

## Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates

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**Abstract.** In this paper, by applying a variation of the backward Euler method, we propose a discrete SIR epidemic model whose discretization scheme preserves the global asymptotic stability of equilibria for a class of corresponding continuous-time SIR epidemic models. Using discrete-time analogue of Lyapunov functionals, the global asymptotic stability of the equilibria is fully determined by the basic reproduction number  $R_0$  when the infection incidence rate has a suitable monotone property.

**Keywords:** Difference equation, global asymptotic stability, SIR epidemic model, basic reproduction number, nonstandard finite discretization.

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## 1 Introduction

The dynamics of epidemic models have received considerable attention. Various mathematical models have been proposed in the literature (see also [1–29] and the references therein). To investigate the dynamical behavior of the transmission of infectious diseases in a long time scale, the following basic SIR model was introduced by Hethcote [8].

$$\begin{cases} \frac{dS(t)}{dt} = \lambda - \mu S(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \end{cases} \quad (1.1)$$

$S(t)$ ,  $I(t)$  and  $R(t)$  denote the proportions of the susceptible, infective and recovered individuals, respectively. It is assumed that all newborns are susceptibles.  $\lambda$  and  $\mu$  are the birth rate and death rate of the population, respectively.

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$\gamma$  is a recovery rate and  $\beta$  denotes the disease transmission coefficient. In general, SIR models have been thought of as an appropriate framework for describing transmission of viral agent diseases such as measles, mumps and smallpox [8].

Since nonlinearity on the incidence rates of several disease transmissions has been observed, many authors have suggested that the standard bilinear incidence rate should be modified into a nonlinear incidence rate. For example, Capasso and Serio [4] introduced an incidence rate which takes a form  $\frac{\beta S(t)I(t)}{1+\alpha I(t)}$  for modeling the spread of cholera in Bari. Brown and Hasibuan [3] developed an infection model of the two-spotted spider mites, *Tetranychus urticae* and introduced an incidence rate which takes a form  $(S(t)I(t))^b$ . Thereafter, in order to study the impact of the nonlinearity on the disease dynamics qualitatively, Korobeinikov and Maini [15] considered a variety of models with an incidence rate of the form  $\phi(S(t))\psi(I(t))$  and constructed Lyapunov functions to analyze their global properties. Korobeinikov [16,17] investigated global properties for a variety of epidemic models with a nonseparable incidence rate which has a more general form. Based on the ideas in Korobeinikov and Maini [15], Huang *et al.* [9] incorporated a time delay which is caused by a latency period of the infection in a vector and studied the following SIR epidemic model with a general nonlinear incidence rate.

$$\begin{cases} \frac{dS(t)}{dt} = \lambda - \mu S(t) - \phi(S(t))\psi(I(t-\tau)), \\ \frac{dI(t)}{dt} = \phi(S(t))\psi(I(t-\tau)) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \quad \tau \geq 0, \end{cases} \quad (1.2)$$

For system (1.2), the rate of new infection is characterized by  $\phi(S(t))\psi(I(t-\tau))$ , which includes some special incidence rates. For instance, if  $\phi(S) = \beta S$  and  $\psi(I) = I$ , then the incidence becomes the standard bilinear form proposed in Cooke [5]. If  $\phi(S) = \beta S$  and  $\psi(I) = \frac{I}{1+\alpha I}$ , then the incidence rate is of the form studied in [4, 20, 28] for a saturated effect. Other specific forms of  $\phi(S)$  can be found in Helmar and Wang [13]. Here,  $\tau$  denotes an incubation time which is needed for infectious agents to develop in a vector (see also [1, 5]). Recently, by using a Lyapunov functional technique in McCluskey [19], the complete global stability results for delayed SIR epidemic models with a nonseparable incidence rates are obtained in [7, 10, 21].

On the other hand, there occur situations such that constructing discrete epidemic models is more appropriate approach to understand disease transmission dynamics because they permit arbitrary time-step units. For example, Zhou *et al.* [29] formulated a discrete mathematical model to investigate the transmission of severe acute respiratory syndrome (SARS) and their simulation results match the statistical data well and indicate that early quarantine and a high quarantine rate are crucial to the control of SARS.

The need for a discretization of continuous models also arises from the fundamental realization. Since nonlinear ordinary differential equations generally do not have analytic solutions expressible in terms of a finite representation of the elementary functions, technical discretization is required to calculate good analytic approximations of the solutions [23].

Jang and Elaydi [14] showed that a nonstandard discretization scheme preserves the global stability of a disease-free equilibrium and the local stability of an endemic equilibrium of the corresponding continuous-time SIS epidemic model.

In addition, Izzo and Vecchio [11] and Izzo *et al.* [12] introduced a variation of the backward Euler discretization called “mixed type” formula and showed that their scheme preserves the positivity and boundedness of the corresponding continuous-time population dynamics model. Based on their ideas, Sekiguchi [24] studied the permanence of a special class of discrete SIR epidemic models and some discrete epidemic models with delays by applying techniques in Wang [27].

However, how to choose the discrete schemes which preserve the global asymptotic stability for equilibria of the corresponding continuous-time epidemic models was an open problem. In fact, it is known that the stability of a fixed point (equilibrium) sometime changes depending on the scheme (see, e.g., Roeger and Barnard [25] and the references therein).

Later, Enatsu *et al.* [6] established complete global stability analysis for a discrete SIR epidemic model with a bilinear incidence rate. Their results agree with those for a corresponding continuous SIR epidemic models in McCluskey [19].

In this paper, we establish the global asymptotic stability of equilibria for a discrete SIR epidemic model with a class of nonlinear incidence rates in which a variation of the backward Euler method is adopted. The main idea of the discretization comes from Enatsu *et al.* [6] and an application of nonstandard finite method given in Mickens [23]. Moreover, for the model, we can formulate a discrete-time analogue of Lyapunov functionals which are used for a class of continuous-time SIR epidemic models in [9, 16, 17, 19, 20]. This is the critical reason why a variation of the backward Euler method is applied and this discretization scheme is different from that of [11, 12, 24].

The organization of this paper is as follows. In Section 2, we introduce a discrete SIR epidemic model with a class of nonlinear incidence rates by applying a variation of backward Euler discretization and establish our main results. In Section 3, we prove the positivity, boundedness of the solution and the unique existence of an endemic equilibrium of the model. In Section 4, we prove the global asymptotic stability of a disease-free equilibrium using a key lemma (see Lemma 4.1). In Section 5, we prove the permanence of the model and the global asymptotic stability of the endemic equilibrium by discrete-time analogue of Lyapunov functional techniques using a key lemma (see Lemma 5.1). Finally, a concluding remark is offered in Section 6.

## 2 Main results

In this paper, by applying a variation of backward Euler discretization, we consider the following model.

$$\begin{cases} S(n+1) - S(n) = \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)), \\ I(n+1) - I(n) = \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - (\mu_2 + \gamma) I(n+1), \\ R(n+1) - R(n) = \gamma I(n+1) - \mu_3 R(n+1), \quad n \geq 0, \end{cases} \quad (2.1)$$

where  $\phi, \psi \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ ,  $\phi(0) = \psi(0) = 0$  and  $\lim_{I \rightarrow +0} (I/\psi(I)) = 1$ . The initial condition of system (2.1) is as follows.

$$S(j) = \varphi_{1,j} \geq 0, \quad I(j) = \varphi_{2,j} \geq 0, \quad R(j) = \varphi_{3,j} \geq 0, \quad j = -m, \dots, 0. \quad (2.2)$$

From a biological meaning, we further assume that  $\varphi_{i,0} > 0$  ( $i = 1, 2, 3$ ). The parameters  $\mu_i$  ( $i = 1, 2, 3$ ) represent the death rates of susceptible, infective and recovered individuals, respectively and  $\gamma$  represents the recovery rate of infectives. The infection rate is given by

$$\phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)),$$

where  $\sum_{j=0}^m f(j) = 1$ ,  $f(j) \geq 0$  for  $0 \leq j \leq m$  and the meaning of  $f(j)$  is derived from the fraction of vector population in which the maximum time taken to become infectious is  $m$  (see, e.g., [1, 9, 18–20]). All the coefficients  $\lambda$ ,  $\gamma$  and  $\mu_i$  ( $i = 1, 2, 3$ ) are assumed to be positive.

For system (2.1), Enatsu *et al.* [6] established a complete stability analysis for a special case  $\phi(S) = \beta S$  and  $\psi(I) = I$ . We note that system (2.1) is a discrete analog of continuous system given in Huang *et al.* [9] with distributed delays.

We define the basic reproduction number  $R_0$  of system (2.1) as follows:

$$R_0 = \frac{\phi(\lambda/\mu_1)}{\mu_2 + \gamma}.$$

$\frac{1}{\mu_2 + \gamma}$  denotes the average infection period and the relation that  $\lim_{I \rightarrow +0} \frac{\phi(\lambda/\mu_1)\psi(I)}{I} = \phi(\lambda/\mu_1)$  implies that  $\phi(\lambda/\mu_1)$  denotes the number of new cases infected per unit time by one infective individual at an initial infection state. Thus,  $R_0$  denotes the expected number of secondary infectious cases generated by one typical primary case in an entirely susceptible and sufficiently large population.  $R_0$  works well as the basic reproduction number for the corresponding continuous epidemic model (see Huang *et al.* [9]).

System (2.1) always has a disease-free equilibrium  $E_0 = (S_0, 0, 0)$ ,  $S_0 = \frac{\lambda}{\mu_1}$  and if  $R_0 > 1$ , system (2.1) may admit a unique endemic equilibrium  $E_* = (S^*, I^*, R^*)$ ,  $S^* > 0$ ,  $I^* > 0$ ,  $R^* > 0$  (see Theorem 3.1 for details).

Our main results are as follows.

**Theorem 2.1.** *Assume that the following conditions hold true.*

(H1)  $\phi(S)$  is strictly monotone increasing on  $S \geq 0$ ,

(H2)  $I/\psi(I)$  is monotone increasing on  $I > 0$ .

*Then, for system (2.1), there is no endemic equilibrium and the disease-free equilibrium  $E_0$  is globally asymptotically stable, if and only if,  $R_0 \leq 1$ .*

**Theorem 2.2.** *Assume that the conditions (H1) and (H2) hold true. Then, there exists a unique endemic equilibrium  $E_*$  for system (2.1) if and only if  $R_0 > 1$ . Furthermore, if the following condition holds true,*

(H3)  $\psi(I)$  is monotone increasing on  $I \geq 0$ ,

*then system (2.1) is permanent and the endemic equilibrium  $E_*$  of system (2.1) is globally asymptotically stable, if and only if,  $R_0 > 1$ .*

The above results indicate that the global asymptotic stability of the equilibria of system (2.1) is determined for any length of time delay under the conditions (H1)-(H3). It is shown that the disease can be eradicated if and only if  $R_0 \leq 1$  and the disease persists in a host population if and only if  $R_0 > 1$ . We further remark that the conditions (H1)-(H3), under which the global dynamics of system (2.1) are determined by the basic reproduction number  $R_0$ , are less restrictive than those in Huang *et al.* [9, Theorem 1].

### 3 Basic properties

For system (2.1), since the variable  $R$  does not appear in the first and the second equations, it is sufficient to consider the following 2-dimensional system.

$$\begin{cases} S(n+1) - S(n) = \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)), \\ I(n+1) - I(n) = \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - (\mu_2 + \gamma) I(n+1). \end{cases} \quad (3.1)$$

For the reduced system (3.1), at first, we show that the solution has positivity for  $n > 0$  and bounded above for sufficiently large  $n$ .

**Lemma 3.1.** *Let  $(S(n), I(n))$  be a solution of system (3.1) with the initial condition (2.2). Then  $S(n) > 0$  and  $I(n) > 0$  for all  $n > 0$ . Furthermore, any solution  $(S(n), I(n))$  of system (3.1) satisfies  $\limsup_{n \rightarrow +\infty} (S(n) + I(n)) \leq \lambda/\underline{\mu}$ , where  $\underline{\mu} = \min\{\mu_1, \mu_2 + \gamma\}$ .*

**Proof.** From the initial condition (2.2) and the first equation of system (3.1), we have

$$S(1) + \mu_1 S(1) + \phi(S(1)) \sum_{j=0}^m f(j) \psi(\varphi_{2,-j}) = \lambda + \varphi_{1,0} > 0.$$

Then, we easily obtain that  $S(1) > 0$ . By the second equation of system (3.1),

$$(1 + \mu_2 + \gamma)I(1) = \varphi_{2,0} + \phi(S(1)) \sum_{j=0}^m f(j) \psi(\varphi_{2,-j}) > 0,$$

which implies that  $I(1) > 0$ . By repeating the above discussion, we obtain that  $S(n) > 0$  and  $I(n) > 0$  for all  $n > 0$ .

We now define  $V(n) = S(n) + I(n)$ . From system (3.1), we have

$$V(n+1) - V(n) = \lambda - \mu_1 S(n+1) - (\mu_2 + \gamma)I(n+1) \leq \lambda - \underline{\mu}V(n+1),$$

from which we have  $\limsup_{n \rightarrow +\infty} V(n) \leq \frac{\lambda}{\underline{\mu}}$ . Hence, the proof is complete.  $\square$

**Remark 3.1.** For any nonnegative initial values  $\varphi_{i,j}$ , for  $i = 1, 2$  and  $j = -m, \dots, 0$ , by a similar method as that used in Lemma 3.1, the following statements are true.

- (i) The solution  $(S(n), I(n))$  of (3.1) exists and  $S(n) > 0$  ( $n > 0$ ),  $I(n) \geq 0$  ( $n \geq 0$ ).
- (ii) If  $\varphi_{2,0} + \sum_{j=0}^m f(j)\varphi_{2,-j} > 0$ , then the solution  $(S(n), I(n))$  of (3.1) exists and  $S(n) > 0$  ( $n > 0$ ),  $I(n) > 0$  ( $n > 0$ ).
- (iii) If  $\varphi_{2,0} + \sum_{j=0}^m f(j)\varphi_{2,-j} = 0$ , then the solution  $(S(n), I(n))$  of (3.1) exists and  $S(n) > 0$  ( $n > 0$ ),  $I(n) = 0$  ( $n \geq 0$ ).

System (3.1) always has a disease-free equilibrium  $\tilde{E}_0 = (S_0, 0)$ . Under the conditions (H1) and (H2), if  $R_0 > 1$ , then system (2.1) has a unique endemic equilibrium  $\tilde{E}_* = (S^*, I^*)$ .

**Theorem 3.1.** *Assume that the conditions (H1) and (H2) hold true. If  $R_0 > 1$ , then system (3.1) has a unique endemic equilibrium  $\tilde{E}_* = (S^*, I^*)$  satisfying*

$$\lambda - \mu_1 S^* - \phi(S^*)\psi(I^*) = 0, \quad \phi(S^*)\psi(I^*) - (\mu_2 + \gamma)I^* = 0. \quad (3.2)$$

*If  $R_0 \leq 1$ , then the disease-free equilibrium is the only equilibrium of system (3.1).*

**Proof.** At a fixed point  $(S, I)$  of system (3.1), the following equations hold.

$$\lambda - \mu_1 S - (\mu_2 + \gamma)I = 0, \quad \phi(S)\psi(I) - (\mu_2 + \gamma)I = 0. \quad (3.3)$$

Substituting the first equation of (3.3) into the second equation of (3.3), we consider the following equation:

$$H(I) := \phi\left(\frac{\lambda}{\mu_1} - \frac{\mu_2 + \gamma}{\mu_1}I\right) \frac{\psi(I)}{I} - (\mu_2 + \gamma) = 0.$$

By the hypothesis (H2),  $H$  is strictly monotone decreasing on  $(0, +\infty)$  satisfying

$$\lim_{I \rightarrow +0} H(I) = \phi\left(\frac{\lambda}{\mu_1}\right) - (\mu_2 + \gamma) = (\mu_2 + \gamma)(R_0 - 1) > 0,$$

and  $H(\frac{\mu_1}{\mu_2 + \gamma}) = -(\mu_2 + \gamma) < 0$  holds. Hence, there exists a unique  $0 < I^* < \frac{\mu_1}{\mu_2 + \gamma}$  such that  $H(I^*) = 0$ . By (3.3), we obtain  $S^* = 1 - \frac{\mu_2 + \gamma}{\mu_1}I^* > 0$ . This implies that (3.1) has a unique positive equilibrium  $\tilde{E}_*$ .  $\square$

## 4 Global stability of the disease-free equilibrium $E_0$ for $R_0 \leq 1$

In this section, in order to prove Theorem 2.1, we show the global stability of the disease-free equilibrium  $\tilde{E}_0$  of the reduced system (3.1) for  $R_0 \leq 1$ . First, we introduce the following lemma which plays a key role such that Lyapunov functional techniques for continuous-time SIR epidemic models in Huang *et al.* [9], Korobeinikov [16, 17] and McCluskey [19, 20] are applicable.

**Lemma 4.1.** *Under the condition (H1), it holds that*

$$\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds \geq \frac{x_2 - x_1}{\phi(x_2)},$$

for any  $x_1 > 0$  and  $x_2 > 0$ .

**Proof.** For the first case  $x_2 \geq x_1$ , we immediately see that  $\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds \geq \int_{x_1}^{x_2} \frac{1}{\phi(x_2)} ds = \frac{x_2 - x_1}{\phi(x_2)}$ . For the second case  $x_2 < x_1$ , we obtain that

$$\int_{x_1}^{x_2} \frac{1}{\phi(s)} ds = - \int_{x_2}^{x_1} \frac{1}{\phi(s)} ds \geq - \int_{x_2}^{x_1} \frac{1}{\phi(x_2)} ds = \frac{x_2 - x_1}{\phi(x_2)},$$

which completes the proof.  $\square$

**Remark 4.1.** By Lemma 4.1, if  $\phi(s) = s$ , then we obtain  $\ln \frac{x_2}{x_1} \geq \frac{x_2 - x_1}{x_2}$ .

**Theorem 4.1.** *Assume that the conditions (H1) and (H2) hold true. If  $R_0 \leq 1$ , then it holds that*

$$\lim_{n \rightarrow +\infty} S(n) = \frac{\lambda}{\mu_1}, \quad \lim_{n \rightarrow +\infty} I(n) = 0, \quad (4.1)$$

and  $\tilde{E}_0$  of system (3.1) is globally asymptotically stable.

**Proof.** From a Lyapunov functional for a continuous-time SIR epidemic model in Huang *et al.* [9], consider the following sequence  $\{U^0(n)\}_{n=0}^{+\infty}$  defined by

$$U^0(n) = U_1^0(n) + I(n) + U_+^0(n), \quad (4.2)$$

where

$$U_1^0(n) = S(n) - S_0 - \int_{S_0}^{S(n)} \frac{\phi(S_0)}{\phi(s)} ds, \quad U_+^0(n) = \phi(S_0) \sum_{j=0}^m f(j) \sum_{k=n-j}^n \psi(I(k)).$$

We now show that  $U^0(n+1) - U^0(n) \leq 0$  for any  $n \geq 0$ . First, we calculate  $U_1^0(n+1) - U_1^0(n)$ . By Lemma 4.1, we obtain

$$\begin{aligned} U_1^0(n+1) - U_1^0(n) &= S(n+1) - S(n) - \int_{S(n)}^{S(n+1)} \frac{\phi(S_0)}{\phi(s)} ds \\ &\leq S(n+1) - S(n) - \phi(S_0) \frac{S(n+1) - S(n)}{\phi(S(n+1))} \\ &= \frac{\phi(S(n+1)) - \phi(S_0)}{\phi(S(n+1))} (S(n+1) - S(n)) \\ &= \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left\{ \lambda - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - \mu_1 S(n+1) \right\}. \end{aligned} \quad (4.3)$$

Substituting  $\lambda = \mu_1 S_0$  into (4.3), we see that

$$\begin{aligned} U_1^0(n+1) - U_1^0(n) &\leq \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left\{ \mu_1 S_0 - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - \mu_1 S(n+1) \right\} \\ &= \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left\{ -\mu_1 (S(n+1) - S_0) - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) \right\}. \end{aligned}$$

Second, calculating  $U_+^0(n+1) - U_+^0(n)$ , we get

$$U_+^0(n+1) - U_+^0(n) = \phi(S_0) \sum_{j=0}^m f(j) \left\{ \sum_{k=n+1-j}^{n+1} \psi(I(k)) - \sum_{k=n-j}^n \psi(I(k)) \right\} = \phi(S_0) \sum_{j=0}^m f(j) \{ \psi(I(n+1)) - \psi(I(n-j)) \}.$$

Therefore, it holds that

$$\begin{aligned}
U^0(n+1) - U^0(n) &\leq -\left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left\{ \mu_1(S(n+1) - S_0) + \phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j)) \right\} \\
&\quad + \phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j)) - (\mu_2 + \gamma)I(n+1) \\
&\quad + \phi(S_0) \sum_{j=0}^m f(j)\{\psi(I(n+1)) - \psi(I(n-j))\} \\
&= -\mu_1 S(n+1) \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left(1 - \frac{S_0}{S(n+1)}\right) + \phi(S_0)\psi(I(n+1)) - (\mu_2 + \gamma)I(n+1) \\
&= -\mu_1 S(n+1) \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left(1 - \frac{S_0}{S(n+1)}\right) + (\mu_2 + \gamma) \left\{ R_0 \frac{\psi(I(n+1))}{I(n+1)} - 1 \right\} I(n+1).
\end{aligned}$$

By the condition (H2), we finally obtain

$$U^0(n+1) - U^0(n) \leq -\mu_1 S(n+1) \left(1 - \frac{\phi(S_0)}{\phi(S(n+1))}\right) \left(1 - \frac{S_0}{S(n+1)}\right) + (\mu_2 + \gamma)(R_0 - 1)I(n+1).$$

By the condition (H1), we have  $(1 - \frac{\phi(S_0)}{\phi(S(n+1))})(1 - \frac{S_0}{S(n+1)}) \leq 0$  with equality if and only if  $S(n+1) = S_0$  and hence  $U^0(n+1) - U^0(n) \leq 0$  holds for any  $n \geq 0$  since we have  $R_0 \leq 1$ . Then,  $\{U^0(n)\}_{n=0}^{+\infty}$  is monotone decreasing sequence which implies that there exists a  $\tilde{u}_0 := \lim_{n \rightarrow +\infty} U^0(n) \geq 0$ . Then,  $\lim_{n \rightarrow +\infty} (U^0(n+1) - U^0(n)) = 0$  holds. For the case  $R_0 < 1$ , we immediately see that  $\lim_{n \rightarrow +\infty} S(n+1) = S_0$  and  $\lim_{n \rightarrow +\infty} I(n+1) = 0$ . On the other hand, for the case  $R_0 = 1$ , we see that  $\lim_{n \rightarrow +\infty} S(n+1) = S_0$ , from which we obtain  $\lim_{n \rightarrow +\infty} I(n-j) = 0$  for  $0 \leq j \leq m$  by (3.1). Hence, it holds that  $\lim_{n \rightarrow +\infty} (S(n), I(n)) = (\frac{\lambda}{\mu_1}, 0)$  if  $R_0 \leq 1$ . Since  $U^0(n) \leq U^0(0)$  for all  $n \geq 0$  holds, we see that  $\tilde{E}_0$  is uniformly stable. This completes the proof.  $\square$

**Lemma 4.2.** (4.1) implies  $R_0 \leq 1$ .

**Proof.** Suppose that  $R_0 > 1$ . Then, by Theorem 3.1, one can see that there exists a positive constant solution  $(S(n), I(n)) = (S^*, I^*)$  of system (3.1), which contradicts the fact that (4.1) holds. Hence, we obtain the conclusion of this lemma.  $\square$

*Proof of Theorem 2.1.* By Theorems 3.1 and 4.1 and Lemma 4.2, we immediately obtain the conclusion of Theorem 2.1.  $\square$

## 5 Global stability of the endemic equilibrium $E_*$ for $R_0 > 1$

In this section, in order to prove Theorem 2.2, we show the global stability of the endemic equilibrium  $\tilde{E}_*$  of the reduced system (3.1) for  $R_0 > 1$ . First, by applying techniques in Enatsu *et al.* [6] and Sekiguchi [24], we show the permanence for  $R_0 > 1$ .

### 5.1 Permanence for $R_0 > 1$

For  $0 < q < \frac{\psi(I^*)}{I^*}$ , we put  $S^\Delta > S^*$  satisfying

$$S^\Delta(1 + \mu_1) + \phi(S^\Delta)qI^* = S^*(1 + \mu_1) + \phi(S^*)\psi(I^*). \quad (5.1)$$

Setting  $F(s) := s(1 + \mu_1) + \phi(s)qI^*$ , it follows that  $F(S^*) = S^*(1 + \mu_1) + \phi(S^*)qI^* < S^*(1 + \mu_1) + \phi(S^*)\psi(I^*)$  and  $\lim_{s \rightarrow +\infty} F(s) = +\infty$ . The above discussion guarantees the existence of  $S^\Delta$ .

We now prove the permanence of (3.1). From Theorem 5.1 below, the disease eventually persists in the host population if  $R_0 > 1$ .

**Theorem 5.1.** Assume that the conditions (H1)-(H3) hold true. If  $R_0 > 1$ , for any solution of system (3.1), it holds that

$$\liminf_{n \rightarrow +\infty} S(n) \geq \underline{v}_1 > 0, \quad \liminf_{n \rightarrow +\infty} I(n) \geq \underline{v}_2 := \left( \frac{1}{1 + \mu_2 + \gamma} \right)^{m+l_0} qI^* > 0, \quad (5.2)$$

where  $\underline{v}_1 > 0$  satisfies  $\lambda - \mu_1 \underline{v}_1 - \phi(\underline{v}_1)\psi(\frac{\lambda}{\mu_1}) = 0$  and  $0 < q < \frac{\psi(I^*)}{I^*}$ ,  $l_0 \geq 1$  satisfy

$$S^* \leq \frac{1}{\mu_1} \left\{ 1 - \left( \frac{1}{1 + \mu_1} \right)^{m+l_0} \right\} \{ \lambda - \phi(S^*)qI^* \}. \quad (5.3)$$

**Proof.** The existence of  $q$  and  $l_0$  is guaranteed, because it follows from (3.2) that  $\frac{1}{\mu_1}\{\lambda - \phi(S^*)qI^*\} = S^* + \frac{\phi(S^*)}{\mu_1}(\psi(I^*) - qI^*) > S^*$ . By the first equation of (3.1) and Lemma 3.1, for any  $\varepsilon > 0$ , there is an integer  $N_\varepsilon \geq 0$  such that

$$\begin{aligned} S(n+1) - S(n) &= \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j)) \\ &\geq \lambda - \mu_1 S(n+1) - \phi(S(n+1))\psi\left(\frac{\lambda}{\mu_1} + \varepsilon\right), \end{aligned} \quad (5.4)$$

for  $n \geq N_\varepsilon + m$ . Let us now consider the auxiliary equation  $\underline{S}(n+1) - \underline{S}(n) = \lambda - \mu_1 \underline{S}(n+1) - \phi(\underline{S}(n+1))\psi(\frac{\lambda}{\mu_1})$ . Then, by the condition (H1), one can immediately obtain that  $\lim_{n \rightarrow +\infty} \underline{S}(n) = \underline{v}_1 > 0$ . Since (5.4) holds for arbitrary  $\varepsilon > 0$  sufficiently small, it follows that  $\liminf_{n \rightarrow +\infty} S(n) \geq \underline{v}_1 > 0$ .

We now show that  $\liminf_{n \rightarrow +\infty} I(n) \geq \underline{v}_2 > 0$  holds. First, we claim that it is impossible that, for any solution  $(S(n), I(n))$  of system (3.1), there exists a nonnegative integer  $p_0 \geq m$  such that  $I(n) \leq qI^*$  for all  $n \geq p_0 - m$ . Suppose to the contrary that there exist a solution  $(S(n), I(n))$  of system (3.1) and a nonnegative integer  $p_0 \geq m$  such that  $I(n) \leq qI^*$  for all  $n \geq p_0 - m$ . From the first equation of system (3.1), one can obtain that,

$$\begin{aligned} S(n+1) &= \frac{1}{1+\mu_1} \left\{ S(n) + \lambda - \phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j)) \right\} \\ &= \frac{1}{1+\mu_1} \left[ \frac{1}{1+\mu_1} \left\{ S(n-1) + \lambda - \phi(S(n)) \sum_{j=0}^m f(j)\psi(I(n-1-j)) \right\} \right] \\ &\quad + \frac{\lambda}{1+\mu_1} - \frac{\phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j))}{1+\mu_1} \\ &= \left( \frac{1}{1+\mu_1} \right)^2 S(n-1) + \lambda \left\{ \frac{1}{1+\mu_1} + \left( \frac{1}{1+\mu_1} \right)^2 \right\} \\ &\quad - \frac{\phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j))}{1+\mu_1} - \frac{\phi(S(n)) \sum_{j=0}^m f(j)\psi(I(n-1-j))}{(1+\mu_1)^2}. \end{aligned}$$

By repeating the above discussion, for  $n \geq p_0$ , we have

$$\begin{aligned} S(n+1) &= \left( \frac{1}{1+\mu_1} \right)^{n-p_0+1} S(p_0) + \frac{\lambda}{\mu_1} \left\{ 1 - \left( \frac{1}{1+\mu_1} \right)^{n-p_0+1} \right\} \\ &\quad - \sum_{l=1}^{n-p_0+1} \left( \frac{1}{1+\mu_1} \right)^l \phi(S(n+2-l)) \sum_{j=0}^m f(j)\psi(I(n+1-l-j)). \end{aligned} \quad (5.5)$$

Here, we suppose that  $S(n) \leq S^*$ , for any  $p_0 + 1 \leq n \leq p_0 + m + l_0$ . Then, noting that  $0 < \psi(I) \leq I$  holds for  $I > 0$  under the condition (H2), we have

$$\begin{aligned} S(p_0 + m + l_0) &> \frac{\lambda}{\mu_1} \left\{ 1 - \left( \frac{1}{1+\mu_1} \right)^{m+l_0} \right\} - \sum_{l=1}^{m+l_0} \left( \frac{1}{1+\mu_1} \right)^l \phi(S^*)\psi(qI^*) \\ &= \frac{1}{\mu_1} \left\{ 1 - \left( \frac{1}{1+\mu_1} \right)^{m+l_0} \right\} \{\lambda - \phi(S^*)\psi(qI^*)\} \\ &\geq \frac{1}{\mu_1} \left\{ 1 - \left( \frac{1}{1+\mu_1} \right)^{m+l_0} \right\} \{\lambda - \phi(S^*)qI^*\} \geq S^*, \end{aligned}$$

which is a contradiction. Therefore, there exists an integer  $\tilde{p}$  such that  $p_0 + 1 \leq \tilde{p} \leq p_0 + m + l_0$  and  $S(\tilde{p}) > S^*$ . By the first equation of (3.1), we have that

$$\begin{aligned} (1+\mu_1)S^* + \phi(S^*)\psi(I^*) &= \lambda + S^* \\ &< \lambda + S(\tilde{p}) \\ &= (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1)) \sum_{j=0}^m f(j)\psi(I(\tilde{p}-j)) \\ &\leq (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1))\psi(qI^*) \\ &\leq (1+\mu_1)S(\tilde{p}+1) + \phi(S(\tilde{p}+1))qI^*, \end{aligned}$$

which is equivalent to  $S(\tilde{p}+1) > S^\Delta > S^*$ . Hence, we obtain that  $S(n) \geq S^\Delta > S^*$ , for any  $n \geq p_0 + m + l_0 + 1$ . Noting that  $I(n) \leq qI^*$  for all  $n \geq p_0 - m$ , define the sequence  $\{w(n)\}_{n=p_0}^{+\infty}$  as

$$w(n) = I(n) + \sum_{j=0}^m f(j) \sum_{k=n-j}^n \phi(S(j+k+1))\psi(I(k)). \quad (5.6)$$

By the conditions (H1) and (H2), we have that

$$\begin{aligned} w(n+1) - w(n) &= I(n+1) - I(n) + \sum_{j=0}^m f(j) \left\{ \sum_{k=n+1-j}^{n+1} \phi(S(j+k+1))\psi(I(k)) - \sum_{k=n-j}^n \phi(S(j+k+1))\psi(I(k)) \right\} \\ &= \phi(S(n+1)) \sum_{j=0}^m f(j)\psi(I(n-j)) - (\mu_2 + \gamma)I(n+1) \\ &\quad + \sum_{j=0}^m f(j) \{ \phi(S(n+2+j))\psi(I(n+1)) - \phi(S(n+1))\psi(I(n-j)) \}. \end{aligned}$$

Then, we obtain that

$$\begin{aligned} w(n+1) - w(n) &= \sum_{j=0}^m f(j)\phi(S(n+2+j))\psi(I(n+1)) - (\mu_2 + \gamma)I(n+1) \\ &> \left\{ \phi(S^\Delta) - (\mu_2 + \gamma) \frac{I(n+1)}{\psi(I(n+1))} \right\} \psi(I(n+1)) \\ &\geq \left\{ \phi(S^\Delta) - (\mu_2 + \gamma) \frac{I^*}{\psi(I^*)} \right\} \psi(I(n+1)) \\ &= \{ \phi(S^\Delta) - \phi(S^*) \} \psi(I(n+1)), \end{aligned} \quad (5.7)$$

for  $n \geq p_0 + m + l_0 - 1$ . Now, we set  $\hat{i} = \min_{\theta \in [-m, 0]} I(\theta + p_0 + 2m + l_0)$  and claim that  $I(n) \geq \hat{i}$  for all  $n \geq p_0 + m + l_0$ . Otherwise, if there is a  $T_1 \geq 0$  such that  $I(n) \geq \hat{i}$  for  $p_0 + m + l_0 \leq n \leq p_0 + 2m + l_0 + T_1$  and  $0 < \underline{i} := I(p_0 + 2m + l_0 + T_1 + 1) < \hat{i}$ , it follows from the condition (H1), (5.3) and the second equation of system (3.1) that, for  $n_1 = p_0 + 2m + l_0 + T_1$ ,

$$\begin{aligned} I(n_1+1) - I(n_1) &= \phi(S(n_1+1)) \sum_{j=0}^m f(j)\psi(I(n_1-j)) - (\mu_2 + \gamma)I(n_1+1) \\ &\geq \phi(S(n_1+1))\psi(I(n_1+1)) - (\mu_2 + \gamma)I(n_1+1) \\ &= \left\{ \phi(S(n_1+1)) - (\mu_2 + \gamma) \frac{I(n_1+1)}{\psi(I(n_1+1))} \right\} \psi(I(n_1+1)) \\ &\geq \left\{ \phi(S^\Delta) - (\mu_2 + \gamma) \frac{I^*}{\psi(I^*)} \right\} \psi(I(n_1+1)) \\ &= \{ \phi(S^\Delta) - \phi(S^*) \} \psi(\underline{i}) > 0. \end{aligned} \quad (5.8)$$

Therefore,  $I(n) \geq \hat{i}$  holds for all  $n \geq p_0 + m + l_0$ . It follows from (5.7) and the condition (H3) that

$$w(n+1) - w(n) > \{ \phi(S^\Delta) - \phi(S^*) \} \psi(\hat{i}) > 0, \text{ for } n \geq p_0 + m + l_0,$$

which implies that  $\lim_{n \rightarrow +\infty} w(n) = +\infty$ . However, by (5.6) and Lemma 3.1, it holds that there is a positive constant  $\bar{p} \geq p_0 + m + l_0$  and  $\bar{w}$  such that  $w(n) \leq \bar{w}$  for any  $n \geq \bar{p}$ , which leads to a contradiction. Hence, the claim holds.

By the above claim, we are left to consider two possibilities.

- (i)  $I(n) \geq qI^*$  for all  $n > 0$  sufficiently large,
- (ii)  $I(n)$  oscillates about  $qI^*$  for all  $n > 0$  sufficiently large.

We now show that  $I(n) \geq v_2$  for all  $n$  sufficiently large. If the first case holds, then we immediately get the conclusion of the theorem. If the second case holds, let  $p_1 < p_2$  be sufficiently large such that

$$I(p_1) > qI^*, \quad I(p_2) > qI^*, \quad \text{and} \quad I(n) \leq qI^*, \quad \text{for any } p_1 < n < p_2.$$

Since, from the second equation of system (2.1), it follows that  $I(n+1) - I(n) \geq -(\mu_2 + \gamma)I(n+1)$ ,  $n \geq p_1$ , we have

$$I(n+1) \geq \frac{1}{1 + \mu_2 + \gamma} I(n), \text{ for any } n \geq p_1,$$

from which we have

$$I(n+1) \geq \left( \frac{1}{1+\mu_2+\gamma} \right)^{n+1-p_1} I(p_1) \geq \left( \frac{1}{1+\mu_2+\gamma} \right)^{n+1-p_1} qI^*, \text{ for any } n \geq p_1.$$

Therefore, if  $p_2 \leq p_1 + m + l_0$ , one can easily obtain that

$$I(n+1) \geq \left( \frac{1}{1+\mu_2+\gamma} \right)^{m+l_0} qI^* = v_2, \text{ for any } p_1 \leq n \leq p_2. \quad (5.9)$$

If  $p_2 > p_1 + m + l_0$ , suppose to the contrary that there exists a  $p_1 + m + l_0 < T_2 \leq p_2$  such that  $I(T_2 + 1) < v_2 \leq I(T_2)$  holds. Then, by the similar discussion to (5.8), this leads to a contradiction. Thus, we obtain that  $I(n+1) \geq v_2$  for  $p_1 + m + l_0 \leq n \leq p_2$ .

Hence, we prove that  $I(n+1) \geq v_2$  for  $p_1 \leq n \leq p_2$ . Since the interval  $[p_1, p_2]$  is arbitrarily chosen, we conclude that  $I(n+1) \geq v_2$  for all  $n \geq p_1$ , which implies that  $\liminf_{n \rightarrow +\infty} I(n) \geq v_2$  holds. This completes the proof.  $\square$

## 5.2 Global asymptotic stability of $E_*$ for $R_0 > 1$

We introduce the following lemma which plays a crucial role to establish Theorem 2.2.

**Lemma 5.1.** *Assume that the conditions (H2) and (H3) hold true. If  $R_0 > 1$ , then it holds that for all  $n \geq 0$ ,*

$$g\left(\frac{I(n)}{I^*}\right) - g\left(\frac{\psi(I(n))}{\psi(I^*)}\right) \geq 0, \quad (5.10)$$

where  $g(x) = x - 1 - \ln x \geq g(1) = 0$  defined on  $x > 0$ .

**Proof.** By the conditions (H2) and (H3), we obtain

$$\left( \frac{\psi(I(n))}{\psi(I^*)} - 1 \right) \left( \frac{I(n)}{I^*} - \frac{\psi(I(n))}{\psi(I^*)} \right) = \frac{\psi(I(n))}{I^* \psi(I^*)} \left( \frac{I(n)}{\psi(I(n))} - \frac{I^*}{\psi(I^*)} \right) (\psi(I(n)) - \psi(I^*)) \geq 0. \quad (5.11)$$

It follows from (5.11) that

$$\begin{aligned} & g\left(\frac{I(n)}{I^*}\right) - g\left(\frac{\psi(I(n))}{\psi(I^*)}\right) \\ &= \frac{I(n)}{I^*} - \frac{\psi(I(n))}{\psi(I^*)} - \ln \frac{I(n)}{I^*} = \frac{I(n)}{I^*} - \frac{\psi(I(n))}{\psi(I^*)} - \frac{\frac{I(n)}{I^*}}{\frac{\psi(I(n))}{\psi(I^*)}} + 1 + \frac{\frac{I(n)}{I^*}}{\frac{\psi(I(n))}{\psi(I^*)}} - 1 - \ln \frac{I(n)}{I^*} \\ &= \frac{\psi(I^*)}{\psi(I(n))} \left( \frac{\psi(I(n))}{\psi(I^*)} - 1 \right) \left( \frac{I(n)}{I^*} - \frac{\psi(I(n))}{\psi(I^*)} \right) + g\left(\frac{I(n)}{\frac{\psi(I(n))}{\psi(I^*)}}\right) \geq 0 \end{aligned}$$

holds for all  $n \geq 0$ . Thus, we get the conclusion of this lemma.  $\square$

We now establish the global asymptotic stability of the endemic equilibrium  $\tilde{E}_*$  of system (3.1).

**Theorem 5.2.** *Assume that the conditions (H1)-(H3) hold true. If  $R_0 > 1$ , then it holds that*

$$\lim_{n \rightarrow +\infty} S(n) = S^*, \quad \lim_{n \rightarrow +\infty} I(n) = I^*, \quad (5.12)$$

and  $\tilde{E}_*$  of system (3.1) is globally asymptotically stable.

**Proof.** Let

$$\tilde{s}_n = \frac{\phi(S(n))}{\phi(S^*)}, \quad i_n = \frac{I(n)}{I^*}, \quad \text{and} \quad \tilde{i}_n = \frac{\psi(I(n))}{\psi(I^*)}.$$

From Lyapunov functionals for continuous-time SIR epidemic models in Huang *et al.* [9], Korobeinikov [16, 17] and McCluskey [19, 20], consider the following sequence  $\{U^*(n)\}_{n=m}^{+\infty}$  defined by

$$U^*(n) = \frac{1}{\phi(S^*)\psi(I^*)} U_1^*(n) + \frac{I^*}{\phi(S^*)\psi(I^*)} U_2^*(n) + U_+^*(n). \quad (5.13)$$

where

$$U_1^*(n) = S(n) - S^* - \int_{S^*}^{S(n)} \frac{\phi(s)}{\phi(s)} ds, \quad U_2^*(n) = g\left(\frac{I(n)}{I^*}\right), \quad U_+^*(n) = \sum_{j=0}^m f(j) \sum_{k=n-j}^n g\left(\frac{\psi(I(k))}{\psi(I^*)}\right). \quad (5.14)$$

Let us show that  $U^*(n+1) - U^*(n) \leq 0$  for any  $n \geq m$ . First, we calculate  $U_1^*(n+1) - U_1^*(n)$ . By using the relation in Lemma 4.1, we obtain

$$\begin{aligned}
U_1^*(n+1) - U_1^*(n) &= S(n+1) - S(n) - \int_{S(n)}^{S(n+1)} \frac{\phi(S^*)}{\phi(s)} ds \\
&\leq S(n+1) - S(n) - \phi(S^*) \frac{S(n+1) - S(n)}{\phi(S(n+1))} \\
&= \frac{\phi(S(n+1)) - \phi(S^*)}{\phi(S(n+1))} (S(n+1) - S(n)) \\
&= \frac{\phi(S(n+1)) - \phi(S^*)}{\phi(S(n+1))} \left\{ \lambda - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - \mu_1 S(n+1) \right\}. \quad (5.15)
\end{aligned}$$

Substituting  $\lambda = \mu_1 S^* + \phi(S^*) \psi(I^*)$  into (5.15), we see that

$$\begin{aligned}
U_1^*(n+1) - U_1^*(n) &\leq \left( 1 - \frac{\phi(S^*)}{\phi(S(n+1))} \right) \{ \mu_1 S^* + \phi(S^*) \psi(I^*) - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - \mu_1 S(n+1) \} \\
&= -\mu_1 \left( 1 - \frac{\phi(S^*)}{\phi(S(n+1))} \right) (S(n+1) - S^*) \\
&\quad + \phi(S^*) \psi(I^*) \sum_{j=0}^m f(j) \left( 1 - \frac{\phi(S^*)}{\phi(S(n+1))} \right) \left( 1 - \frac{\phi(S(n+1))}{\phi(S^*)} \cdot \frac{\psi(I(n-j))}{\psi(I^*)} \right) \\
&= -\mu_1 S(n+1) \left( 1 - \frac{\phi(S^*)}{\phi(S(n+1))} \right) \left( 1 - \frac{S^*}{S(n+1)} \right) \\
&\quad + \phi(S^*) \psi(I^*) \sum_{j=0}^m f(j) \left( 1 - \frac{1}{\tilde{s}_{n+1}} \right) (1 - \tilde{s}_{n+1} \tilde{i}_{n-j}).
\end{aligned}$$

Second, we similarly calculate  $U_2^*(n+1) - U_2^*(n)$ . By using the relation given in Remark 4.1, we obtain that

$$\begin{aligned}
U_2^*(n+1) - U_2^*(n) &= \frac{I(n+1) - I(n)}{I^*} - \ln \frac{I(n+1)}{I(n)} \\
&\leq \frac{I(n+1) - I(n)}{I^*} - \frac{I(n+1) - I(n)}{I(n+1)} \\
&= \frac{1}{I^*} \frac{I(n+1) - I^*}{I(n+1)} (I(n+1) - I(n)) \\
&= \frac{1}{I^*} \frac{I(n+1) - I^*}{I(n+1)} \{ \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - (\mu_2 + \gamma) I(n+1) \}.
\end{aligned}$$

Since we have  $\mu_2 + \gamma = \frac{\phi(S^*) \psi(I^*)}{I^*}$ , it follows that

$$\begin{aligned}
U_2^*(n+1) - U_2^*(n) &\leq \frac{1}{I^*} \frac{I(n+1) - I^*}{I(n+1)} \left\{ \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) - \frac{\phi(S^*) \psi(I^*)}{I^*} I(n+1) \right\} \\
&= \frac{\phi(S^*) \psi(I^*)}{I^*} \sum_{j=0}^m f(j) \left( 1 - \frac{I^*}{I(n+1)} \right) \left( \frac{\phi(S(n+1))}{\phi(S^*)} \cdot \frac{\psi(I(n-j))}{\psi(I^*)} - \frac{I(n+1)}{I^*} \right) \\
&= \frac{\phi(S^*) \psi(I^*)}{I^*} \sum_{j=0}^m f(j) \left( 1 - \frac{1}{\tilde{i}_{n+1}} \right) (\tilde{s}_{n+1} \tilde{i}_{n-j} - i_{n+1}).
\end{aligned}$$

Finally, calculating  $U_+^*(n+1) - U_+^*(n)$ , we get that

$$\begin{aligned}
U_+^*(n+1) - U_+^*(n) &= \sum_{j=0}^m f(j) \left\{ \sum_{k=n+1-j}^{n+1} g\left( \frac{\psi(I(k))}{\psi(I^*)} \right) - \sum_{k=n-j}^n g\left( \frac{\psi(I(k))}{\psi(I^*)} \right) \right\} \\
&= \sum_{j=0}^m f(j) \left\{ g\left( \frac{\psi(I(n+1))}{\psi(I^*)} \right) - g\left( \frac{\psi(i(n-j))}{\psi(i^*)} \right) \right\} \\
&= \sum_{j=0}^m f(j) g(\tilde{i}_{n+1}) - \sum_{j=0}^m f(j) g(\tilde{i}_{n-j}).
\end{aligned}$$

Then, we have

$$\begin{aligned}
U^*(n+1) - U^*(n) &\leq -\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)} \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \left(1 - \frac{S^*}{S(n+1)}\right) \\
&\quad - \sum_{j=0}^m f(j) \left\{ -\left(1 - \frac{1}{\tilde{s}_{n+1}}\right) (1 - \tilde{s}_{n+1} \tilde{i}_{n-j}) \right. \\
&\quad \left. - \left(1 - \frac{1}{\tilde{i}_{n+1}}\right) (\tilde{s}_{n+1} \tilde{i}_{n-j} - i_{n+1}) - g(\tilde{i}_{n+1}) + g(\tilde{i}_{n-j}) \right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&-\left(1 - \frac{1}{\tilde{s}_{n+1}}\right) (1 - \tilde{s}_{n+1} \tilde{i}_{n-j}) - \left(1 - \frac{1}{\tilde{i}_{n+1}}\right) (\tilde{s}_{n+1} \tilde{i}_{n-j} - i_{n+1}) - g(\tilde{i}_{n+1}) + g(\tilde{i}_{n-j}) \\
&= -\left(1 - \tilde{s}_{n+1} \tilde{i}_{n-j} - \frac{1}{\tilde{s}_{n+1}} + \tilde{i}_{n-j}\right) - \left(\tilde{s}_{n+1} \tilde{i}_{n-j} - i_{n+1} - \frac{\tilde{s}_{n+1} \tilde{i}_{n-j}}{\tilde{i}_{n+1}} + 1\right) - \tilde{i}_{n+1} + \tilde{i}_{n-j} + \ln \tilde{i}_{n+1} - \ln \tilde{i}_{n-j} \\
&= -2 + \frac{1}{\tilde{s}_{n+1}} + \frac{\tilde{s}_{n+1} \tilde{i}_{n-j}}{\tilde{i}_{n+1}} + \ln \tilde{i}_{n+1} - \ln \tilde{i}_{n-j},
\end{aligned}$$

we obtain that

$$\begin{aligned}
U^*(n+1) - U^*(n) &\leq -\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)} \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \left(1 - \frac{S^*}{S(n+1)}\right) \\
&\quad - \sum_{j=0}^m f(j) \left(-2 + \frac{1}{\tilde{s}_{n+1}} + \frac{\tilde{s}_{n+1} \tilde{i}_{n-j}}{\tilde{i}_{n+1}} + \ln \tilde{i}_{n+1} - \ln \tilde{i}_{n-j}\right) \\
&= -\frac{\mu_1 S(n+1)}{\phi(S^*)\psi(I^*)} \left(1 - \frac{\phi(S^*)}{\phi(S(n+1))}\right) \left(1 - \frac{S^*}{S(n+1)}\right) \\
&\quad - \sum_{j=0}^m f(j) \left\{ g\left(\frac{1}{\tilde{s}_{n+1}}\right) + g\left(\frac{\tilde{s}_{n+1} \tilde{i}_{n-j}}{\tilde{i}_{n+1}}\right) + g(i_{n+1}) - g(\tilde{i}_{n+1}) \right\}.
\end{aligned}$$

By the condition (H1), we have  $(1 - \frac{\phi(S^*)}{\phi(S(n+1))})(1 - \frac{S^*}{S(n+1)}) \leq 0$  with equality if and only if  $S(n+1) = S^*$  and it follows from Lemma 5.1 that  $U^*(n+1) - U^*(n) \leq 0$  for any  $n \geq m$ . Since  $\{U^*(n)\}_{n=m}^{+\infty}$  is monotone decreasing sequence, there exists a  $\tilde{u}_* := \lim_{n \rightarrow +\infty} U^*(n) \geq 0$ . Then,  $\lim_{n \rightarrow +\infty} (U^*(n+1) - U^*(n)) = 0$ , from which we obtain that  $\lim_{n \rightarrow +\infty} S(n+1) = S^*$  and  $\lim_{n \rightarrow +\infty} \frac{\tilde{i}_{n-j}}{\tilde{i}_{n+1}} = 1$ , that is,

$$\lim_{n \rightarrow +\infty} \frac{\psi(I(n-j))}{I(n+1)} = \frac{\psi(I^*)}{I^*}.$$

if  $f(j) > 0$ ,  $j = 0, 1, \dots, m$ . Then, by the first equation of (3.1), we have that for  $n \geq m$ ,

$$\begin{aligned}
S(n+1) - S(n) &= \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \sum_{j=0}^m f(j) \psi(I(n-j)) \\
&= \lambda - \mu_1 S(n+1) - \phi(S(n+1)) \frac{\sum_{j=0}^m f(j) \psi(I(n-j))}{I(n+1)} I(n+1),
\end{aligned}$$

which implies

$$I(n+1) = \frac{\lambda - (1 + \mu_1)S(n+1) + S(n)}{\phi(S(n+1)) \frac{\sum_{j=0}^m f(j) \psi(I(n-j))}{I(n+1)}}.$$

Using the relations  $\lim_{n \rightarrow +\infty} (\lambda - (1 + \mu_1)S(n+1) + S(n)) = \lambda - \mu_1 S^* = \phi(S^*)\psi(I^*) > 0$ , and

$$\lim_{n \rightarrow +\infty} \phi(S(n+1)) \frac{\sum_{j=0}^m f(j) \psi(I(n-j))}{I(n+1)} = \phi(S^*) \frac{\psi(I^*)}{I^*} > 0,$$

we obtain  $\lim_{n \rightarrow +\infty} I(n+1) = I^*$ . Thus,  $\lim_{n \rightarrow +\infty} (S(n), I(n)) = (S^*, I^*)$ . Since  $U^*(n) \leq U^*(m)$  for all  $n \geq m$  and  $g(x) \geq 0$  with equality if and only if  $x = 1$ ,  $\tilde{E}_*$  is uniformly stable. Hence, the proof is complete.  $\square$

*Proof of Theorem 2.2.* By Theorems 3.1, 5.1 and 5.2, we obtain the conclusion of Theorem 2.2.  $\square$

## 6 Conclusion and future directions

In this paper, we propose a discrete SIR epidemic model with a class of nonlinear incidence rate and a distributed latent period by applying a variation of Euler Backward discretization. We show that the disease-free equilibrium is globally asymptotically stable if and only if  $R_0 \leq 1$  and the system is permanent if and only if  $R_0 > 1$ . This implies that the disease can be eradicated completely for  $R_0 \leq 1$  and the disease will persist for  $R_0 > 1$ . We also establish that our model admits a unique endemic equilibrium which is globally asymptotically stable for  $R_0 > 1$ . Therefore, the discretization scheme preserves the global asymptotic stability of equilibria for corresponding continuous-time SIR epidemic models as well as positivity and boundedness of the solution (cf. Huang *et al.* [9]).

The discretization scheme is same as that in [6] and different from those in [11, 12, 24] due to the problem of the global stability for the endemic equilibrium. In fact, it is still difficult to analyze the global dynamics when their model admits an endemic equilibrium. However, a variation of Euler Backward discretization, we choose in this paper, enables us to construct a discrete-time analogue of Lyapunov functionals used in Huang *et al.* [9], Korobeinikov [16, 17] and McCluskey [19, 20] for continuous-time epidemic models and we can conclude that the endemic equilibrium is globally asymptotically stable whenever it exists. We also introduced Lemmas 4.1 and 5.1 to show that the each sequence for the discrete-time analogue of Lyapunov functionals is non-increasing.

For the nonlinear incidence rate, we offer conditions (H1)-(H3) to establish the global threshold property. The conditions (H1)-(H3) are more essential and less restrictive than those in Huang *et al.* [9, Theorem 1]. The conditions (H1)-(H3) are satisfied with various special incidence rates (see, e.g., [4, 13, 20, 28]). It is interesting that the condition that  $I/\psi(I)$  is monotone increasing leads to a justification of crowding (saturation) effects for the case that the proportion of infectives individuals in a host population is very high (see, for example, [4, 20, 28]). Applying the present techniques to stability analysis for other types of epidemic models will become our future works.

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