_1 such that $S(t_1) = 0$ and $S(t) > 0$, $I(t) > 0$, $R(t) > 0$ for any $0 \leq t < t_1$. By the first equation of (1.3), we have that $\frac{dS}{dt}(t_1) \geq b > 0$ which is a contradiction to the fact that $S(t) > 0 = S(t_1)$ for any $0 \leq t < t_1$. Thus, if there exists a positive t_1 such that $S_k(t) > 0$, $I_k(t) > 0$ and $R_k(t) > 0$ for any $0 \leq t < t_1$, then $S(t_1) > 0$.

Moreover, by the second and third equations of (1.3), we have that

$$\begin{aligned} I(t) =& \mathrm{e}^{-(\mu_2+\gamma)t} I(0) + \mathrm{e}^{-(\mu_2+\gamma)t} \int_0^t \mathrm{e}^{(\mu_2+\gamma)u} \beta \mathrm{e}^{-\mu_1\tau_1} S(u) I(u-\tau_1) du, \\ R(t) =& \mathrm{e}^{-(\mu_3+\nu)t} R(0) + \mathrm{e}^{-(\mu_3+\nu)t} \int_0^t \mathrm{e}^{(\mu_3+\nu)u} \gamma I(u) du, \quad \text{for } t > 0, \end{aligned}$$

which implies that if there exists a positive t_1 such that $S(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for any $0 \leq t < t_1$, then $I(t_1) > 0$ and $R(t_1) > 0$. Thus, we obtain (2.1).

By (1.3) and (1.4),

$$\begin{cases} \dfrac{d}{dt} N(t) = \dfrac{d}{dt} \{S(t) + I(t) + R(t)\} \\ \qquad\qquad = b - \mu_1 S(t) - \mu_2 I(t) - \mu_3 R(t) + \nu\{R(t-\tau_2) - R(t)\} \\ \qquad\qquad \leq b - \mu_1 N(t) + \nu N(t-\tau_2), \\ N(\theta) = \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta) \geq 0, \ \theta \in [-\bar{\tau}, 0], \\ \text{with } N(0) = \phi_1(0) + \phi_2(0) + \phi_3(0) > 0. \end{cases} \tag{2.3}$$

We now consider the following auxiliary equation (see for example, Kuang [9] or Song *et al.* [22]):

$$\begin{cases} \dfrac{du(t)}{dt} = b - \mu_1 u(t) + \nu u(t-\tau_2), \\ u(\theta) = \phi_1(\theta) + \phi_2(\theta) + \phi_3(\theta) \geq 0, \ \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{2.4}$$

One can see that (2.4) has a unique positive equilibrium $u^* = \frac{b}{\mu_1 - \nu}$. We define the following functional:

$$V_u(t) = \frac{\{u(t) - u^*\}^2}{2} + \frac{\nu}{2} \int_{t-\tau_2}^t \{u(\theta) - u^*\}^2 d\theta. \tag{2.5}$$

By differentiating V_u along the solution $u(t)$ of (2.4),

$$\frac{dV_u(t)}{dt} = \{u(t) - u^*\} \frac{du(t)}{dt} + \frac{\nu}{2} [\{u(t) - u^*\}^2 - \{u(t-\tau_2) - u^*\}^2]. \tag{2.6}$$

From the equilibrium condition $b = (\mu_1 - \nu)u^*$, we have that

$$\begin{aligned} \{u(t) - u^*\} \frac{du(t)}{dt} &= \{u(t) - u^*\}\{b - \mu_1 u(t) + \nu u(t-\tau_2))\} \\ &= \{u(t) - u^*\}[-\mu_1\{u(t) - u^*) + \nu\{u(t-\tau_2) - u^*\}] \\ &= -\mu_1\{u(t) - u^*\}^2 + \nu\{u(t) - u^*\}\{u(t-\tau_2) - u^*\} \\ &\leq -\mu_1\{u(t) - u^*\}^2 + \frac{\nu}{2}[\{u(t) - u^*\}^2 + \{u(t-\tau_2) - u^*\}^2]. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dV_u(t)}{dt} \leq& [-\mu_1\{u(t) - u^*\}^2 + \frac{\nu}{2}[\{u(t) - u^*\}^2 + \{u(t-\tau_2) - u^*\}^2] \\ &+ \frac{\nu}{2}[\{u(t) - u^*\}^2 - \{u(t-\tau_2) - u^*\}^2] \\ =& -(\mu_1 - \nu)\{u(t) - u^*\}^2 \\ \leq& \, 0. \end{aligned} \tag{2.7}$$

Then, the nonnegative functional V_u is monotonically decreasing and there exists a nonnegative constant $\hat{V}_u$ such that $\lim_{t\to+\infty} V_u(t) = \hat{V}_u$. Since the equality of (2.7) holds if and only if $u(t) = u^*$, we have $\hat{V}_u = 0$, which yields $\lim_{t\to+\infty} u(t) = u^* = b/(\mu_1 - \nu)$. Thus, by the above discussion and the comparison principle, we obtain $\limsup_{t\to+\infty} N(t) \le b/(\mu_1 - \nu)$. Hence, the proof is complete. □

Lemma 2.2. *If $R_0 < 1$, then the virus-free equilibrium E^0 of system (1.3) is locally asymptotically stable. Furthermore, the virus-free equilibrium E^0 is unstable if $R_0 > 1$.*

Proof. The characteristic equation to the linearized equation of system (1.3) at the virus-free equilibrium E^0 is of the form

$$\det \begin{vmatrix} -\mu_1 - \lambda & -\beta b e^{(-\mu_1-\lambda)\tau_1}/\mu_1 & \nu e^{-\lambda\tau_2} \\ 0 & \beta b e^{(-\mu_1-\lambda)\tau_1}/\mu_1 - (\mu_2+\gamma) - \lambda & 0 \\ 0 & \gamma & -(\mu_3+\nu) - \lambda \end{vmatrix} = 0, \tag{2.8}$$

which is equivalent to

$$(-\mu_1 - \lambda)\{(\mu_2+\gamma)(R_0 e^{-\lambda\tau_1} - 1) - \lambda\}\{-(\mu_3+\nu) - \lambda\} = 0. \tag{2.9}$$

One can see that $\lambda = -\mu_1$ and $\lambda = -(\mu_3+\nu)$ are always roots of the characteristic equation (2.9). The other roots of (2.9) are determined by

$$f(\lambda) \equiv (\mu_2+\gamma)(R_0 e^{-\lambda\tau_1} - 1) - \lambda = 0. \tag{2.10}$$

We note that $\lambda = 0$ is not a root of (2.10) if $R_0 \neq 1$. When $\tau_1 = 0$, (2.10) has the form

$$(\mu_2+\gamma)(R_0 - 1) - \lambda = 0. \tag{2.11}$$

If $R_0 < 1$, then (2.11) has a negative real root. Therefore, the virus-free equilibrium E^0 is locally asymptotically stable when $\tau_1 = 0$. Suppose that $\lambda = iw$ $w > 0$, is a root of (2.9). Then by separating real and imaginary parts, we derive

$$(\mu_2+\gamma)\{R_0\cos(w\tau_1) - 1\} = 0, \ (\mu_2+\gamma)R_0\sin(w\tau_1) - w = 0. \tag{2.12}$$

From the first equation of (2.12), we obtain

$$(\mu_2+\gamma)\{R_0\cos(w\tau_1) - 1\} \le (\mu_2+\gamma)(R_0 - 1) < 0,$$

for all $w > 0$, which is a contradiction. It follows that the real parts of all the eigenvalues of (2.9) are negative for all $\tau_1 \ge 0$. Therefore, if $R_0 < 1$, the virus-free equilibrium E^0 of system (1.3) is locally asymptotically stable for all $\tau_1 \ge 0$.

Assume $R_0 > 1$. Then $f(0) = (\mu_2+\gamma)(R_0 - 1) > 0$ holds and $\lim_{\lambda\to+\infty} f(\lambda) = -\infty$, for $\lambda \in \mathbb{R}$. Therefore, (2.10) has at least one positive real root. Hence, if $R_0 > 1$, the virus-free equilibrium E^0 is unstable. □

We now prove the global asymptotic stability of the virus equilibrium E^0 of system (1.3) for $R_0 < 1$ by Lyapunov functional techniques.

Proof of Theorem 1.1. Assume that $R_0 < 1$, $\tau_2 > 0$ and $0 < \nu < \mu_1$. By introducing the variables

$$x(t) = S(t) - S^0, \ y(t) = I(t), \ z(t) = R(t), \tag{2.13}$$

system (1.3) is centered at the virus-free equilibrium E^0 and is rewritten as

$$\begin{cases} \dot{x} = -\mu_1 x - \beta(S^0 + x)e^{-\mu\tau_1} y(t-\tau_1) + \nu z(t-\tau_2), \\ \dot{y} = \beta S^0 e^{-\mu\tau_1} y(t-\tau_1) - (\mu_2+\gamma)y, \\ \dot{z} = \gamma y - (\mu_3+\nu)z. \end{cases} \tag{2.14}$$

By introducing the functional

$$W(t) = \frac{y^2}{2} + \frac{\mu_2+\gamma}{2}\int_{t-\tau_1}^{t} y^2(\theta)d\theta, \tag{2.15}$$

the derivative of W becomes

$$\begin{aligned} \dot{W}(t) =& y\dot{y} + \frac{\mu_2+\gamma}{2}\{y^2 - y^2(t-\tau_1)\} \\ =& y\{\beta S^0 e^{-\mu_1\tau_1} y(t-\tau_1) - (\mu_2+\gamma)y\} + \frac{\mu_2+\gamma}{2}\{y^2 - y^2(t-\tau_1)\}. \end{aligned}$$

Applying Cauchy-Schwartz inequality, we have

$$\begin{aligned}\dot{W}(t) \leq& \beta S^0 \mathrm{e}^{-\mu_1\tau_1} \frac{y^2 + y^2(t-\tau_1)}{2} - (\mu_2+\gamma)\frac{y^2+y^2(t-\tau_1)}{2}\\ =&\{\beta S^0 \mathrm{e}^{-\mu_1\tau_1} - (\mu_2+\gamma)\}\frac{y^2+y^2(t-\tau_2)}{2}\\ =&(\mu_2+\gamma)(R_0-1)\frac{y^2+y^2(t-\tau_2)}{2}\\ \leq& 0. \end{aligned} \tag{2.16}$$

By Lemma 2.1, the nonnegative function $W(t)$ is monotonically decreasing and there exists a nonnegative constant $\hat{W}$ such that $\lim_{t\to+\infty} W(t) = \hat{W}$. By $R_0 < 1$ and the continuity of y, we conclude that $\lim_{t\to+\infty} y(t) = 0$ and $\hat{W} = 0$. From the first and third equations of (2.14), we obtain $\lim_{t\to+\infty} z(t) = 0$. Applying Lyapunov-LaSalle asymptotic stability theorem, E^0 is globally asymptotically stable. □

3 Existence of the virus equilibrium for $R_0 > 1$

We obtain the following basic lemma which ensures the unique existence of a virus equilibrium E^* of system (1.3) for $R_0 > 1$ (see also Enatsu *et al.* [3, Lemma 22]).

Lemma 3.1. *If $R_0 > 1$, then system (1.3) has a unique virus equilibrium $E^* = (S^*, I^*, R^*)$.*

Proof. Assume that $R_0 > 1$. From the second and the third equations of (1.8), it follows that

$$S^* = \frac{\mu_2+\gamma}{\beta \mathrm{e}^{-\mu_1\tau_1}}, \text{ and } R^* = \frac{\gamma I^*}{\mu_3+\nu}. \tag{3.1}$$

After substituting (3.1) into the first equation of (1.8), we obtain that

$$H(I^*) = 0,$$

where

$$H(I) \equiv b - \frac{\mu_1(\mu_2+\gamma)}{\beta \mathrm{e}^{-\mu_1\tau_1}} - (\mu_2+\gamma)I + \nu\frac{\gamma I}{\mu_3+\nu} = b\left(1-\frac{1}{R_0}\right) - \left(\mu_2 + \frac{\mu_3}{\mu_3+\nu}\gamma\right)I.$$

One can see that H is strictly monotonically decreasing on $(0, +\infty)$ satisfying

$$\lim_{I\to+0} H(I) = b\left(1-\frac{1}{R_0}\right) > 0,$$

and $H(I) < 0$ for any $I > b(1-1/R_0)/\{\mu_2 + \mu_3\gamma/(\mu_3+\nu)\} > 0$. Hence, there exists a unique positive $I^* > 0$ such that $H(I^*) = 0$. By (3.1), there exists a unique virus equilibrium $E^* = (S^*, I^*, R^*)$. □

3.1 Permanence for $R_0 > 1$

First, we give a basic lemma whose proof is similar to that by Enatsu *et al.* [3, Proof of Lemma 3.2].

Lemma 3.2. *Assume that $I(s) \leq I^*$ for any s such that $t - \tau_1 \leq s < t$. If $I(t) < I(s)$ for any s such that $t-\tau_1 \leq s < t$ then $S(t) < S^*$. Inversely, if $S(t) \geq S^*$, then there exists an $s_t \in [t-\tau_1, t)$ such that $I(t) \geq I(s_t)$.*

By applying Lemma 3.2, we offer a proof for the permanence of system (1.3) (cf. Xu and Ma [24]).

Lemma 3.3. *If $R_0 > 1$, then for any solution of system (1.3) with initial condition (1.4), it holds that*

$$\begin{cases} \liminf_{t\to+\infty} S(t) \geq v_1 := \frac{b}{\mu_1 + \beta \mathrm{e}^{-\mu_1\tau_1} b/(\mu_1-\nu)} > 0, \\ \liminf_{t\to+\infty} I(t) \geq v_2(q) := qI^* \exp\{-(\mu_2+\gamma)\rho(q)\} > 0, \\ \liminf_{t\to+\infty} R(t) \geq v_3(q) := \frac{\gamma}{\mu_3+\nu} v_2(q), \end{cases}$$

where for any $0 < q < \frac{b\beta \mathrm{e}^{-\mu_1\tau_1} I^* - \mu_1\nu R^*}{\beta I^*(b+\nu R^*)}$, $\rho(q) > 0$ is a constant such that

$$S^* < S^{\triangle} := \frac{b}{r}(1 - \exp(-r\rho(q))), \quad \textit{and} \quad r = \mu_1 + \beta \mathrm{e}^{-\mu_1\tau_1} q I^*. \tag{3.2}$$

Proof. Let $(S(t), I(t), R(t))$ be a solution of system (1.3) with initial condition (1.4). By Lemma 2.1, it holds that

$$\limsup_{t\to+\infty} I(t) \le \frac{b}{\mu_1 - \nu}.$$

For a sufficiently small $\epsilon > 0$, there is a positive real T_1 such that $I(t) < b/(\mu_1 - \nu) + \epsilon$ for $t > T_1$. Then, from the first equation of (1.3), we derive that

$$S'(t) \ge b - \left\{\mu_1 + \beta e^{-\mu_1\tau_1}\left(\frac{b}{\mu_1 - \nu} + \epsilon\right)\right\} S(t),$$

which implies that

$$\liminf_{t\to+\infty} S(t) \ge \frac{b}{\mu_1 + \beta e^{-\mu_1\tau_1}\left(b/(\mu_1 - \nu) + \epsilon\right)}.$$

Since the above inequality holds for arbitrary sufficiently small $\epsilon > 0$, it follows that

$$\liminf_{t\to+\infty} S(t) \ge \frac{b}{\mu_1 + \beta e^{-\mu_1\tau_1} b/(\mu_1 - \nu)} = v_1.$$

We now show that $\liminf_{t\to+\infty} I(t) \ge v_2(q)$, for any $0 < q < \frac{b\beta e^{-\mu_1\tau_1} I^* - \mu_1\nu R^*}{\beta I^*(b+\nu R^*)}$. For any $0 < q < \frac{b\beta e^{-\mu_1\tau_1} I^* - \mu_1\nu R^*}{\beta I^*(b+\nu R^*)}$, one can see that

$$S^* = \frac{b + \nu R^*}{\mu_1 + \beta e^{-\mu_1\tau_1} I^*} < \frac{b}{\mu_1 + \beta e^{-\mu_1\tau_1} q I^*} = \frac{b}{r},$$

because

$$\begin{aligned} b\beta e^{-\mu_1\tau_1} I^* - \mu_1\nu R^* &= b\beta e^{-\mu_1\tau_1} I^* - \mu_1\nu\frac{\gamma}{\mu_3 + \nu} I^* \\ &= \mu_1(\mu_2 + \gamma)\left(R_0 - \frac{\gamma}{\mu_2 + \gamma}\frac{\nu}{\mu_3 + \nu}\right) I^* \\ &> \mu_1(\mu_2 + \gamma)(R_0 - 1) I^* > 0, \end{aligned}$$

and we have

$$\begin{aligned} S^* =& \frac{b + \nu R^*}{\mu_1 + \beta e^{-\mu_1\tau_1} I^*} \\ =& \frac{b}{b(\mu_1 + \beta e^{-\mu_1\tau_1} I^*)/(b + \nu R^*)} \\ =& \frac{b}{\mu_1 + (b\beta e^{-\mu_1\tau_1} I^* - \mu_1\nu R^*)/(b + \nu R^*)} \\ <& \frac{b}{\mu_1 + \beta e^{-\mu_1\tau_1} q I^*}, \end{aligned}$$

Thus, there exists a positive constant $\rho(q)$ such that (3.2) holds. The rest of the proof is similar to that by Enatsu *et al.* [3, Proof of Lemma 2.4]. ☐

Remark 3.1 Likewise in the above proof, a correction is needed in Enatsu *et al.* [3, Lemma 2.3 and its proof]. The condition "for any $0 < q < 1$" must be replaced by a more restricted condition "for any $0 < q < \frac{B\beta G(I^*) - \mu_1\delta R^*}{\beta G(I^*)(B+\delta R^*)}$" to guarantee the necessary inequality on q in the step $S^* = \frac{B+\delta R^*}{\mu_1+\beta G(I^*)} < \frac{B}{\mu_1+\beta q G(I^*)} = \frac{B}{r}$.

By Lemmas 2.1 and 3.3, we obtain the permanence of system (1.3) for $R_0 > 1$.

4 Stability analysis by monotone iterative techniques for $R_0 > 1$

In this section, we consider the global stability of the virus equilibrium of system (1.3) applying monotone iterative techniques (cf. Muroya *et al.* [15, Lemmas 4.1-4.4 and Corollary 1.1]).

By Lemmas 2.1 and 3.3, we may put

$$\begin{cases} \liminf\limits_{t\to+\infty} S(t) = \underline{\hat{S}} \ge v_1, \quad \liminf\limits_{t\to+\infty} I(t) = \underline{\hat{I}} \ge v_2, \quad \liminf\limits_{t\to+\infty} R(t) = \underline{\hat{R}} \ge v_3, \\ \limsup\limits_{t\to+\infty} S(t) = \hat{\hat{S}} \le \dfrac{b}{\mu_1 - \nu}, \quad \limsup\limits_{t\to+\infty} I(t) = \hat{\hat{I}} \le \dfrac{b}{\mu_1 - \nu}, \\ \limsup\limits_{t\to+\infty} R(t) = \hat{\hat{R}} \le \dfrac{b}{\mu_1 - \nu}, \quad \liminf\limits_{t\to+\infty} N(t) = \underline{\hat{N}}, \text{ and } \limsup\limits_{t\to+\infty} N(t) = \hat{\hat{N}}. \end{cases} \tag{4.1}$$

Then, we have the following lemmas.

Lemma 4.1. *It holds*

$$\underline{\hat{N}} - \underline{\hat{I}} - \hat{\bar{R}} > 0, \quad \textit{and} \quad \hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}} > 0. \tag{4.2}$$

Proof. Suppose that $\hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}} \leq 0$. Then, by (4.1), there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} I(t_n) = \hat{\bar{I}}$. Since $\liminf_{n\to+\infty} R(t_n) \geq \underline{\hat{R}}$, by $S(t) = N(t) - I(t) - R(t)$, we have

$$0 < \limsup_{n\to+\infty} S(t_n) \leq \limsup_{n\to\infty} N(t_n) - \liminf_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \leq \hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}} \leq 0,$$

which is a contradiction. Thus, we have $\hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}} > 0$. Similarly, we can prove that $\underline{\hat{N}} - \underline{\hat{I}} - \hat{\bar{R}} > 0$ holds. $\square$

Lemma 4.2. *It holds that*

$$\begin{cases} 0 \geq b - \mu_1 \underline{\hat{S}} - \beta \underline{\hat{S}} \hat{\bar{I}} + \nu \underline{\hat{R}}, \\ 0 \geq \beta(\underline{\hat{N}} - \underline{\hat{I}} - \hat{\bar{R}}) \underline{\hat{I}} - (\mu_2 + \gamma) \underline{\hat{I}}, \\ 0 \geq \gamma \underline{\hat{I}} - (\mu_3 + \nu) \underline{\hat{R}}, \\ 0 \geq b - \mu_1 \underline{\hat{N}} - (\mu_2 - \mu_1) \hat{\bar{I}} - (\mu_3 - \mu_1) \hat{\bar{R}} + \nu(\underline{\hat{R}} - \hat{\bar{R}}), \end{cases} \tag{4.3}$$

and

$$\begin{cases} 0 \leq b - \mu_1 \hat{\bar{S}} - \beta \hat{\bar{S}} \underline{\hat{I}} + \nu \hat{\bar{R}}, \\ 0 \leq \beta(\hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}}) \hat{\bar{I}} - (\mu_2 + \gamma) \hat{\bar{I}}, \\ 0 \leq \gamma \hat{\bar{I}} - (\mu_3 + \nu) \hat{\bar{R}}, \\ 0 \leq b - \mu_1 \hat{\bar{N}} - (\mu_2 - \mu_1) \underline{\hat{I}} - (\mu_3 - \mu_1) \underline{\hat{R}} + \nu(\hat{\bar{R}} - \underline{\hat{R}}). \end{cases} \tag{4.4}$$

Proof. Assume that $I(t)$ is eventually monotonically decreasing for $t \geq 0$. By Lemma 3.3, there exists $\lim_{t\to+\infty} I(t) = \hat{\bar{I}} = \underline{\hat{I}} > 0$. Then, by the third equation of (1.3), we obtain that there exists $\lim_{t\to+\infty} R(t) = \hat{\bar{R}} = \underline{\hat{R}} > 0$. By the first equation of (1.3), there exists $\lim_{t\to+\infty} S(t) = \hat{\bar{S}} = \underline{\hat{S}} > 0$. Then, by $N(t) = S(t) + I(t) + R(t)$, we obtain that $\lim_{t\to\infty} N(t) = \hat{\bar{N}} = \underline{\hat{N}}$. Since the positive equilibrium $E^* = (S^*, I^*, R^*)$ is unique, we have that $\hat{\bar{S}} = \underline{\hat{S}} = S^*$, $\hat{\bar{I}} = \underline{\hat{I}} = I^*$ and $\bar{\bar{R}} = \underline{\hat{R}} = R^*$. By the equilibrium condition (1.8), we have (4.4).

Now, suppose that $I(t)$ is not eventually monotonically decreasing for $t \geq 0$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to+\infty} I'(t_n) \geq 0 \text{ and } \lim_{n\to+\infty} I(t_n) = \hat{\bar{I}}.$$

Since

$$\limsup_{n\to+\infty} S(t_n) \leq \limsup_{n\to\infty} N(t_n) - \lim_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \leq \hat{\bar{N}} - \hat{\bar{I}} - \underline{\hat{R}},$$

we obtain the second equation of (4.4). From (1.3) and (2.3), we similarly get the first, second and the last equations of (4.4) and obtain (4.3). $\square$

Putting

$$c = \mu_2 - \mu_1 + (\mu_3 + \nu - \mu_1)\frac{\gamma}{\mu_3 + \nu}, \tag{4.5}$$

we obtain the following lemma by Lemma 4.2:

Lemma 4.3. *It holds that*

$$\begin{cases} \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3 + \nu)}\right)\underline{\hat{I}} + \dfrac{\mu_2 + \gamma}{\beta} \geq \dfrac{b}{\mu_1} - \left(\dfrac{\mu_3 + \nu}{\mu_1}\dfrac{\gamma}{\mu_3 + \nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\hat{\bar{I}}, \\ \underline{\hat{S}} \geq \dfrac{b + \frac{\nu\gamma}{\mu_3+\nu}\underline{\hat{I}}}{\mu_1 + \beta\hat{\bar{I}}}, \quad \underline{\hat{R}} \geq \dfrac{\gamma}{\mu_3 + \nu}\underline{\hat{I}}, \quad \underline{\hat{N}} \geq \dfrac{b}{\mu_1} - \dfrac{c}{\mu_1}\hat{\bar{I}} + \dfrac{\nu\gamma}{\mu_1(\mu_3 + \nu)}\underline{\hat{I}}, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3 + \nu)}\right)\hat{\bar{I}} + \dfrac{\mu_2 + \gamma}{\beta} \leq \dfrac{b}{\mu_1} - \left(\dfrac{\mu_3 + \nu}{\mu_1}\dfrac{\gamma}{\mu_3 + \nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\underline{\hat{I}}, \\ \hat{\bar{S}} \leq \dfrac{b + \frac{\nu\gamma}{\mu_3+\nu}\hat{\bar{I}}}{\mu_1 + \beta\underline{\hat{I}}}, \quad \hat{\bar{R}} \leq \dfrac{\gamma}{\mu_3 + \nu}\hat{\bar{I}}, \quad \hat{\bar{N}} \leq \dfrac{b}{\mu_1} - \dfrac{c}{\mu_1}\underline{\hat{I}} + \dfrac{\nu\gamma}{\mu_1(\mu_3 + \nu)}\hat{\bar{I}}. \end{cases} \tag{4.7}$$

We now consider the following six sequences $\bar{S}_n$, $\bar{I}_n$, $\bar{R}_n$, $\underline{S}_n$, $\underline{I}_n$ and $\underline{R}_n$, $(n = 1, 2, \cdots)$ as follows (cf. Xu and Ma [24, (3.3)]).

$$\begin{cases} 0 \le \underline{I}_0 \le \liminf_{t\to+\infty} I(t), \\ \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)\bar{I}_n + \dfrac{\mu_2+\gamma}{\beta} = \dfrac{b}{\mu_1} - \left(\dfrac{\mu_3+\nu}{\mu_1}\dfrac{\gamma}{\mu_3+\nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\underline{I}_{n-1}, \\ \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)\underline{I}_n + \dfrac{\mu_2+\gamma}{\beta} = \dfrac{b}{\mu_1} - \left(\dfrac{\mu_3+\nu}{\mu_1}\dfrac{\gamma}{\mu_3+\nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\bar{I}_n, \ n = 1, 2, 3, \cdots \end{cases} \tag{4.8}$$

and

$$\begin{cases} \underline{S}_n = \dfrac{b + \frac{\nu\gamma}{\mu_3+\nu}\underline{I}_n}{\mu_1 + \beta\bar{I}_n}, \quad \underline{R}_n = \dfrac{\gamma}{\mu_3+\nu}\underline{I}_n, \\ \bar{S}_n = \dfrac{b + \frac{\nu\gamma}{\mu_3+\nu}\bar{I}_n}{\mu_1 + \nu + \beta\underline{I}_n}, \quad \bar{R}_n = \dfrac{\gamma}{\mu_3+\nu}\bar{I}_n. \end{cases} \tag{4.9}$$

By Lemma 3.3, (4.7) and (4.8), we have

$$\underline{I}_0 \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \bar{I}_1. \tag{4.10}$$

Lemma 4.4. *For the sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\bar{R}_n\}_{n=1}^{\infty}$, $\{\underline{R}_n\}_{n=1}^{\infty}$ defined by* (4.8) *and* (4.9)*, assume $\underline{I}_0 < \bar{I}_1$. Then, it holds that*

$$\underline{I}_0 < \underline{I}_1 < \bar{I}_1, \tag{4.11}$$

if and only if,

$$\frac{\mu_3+2\nu}{\mu_1}\frac{\gamma}{\mu_3+\nu} + \frac{\mu_2}{\mu_1} < 2. \tag{4.12}$$

Moreover, in this case, the three sequences $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\underline{R}_n\}_{n=1}^{\infty}$ are strongly monotonically increasing and converge to I^*, S^* and R^*, respectively, and the three sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\bar{R}_n\}_{n=1}^{\infty}$ are strongly monotonically decreasing and converge to I^*, S^* and R^*, respectively, as n tends to $+\infty$.

Proof. By (4.8), we have that for $\underline{I}_n < \bar{I}_n$ and $n = 1, 2, 3, \cdots$,

$$\left(1 - \frac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)(\bar{I}_n - \underline{I}_n) = \left(\frac{\mu_3+\nu}{\mu_1}\frac{\gamma}{\mu_3+\nu} + \frac{\mu_2}{\mu_1} - 1\right)(\bar{I}_n - \underline{I}_{n-1}).$$

Hence, we obtain that for $\underline{I}_n < \bar{I}_n$,

$$\left(1 - \frac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)(\bar{I}_n - \underline{I}_n) = \left(\frac{\mu_3+\nu}{\mu_1}\frac{\gamma}{\mu_3+\nu} + \frac{\mu_2}{\mu_1} - 1\right)(\bar{I}_n - \underline{I}_{n-1}), \ n = 1, 2, 3, \cdots,$$

from which one can see that (4.11) holds, if and only if,

$$1 - \frac{\nu\gamma}{\mu_1(\mu_3+\nu)} > \frac{\mu_3+\nu}{\mu_1}\frac{\gamma}{\mu_3+\nu} + \frac{\mu_2}{\mu_1} - 1, \tag{4.13}$$

which is equivalent to (4.12). Then, by the monotonicity and (4.8), we can prove that $\underline{I}_{n-1} < \underline{I}_n < \bar{I}_n < \bar{I}_{n-1}$, $n = 2, 3, \cdots$, and the three sequences $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\underline{R}_n\}_{n=1}^{\infty}$ are strongly monotonically increasing and converge to $\underline{I}^*$, $\underline{S}^*$ and $\underline{R}^*$, respectively, and the three sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\bar{R}_n\}_{n=1}^{\infty}$ are strongly monotonically decreasing and converge to $\bar{I}^*$, $\bar{S}^*$ and $\bar{R}^*$, respectively, as n tends to $+\infty$. Moreover, we have

$$\begin{cases} \lim_{n\to+\infty} \underline{I}_n = \underline{I}^* \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \lim_{n\to+\infty} \bar{I}_n = \bar{I}^*, \\ \lim_{n\to+\infty} \underline{S}_n = \underline{S}^* \le \liminf_{t\to+\infty} S(t) \le \limsup_{t\to+\infty} S(t) \le \lim_{n\to+\infty} \bar{S}_n = \bar{S}^*, \\ \lim_{n\to+\infty} \underline{R}_n = \underline{R}^* \le \liminf_{t\to+\infty} R(t) \le \limsup_{t\to+\infty} R(t) \le \lim_{n\to+\infty} \bar{R}_n = \bar{R}^*, \end{cases} \tag{4.14}$$

and

$$\begin{cases} \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)\bar{I}^* + \left(\dfrac{\mu_3+\nu}{\mu_1}\dfrac{\gamma}{\mu_3+\nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\underline{I}^* + \dfrac{\mu_2+\gamma}{\beta} = \dfrac{b}{\mu_1}, \\ \left(1 - \dfrac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)\underline{I}^* + \left(\dfrac{\mu_3+\nu}{\mu_1}\dfrac{\gamma}{\mu_3+\nu} + \dfrac{\mu_2}{\mu_1} - 1\right)\bar{I}^* + \dfrac{\mu_2+\gamma}{\beta} = \dfrac{b}{\mu_1}. \end{cases} \tag{4.15}$$

We thus obtain

$$\left(1 - \frac{\nu\gamma}{\mu_1(\mu_3+\nu)}\right)(\bar{I}^* - \underline{I}^*) = \left(\frac{\mu_3+\nu}{\mu_1}\frac{\gamma}{\mu_3+\nu} + \frac{\mu_2}{\mu_1} - 1\right)(\bar{I}^* - \underline{I}^*).$$

By the condition (4.13), we obtain $\bar{I}^* = \underline{I}^* = I^*$, from which by (1.8) and (4.9), we can derive $\bar{S}^* = \underline{S}^* = S^*$ and $\bar{R}^* = \underline{R}^* = R^*$. $\square$

Proof of Theorem 1.2. By Lemma 3.1, if $R_0 > 1$, there exists a unique virus equilibrium E^*. Here the condition (4.12) is equivalent to

$$(2\mu_1 - \mu_2 - 2\gamma)\nu > \{\gamma - (2\mu_1 - \mu_2)\}\mu_3. \tag{4.16}$$

The conditions (1.10) and (1.11) in Theorem 1.2 satisfy (4.16) and if $2\mu_1 - \mu 2 \le \gamma$, then (4.16) does not holds. Thus, by Lemma 4.4, we obtain the second part of Theorem 1.2, that is, under the conditions (1.10) and (1.11), E^* is globally asymptotically stable for any $\tau_2 > 0$. For the case $\mu_1 = \mu_2 = \mu_3 = \mu$, the conditions (1.10) and (1.11) have the form

$$2\gamma \le \mu, \tag{4.17}$$

and

$$\gamma < \mu < 2\gamma, \quad \text{and} \quad \frac{\mu - \gamma}{2\gamma - \mu} < \nu < \mu, \tag{4.18}$$

, respectively, from which we can conclude the remaining part of this theorem. $\square$

5 Stability analysis by Lyapunov functional techniques for $R_0 > 1$

In this section, we consider the global stability of the virus equilibrium of system (1.3) by Lyapunov functional techniques for $R_0 > 1$ (cf. Ren *et al.* [19, Proof of Theorem 3.3]).

By introducing the variables

$$x(t) = S(t) - S^*, \ y(t) = I(t) - I^*, \ z(t) = R(t) - R^*, \tag{5.1}$$

system (1.3) is centered at the virus equilibrium E^* and is rewritten as

$$\begin{cases} \dot{x} = -\mu_1 x - \beta I^* \mathrm{e}^{-\mu\tau_1} x - \beta S^* \mathrm{e}^{-\mu\tau_1} y(t-\tau_1) + \nu z(t-\tau_2), \\ \dot{y} = \beta I^* \mathrm{e}^{-\mu\tau_1} x + \beta S^* \mathrm{e}^{-\mu\tau_1} y(t-\tau_1) - (\mu_2 + \gamma) y, \\ \dot{z} = \gamma y - (\mu_3 + \nu) z. \end{cases} \tag{5.2}$$

Proof of Theorem 1.3. Let us introduce the following function:

$$V(x,y,z) = \omega \frac{(x+y)^2}{2} + \frac{y^2 + z^2}{2}, \tag{5.3}$$

where $\omega > 0$ is a real constant. The derivative of V along the solutions of system (5.2) is

$$\begin{aligned} \dot{V}(x,y,z) &= \omega(x+y)(\dot{x}+\dot{y}) + y\dot{y} + z\dot{z} \\ &= \omega(x+y)\{-\mu_1 x + \nu z(t-\tau_2) - (\mu_2+\gamma)y\} \\ &\quad + y\{\beta I^* \mathrm{e}^{-\mu_1\tau_1} x + \beta S^* \mathrm{e}^{-\mu_1\tau_1} y(t-\tau_1) - (\mu_2+\gamma)y\} + z\{\gamma y - (\mu_3+\nu)z\} \\ &= -\omega\mu_1 x^2 - (\omega+1)(\mu_2+\gamma)y^2 - (\mu_3+\nu)z^2 \\ &\quad - \{\omega(\mu_1+\mu_2+\gamma) - \beta I^* \mathrm{e}^{-\mu_1\tau_1}\}xy + \gamma yz \\ &\quad + \omega\nu y z(t-\tau_2) + \beta S^* \mathrm{e}^{-\mu_1\tau_1} y y(t-\tau_1) + \omega\nu x z(t-\tau_2). \end{aligned}$$

Now, set $\omega(\mu_1+\mu_2+\gamma) - \beta I^* \mathrm{e}^{-\mu_1\tau_1} = 0$, that is, $\omega = \frac{\beta I^* \mathrm{e}^{-\mu_1\tau_1}}{\mu_1+\mu_2+\gamma}$. Then, by the equation $\beta S^* \mathrm{e}^{-\mu_1\tau_1} = \mu_2 + \gamma$, we have

$$\begin{aligned} \dot{V}(x,y,z) = &-\omega\mu_1 x^2 - (\omega+1)(\mu_2+\gamma)y^2 - (\mu_3+\nu)z^2 + \gamma yz \\ &+ \omega\nu y z(t-\tau_2) + (\mu_2+\gamma) y y(t-\tau_1) + \omega\nu x z(t-\tau_2). \end{aligned}$$

This plus Cauchy-Schwartz inequality yields

$$\begin{aligned} \dot{V}(x,y,z) &\le -\omega\mu_1 x^2 - (\omega+1)(\mu_2+\gamma)y^2 - (\mu_3+\nu)z^2 + \frac{\gamma(y^2+z^2)}{2} \\ &\quad + \frac{\omega\nu\{y^2 + z^2(t-\tau_2)\}}{2} + \frac{(\mu_2+\gamma)\{y^2 + y^2(t-\tau_1)\}}{2} + \frac{\omega\nu\{x^2 + z(t-\tau_2)\}}{2} \\ &= -\omega\left(\mu_1 - \frac{\nu}{2}\right)x^2 - \left\{(\omega+1)(\mu_2+\gamma) - \frac{\omega\nu+\gamma}{2} - \frac{\mu_2+\gamma}{2}\right\}y^2 \\ &\quad - \left(\mu_3 + \nu - \frac{\gamma}{2}\right)z^2 + \frac{\omega\nu}{2}z^2(t-\tau_2) + \frac{\mu_2+\gamma}{2}y^2(t-\tau_1). \end{aligned}$$

We choose the Lyapunov functional of the form

$$V_1(x,y,z) = V(x,y,z) + \frac{\omega\nu}{2}\int_{t-\tau_2}^{t} z^2(\theta)d\theta + \frac{\mu_2+\gamma}{2}\int_{t-\tau_1}^{t} y^2(\theta)d\theta.$$

Therefore,

$$\begin{aligned}
\dot V_1(x,y,z) &= \dot V(x,y,z) + \frac{\omega\nu}{2}\{z^2 - z^2(t-\tau_2)\} + \frac{\mu_2+\gamma}{2}\{y^2 - y^2(t-\tau_1)\} \\
&\le -\omega\Big(\mu_1 - \frac{1}{2}\nu\Big)x^2 - \Big\{(\omega+1)(\mu_2+\gamma) - \frac{1}{2}(\omega\nu+\gamma) - \frac{\mu_2+\gamma}{2}\Big\}y^2 \\
&\quad - \Big(\mu_3 + \nu - \frac{1}{2}\gamma\Big)z^2 + \frac{1}{2}\omega\nu z^2 + \frac{\mu_2+\gamma}{2}y^2 \\
&= -\omega\Big(\mu_1 - \frac{1}{2}\nu\Big)x^2 - \Big\{(\omega+1)(\mu_2+\gamma) - \frac{1}{2}(\omega\nu+\gamma) - (\mu_2+\gamma)\Big\}y^2 - \Big(\mu_3+\nu-\frac{1}{2}\gamma - \frac{1}{2}\omega\nu\Big)z^2 \\
&= -\frac{\omega(2\mu_1-\nu)}{2}x^2 - \frac{\omega(2\mu_2+2\gamma-\nu)-\gamma}{2}y^2 - \frac{(2\mu_3+2\nu-\gamma)-\omega\nu}{2}z^2.
\end{aligned}$$

By (1.5), $2\mu_1 - \nu > 0$, and further if

$$\omega(2\mu_2+2\gamma-\nu)-\gamma > 0, \text{ and } 2\mu_3 + 2\nu - \gamma - \omega\nu > 0, \tag{5.4}$$

then $\dot V_1(x,y,z) \le 0$. By Lemma 3.3 and Lyapunov-LaSalle asymptotic stability theorem, we derive $\lim_{x\to+\infty} x(t) = 0$, $\lim_{y\to+\infty} y(t) = 0$ and $\lim_{z\to+\infty} z(t) = 0$.

Assume that $2\mu_3 - \gamma > 0$. If $\omega \le 2$, then $2\mu_3 + 2\nu - \gamma - \omega\nu = 2\mu_3 - \gamma + (2-\omega)\nu > 0$. Hence the condition (5.4) has the form $\nu < \frac{2\omega(\mu_2+\gamma)-\gamma}{\omega}$. If $\omega > 2$, then $\nu < \min(\frac{2\omega(\mu_2+\gamma)-\gamma}{\omega}, \frac{2\mu_3-\gamma}{\omega-2})$. Thus, if

$$\begin{cases} \nu < \dfrac{2\omega(\mu_2+\gamma)-\gamma}{\omega}, & \text{if } \omega \le 2, \\ \nu < \min\Big(\dfrac{2\omega(\mu_2+\gamma)-\gamma}{\omega}, \dfrac{2\mu_3-\gamma}{\omega-2}\Big), & \text{if } \omega > 2, \end{cases} \tag{5.5}$$

then (5.4) is satisfied. Moreover, it holds that

$$\begin{cases} \dfrac{2\omega(\mu_2+\gamma)-\gamma}{\omega} - \mu_1 = \dfrac{\omega\{2(\mu_2+\gamma)-\mu_1\}-\gamma}{\omega}, \\ \dfrac{2\mu_3-\gamma}{\omega-2} - \mu_1 = \dfrac{2+(2\mu_3-\gamma)/\mu_1-\omega}{\mu_1(\omega-2)}. \end{cases}$$

Hence, if (1.14) holds, the inequalities (5.4) hold, too. If $\mu_1 = \mu_2 = \mu_3 = \mu$, the condition (1.14) coincides (1.15). $\square$

6 Numerical examples

In this section, we use Matlab to investigate our results by plotting the solutions $(S(t), I(t), R(t))$ for some initial conditions.

The first example shows that $R(t)$ becomes negative at some $t > 0$. Therefore there are cases that system (1.1) does not generate a positive solution (see Lemma A.1 in Appendix).

Example 6.1. By taking $b = 4$, $\beta = 1$, $\nu = 1.2$, $\gamma = 2$, $\mu_1 = 1$, $\mu_2 = 1.5$, $\mu_3 = 1$, $\tau_1 = 0.5$, $\tau_2 = 1.5$ for (1.1), we obtain Figure 1 indicating $R(t)$ becomes negative for some $t > 0$.

Example 6.2 illustrates a case where $R_0 < 1$ for Theorem 1.1.

Example 6.2. We take $b = 5$, $\beta = 1$, $\nu = 1.2$, $\gamma = 2$, $\mu_1 = 1$, $\mu_2 = 1.5$, $\mu_3 = 1$, $\tau_1 = 0.5$, $\tau_2 = 1.5$. Then, $R_0 = 0.8664724\ldots < 1$ and by Theorem 1.1, the virus-free equilibrium $E^0 = (5,0,0)$ is globally asymptotically stable (see Figure 2).

Next, in Examples 6.3 and 6.4, we investigate the sufficient conditions in Theorem 1.3 under which E^* is globallly asymptotically stable.

Example 6.3. We take $b = 17$, $\beta = 1.2$, $\nu = 0.8$, $\gamma = 1.5$, $\mu_1 = \mu_2 = \mu_3 = 1.2$, $\tau_1 = 1.0$ and $\tau_2 = 1.5$. We have $R_0 = 1.8964\cdots > 1$, $0 < \gamma = 1.5 < \mu + \frac{2\omega(\mu+\gamma)-\gamma}{\omega} = 1.1701\cdots$ and $\frac{\gamma}{2(\mu+\gamma)} = 0.2777\cdots < \omega = 0.3546\cdots < \frac{\gamma}{\mu+2\gamma} = 0.3571\cdots$. In this case, the first part of (1.15) is satisfied. Thus, by Theorem 1.3, the virus equilibrium $E^* = (S^*, I^*, R^*) = (7.4702\cdots, 3.8265\cdots, 2.8698\cdots)$ is globally asymptotically stable (see Figure 3).

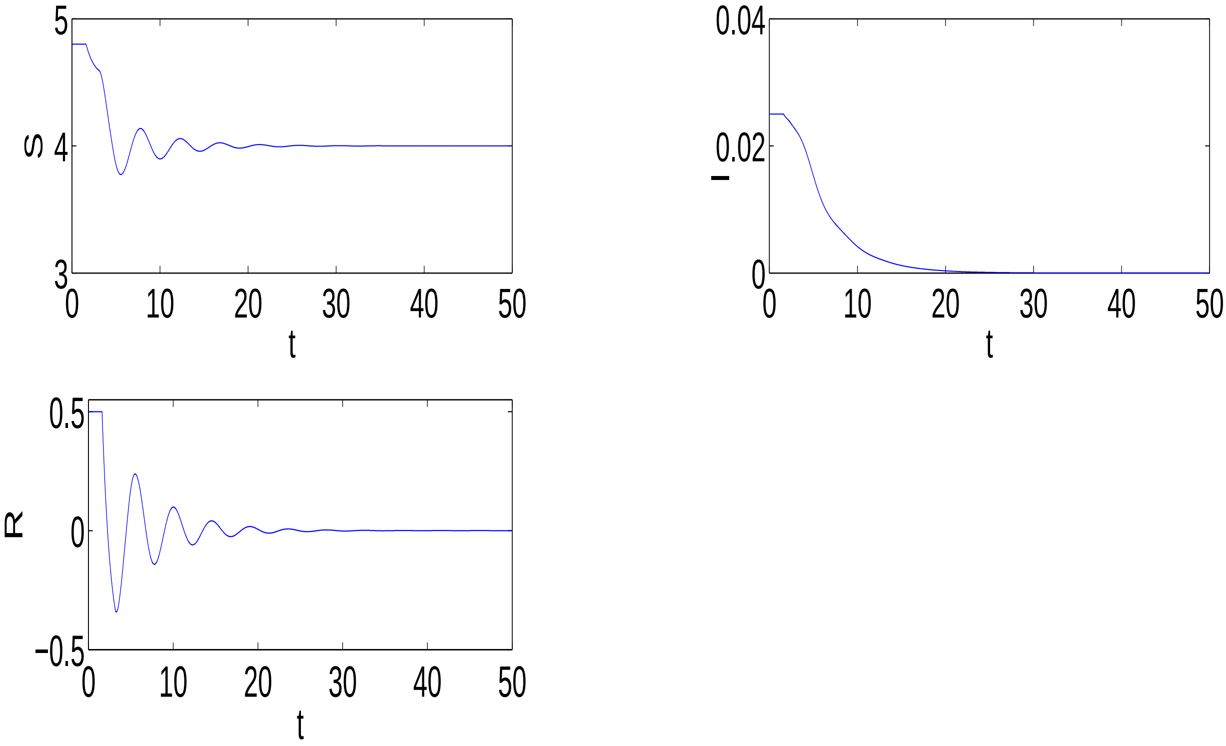

Figure 1: The graphs of $S(t)$, $I(t)$ and $R(t)$ in Example 6.1. Here $R(t)$ becomes negative for some $t > 0$.

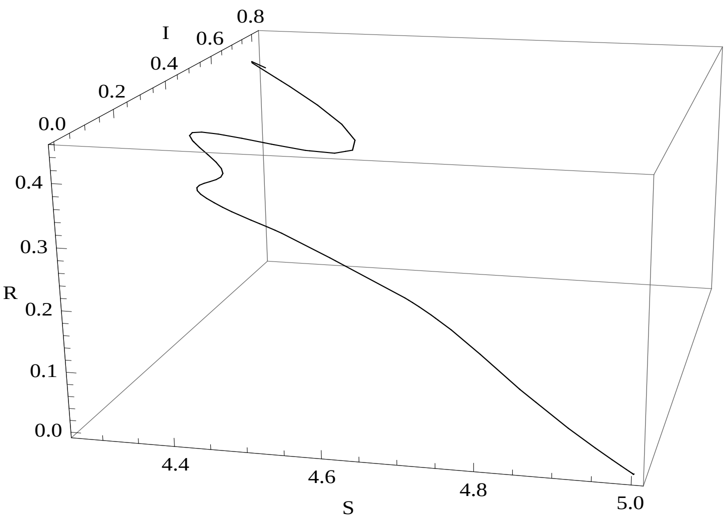

Figure 2: Phase portrait of system (1.3) for $R_0 < 1$. By Theorem 1.1, the virus-free equilibrium $E^0 = (5, 0, 0)$ is globally asymptotically stable.

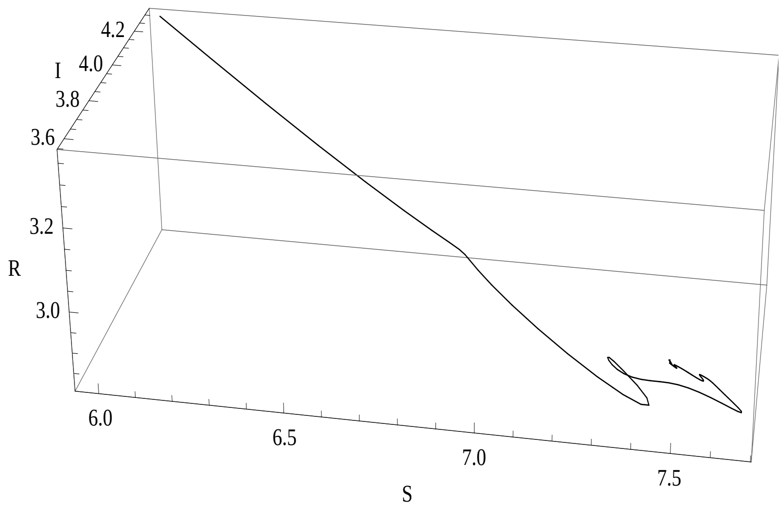

Figure 3: Phase portrait of system (1.3) for $R_0 > 1$. Since the first part of (1.15) is satisfied, by Theorem 1.3, the virus equilibrium $E^* = (7.4702\cdots, 3.8265\cdots, 2.8698\cdots)$ is globally asymptotically stable.

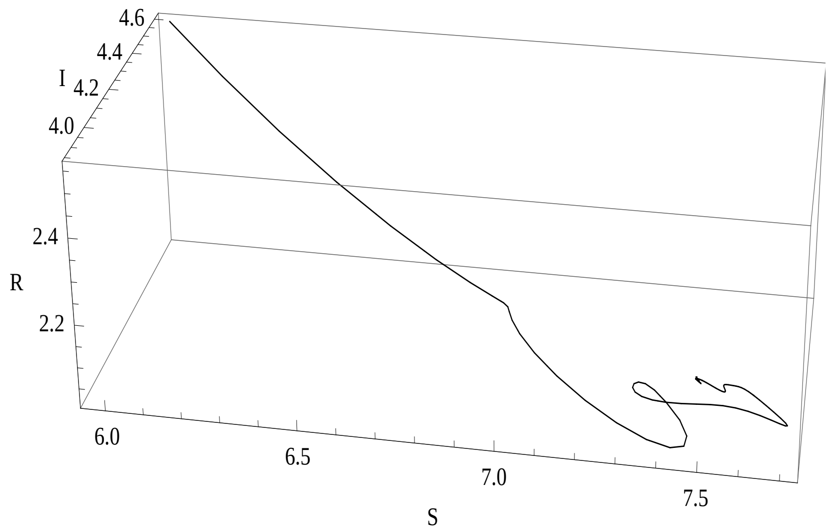

Figure 4: Phase portrait of system (1.3) for $R_0 > 1$. For the case that the sufficient conditions (1.14) in Theorem 1.3 are not satisfied, it seems that the virus equilibrium $E^* = (7.4702\cdots, 4.1208\cdots, 2.0604\cdots)$ is globally asymptotically stable.

Example 6.4. We take $b = 17$, $\beta = 1.2$, $\nu = 1.5$, $\gamma = 1.5$, $\mu_1 = 1.2$, $\mu_2 = 1.2$, $\mu_3 = 1.5$, $\tau_1 = 1.0$, $\tau_2 = 1.5$. Then, $R_0 = 1.8964\cdots > 1$, $\omega = 0.0381\cdots$ and it does not satisfy (1.14) in Theorem 1.3. However, Figure 4 seems to show that the virus equilibrium $E^* = (S^*, I^*, R^*) = (7.4702\cdots, 4.1208\cdots, 2.0604\cdots)$ is globally asymptotically stable.

Finally, by Examples 6.5 and 6.6, we investigate the asymptotic behavior of system (1.3) for the case that the sufficient conditions in Theorems 1.2 and 1.3 are not satisfied.

Example 6.5. We take $b = 17$, $\beta = 1.2$, $\nu = 0.8$, $\gamma = 1.5$, $\mu_1 = 1.2$, $\mu_2 = 2.5$, $\mu_3 = 1.2$, $\tau_1 = 1.0$, $\tau_2 = 1.5$. Then, $R_0 = 1.2800\cdots > 1$ and it does not satisfy $2\mu_1 - \mu_2 > 0$ in Theorem 1.2. However, Figure 5 seems to show that the virus equilibrium $E^* = (S^*, I^*, R^*) = (11.0671\cdots, 1.0939\cdots, 8.2048\cdots)$ is globally asymptotically stable.

Example 6.6. We take $b = 17$, $\beta = 1.2$, $\nu = 0.8$, $\gamma = 2.1$, $\mu_1 = \mu_2 = \mu_3 = 1$, $\tau_1 = 1.0$, $\tau_2 = 1.5$. Then, $R_0 = 2.4208\cdots > 1$, $\omega = 0.0495\cdots$ and it does not satisfy $\frac{1}{2}\gamma < \mu$ in Theorem 1.3. However, Figure 6 seems to show that the virus equilibrium $E^* = (S^*, I^*, R^*) = (7.0222\cdots, 4.6051\cdots, 5.3726\cdots)$ is globally asymptotically stable.

We leave it as an open question to analyze the phenomena in Figures 4-6.

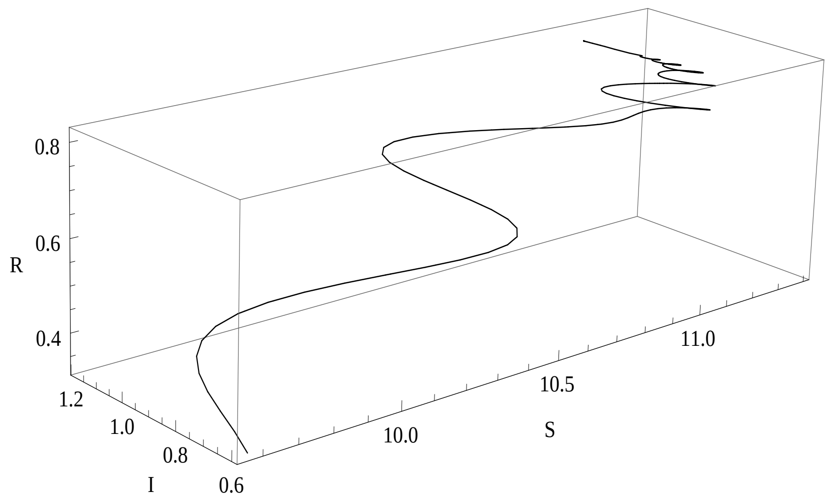

Figure 5: Phase portrait of system (1.3) for $R_0 > 1$ and the condition that $2\mu_1 - \mu_2 > 0$ does not hold. For the case that the sufficient conditions (1.10) and (1.11) in Theorem 1.2 are not satisfied, it seems that the virus equilibrium $E^* = (11.0671\cdots, 1.0939\cdots, 8.2048\cdots)$ is globally asymptotically stable.

7 Concluding remarks

By introducing a force of infection into the transmission SIR model, Ren *et al.* [19] and Han and Tan [5] proposed SIRS computer virus propagation models with delays. However, their models do not ensure positiveness of the solutions. We have proposed a revised SIRS computer virus propagation model with delays.

The new system always has a positive solution. However there occurs a problem; we need some additional restrictions on $\nu > 0$ and γ for eventual boundedness property of solutions and the global stability.

For $R_0 < 1$, we established the global asymptotic stability of the virus-free equilibrium (see Theorem 1.1), and for $R_0 > 1$, we obtained the sufficient conditions of the global asymptotic stability of virus equilibrium by applying monotone iterative techniques of Muroya *et al.* [15] (see Theorem 1.2).

Moreover, for $R_0 > 1$, $\nu > 0$ and $\tau_2 > 0$, by applying the similar Lyapunov functional approach as by Ren et al. [19, Theorem 3.3], we also obtained another sufficient conditions for the global asymptotic stability of virus equilibrium of (1.3) (see Theorem 1.3).

By applying monotone iterative techniques and Lyapunov functional techniques, we derived some answers on the global dynamics of a computer virus propagation model, but there still remains many questions on computer virus models. How to develop our results for such models will become one of our future research.

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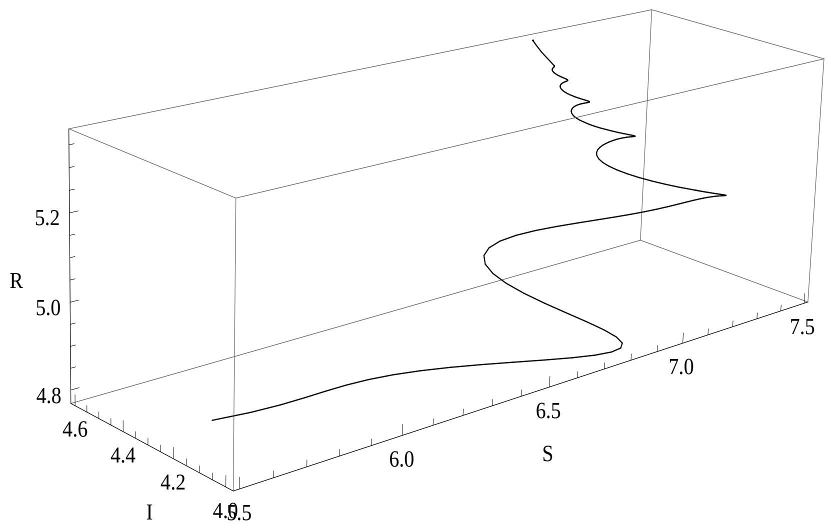

Figure 6: Phase portrait of system (1.3) for $R_0 > 1$ and the condition that $\frac{1}{2}\gamma < \mu$ does not hold. For the case that the sufficient conditions (1.15) in Theorem 1.3 are not satisfied, it seems that the virus equilibrium $E^* = (7.0222\cdots, 4.6051\cdots, 5.3726\cdots)$ is globally asymptotically stable.

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Appendix

We consider the following model given in Ren *et al.* [19]:

$$\begin{cases} \dfrac{dS(t)}{dt} = b - \beta S(t)I(t-\tau_1)\mathrm{e}^{-\mu\tau_1} + \nu R(t-\tau_2) - \mu S(t), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau_1)\mathrm{e}^{-\mu\tau_1} - (\mu+\gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \nu R(t-\tau_2) - \mu R(t), \end{cases} \tag{A.1}$$

with the initial conditions

$$\begin{aligned} &S(\theta) = \phi_1(\theta) \ge 0,\ I(\theta) = \phi_2(\theta) \ge 0,\ R(\theta) = \phi_3(\theta) \ge 0,\ \theta \in [-\bar{\tau}, 0], \\ &\bar{\tau} = \max(\tau_1, \tau_2),\ \text{with } \phi_1(0) > 0,\ \phi_2(0) > 0,\ \phi_3(0) > 0. \end{aligned} \tag{A.2}$$

Lemma A.1 *For (A.1), assume that $R_0 = \frac{\beta \mathrm{e}^{-\mu\tau_1} b}{\mu(\mu+\gamma)} < 1$. Then, there exist positive functions $\phi_2(\theta)$ and $\phi_3(\theta)$ in (A.2) and positive constants $W(0)$ and τ_2 such that*

$$\begin{cases} 2W(0) = \phi_2^2(0) + (\mu+\gamma)\displaystyle\int_{-\tau_1}^{0} \phi_2(\theta)d\theta < \dfrac{\nu^2\phi_3^2(0)}{\gamma^2}, \\ \phi_3(\theta) \ge \phi_3(0) > 0 \quad \textit{for any } -\tau_2 \le \theta \le 0, \\ \mathrm{e}^{-\mu\tau_2} < \dfrac{\tilde{\nu}}{\mu+\tilde{\nu}}, \quad \textit{for} \quad \tilde{\nu} = \nu - \dfrac{\gamma\sqrt{2W(0)}}{\phi_3(0)} > 0, \end{cases} \tag{A.3}$$

and the solution $R(t)$ in (A.1) has the form

$$R(\tau_2) < 0. \tag{A.4}$$

Proof. Assume $R_0 < 1$ and suppose that (A.3) holds. Then we introduce the following functional:

$$W(t) = \frac{I(t)^2}{2} + \frac{\mu+\gamma}{2}\int_{t-\tau_1}^{t} I^2(\theta)d\theta. \tag{A.5}$$

Similar to (2.16) in the proof of Theorem 1.1, we obtain

$$I(t) \le \sqrt{2W(t)}, \quad \text{and} \quad \frac{dW(t)}{dt} \le 0,\ t \ge 0.$$

Since $R_0 < 1$, $W(t)$ is a monotone decreasing function of $t \ge 0$ and hence, $I(t) \le \sqrt{2W(t)} \le \sqrt{2W(0)}$ for any $t \ge 0$. From the third equation of (A.1) and the initial condition $R(\theta) = \phi_3(\theta)$, $\theta \in [-\tau_2, 0]$ in (A.3), we have that

$$\begin{aligned} \frac{dR(t)}{dt} =& \gamma I(t) - \nu\phi_3(t-\tau_2) - \mu R(t) \\ \le& \gamma\sqrt{2W(0)} - \nu\phi_3(0) - \mu R(t) \\ =& -\tilde{\nu}\phi_3(0) - \mu R(t),\ \text{for } t \in [0, \tau_2]. \end{aligned}$$

Then, by the comparison theorem,

$$
\begin{aligned}
R(t) &\leq -\frac{\tilde{\nu}\phi_3(0)}{\mu} + \left(\phi_3(0) + \frac{\tilde{\nu}\phi_3(0)}{\mu}\right)\mathrm{e}^{-\mu t} \\
&= \left\{-\frac{\tilde{\nu}}{\mu} + \left(1 + \frac{\tilde{\nu}}{\mu}\right)\mathrm{e}^{-\mu t}\right\}\phi_3(0), \\
&= \frac{1}{\mu}\{-\tilde{\nu} + (\mu + \tilde{\nu})\mathrm{e}^{-\mu t}\}\phi_3(0), \quad \text{for} \quad 0 \leq t \leq \tau_2.
\end{aligned}
$$

By the condition (A3) for $\tau_2 > 0$, we obtain (A.4). $\square$