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Global stability of a delayed SIRS epidemic model with a non-monotonic incidence rate

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Abstract. In this paper, we investigate a disease transmission model of SIRS type with latent period $\tau \geq 0$ and the specific nonmonotone incidence rate $\frac{k\exp(-d\tau)S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)}$. For the basic reproduction number $R_0 > 1$, applying monotone iterative techniques, we establish sufficient conditions for the global asymptotic stability of endemic equilibrium of system which become partial answers to the open problem in [Hai-Feng Huo and Zhan-Ping Ma, Dynamics of a delayed epidemic model with non-monotonic incidence rate, *Commun. Nonlinear Sci. Numer. Simulat.* **15** (2010), 459-468]. Moreover, combining both monotone iterative techniques and the Lyapunov functional techniques to an SIR model by perturbation, we derive another type of sufficient conditions for the global asymptotic stability of the endemic equilibrium.

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1 Introduction

Mathematical models which describe the dynamics of infectious diseases have recently played a crucial role in the disease control in epidemiological aspect. Many authors have proposed various kinds of epidemic models to understand the mechanism of disease transmission (see [1–9] and references therein).

On the global asymptotic stability of an endemic equilibrium of endemic models with a non-monotonic incidence rate, there are papers recently appears in the literature. Xiao and Ruan [6] first proposed a non-monotonic incident rate $\frac{kSI}{1+\alpha I^2}$, where kI measure the infection force of the disease and $\frac{1}{1+\alpha I^2}$ describes the psychological or inhibitory effect from the behavioral change of the susceptible individuals when the number of infective individuals is very large. By applying Dulac function, Xiao and Ruan [6] established that the endemic equilibrium of an SIRS model with this non-monotonic incidence rate and no delays, is globally asymptotically stable.

Recently, for the following SIRS model with a non-monotonic incident rate with latency, Huo and Ma [4] investigated the existence of equilibria and global asymptotic stability of the disease-free equilibrium of system, permanence and local stability of the endemic equilibrium of system, and conjectured with numerical simulations that there are cases that the global stability of the endemic equilibrium can occur.

$$\begin{cases} \dfrac{dS(t)}{dt} = b - dS(t) - \dfrac{k\exp(-d\tau)S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)} + \gamma R(t), \\ \dfrac{dI(t)}{dt} = \dfrac{k\exp(-d\tau)S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)} - (d+\mu)I(t), \\ \dfrac{dR(t)}{dt} = \mu I(t) - (d+\gamma)R(t), \end{cases} \tag{1.1}$$

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where $S(t)$, $I(t)$ and $R(t)$ denote the numbers of susceptible to the disease, infective and recovered individuals at time t, respectively. b is the recruitment rate of the population, d is the natural death rate of the population. The positive constant k and τ is the average number of contacts per infective per day and latent period, respectively. μ is the natural recovery rare of the infective individuals, γ is the rate at which recovered individuals lose immunity and return to the susceptible class, α is the parameter measures the psychological or inhibitory effect. We assume that the force of infection at any time t is given by $\frac{k\exp(-d\tau)S(t)I(t-\tau)}{1+\alpha I^2(t-\tau)}$ (see Capasso [2]), since those infected at time $t-\tau$ become infectious at time τ latter. The term $0<\exp(-d\tau)\leq 1$ represents the survival of vector population in which the time taken to become infectious is τ.

The initial conditions of system (1.1) take the form

$$\begin{cases} S(\theta)=\phi_1(\theta), \ \ I(\theta)=\phi_2(\theta), \ \ R(\theta)=\phi_3(\theta), \\ \phi_i(\theta)\geq 0, \ \ \theta\in[-h,0], \ \phi_i(0)>0, \ \ \phi_i\in C([-h,0],\mathbb{R}_{+0}), \ \ i=1,2,3, \end{cases} \tag{1.2}$$

where $\mathbb{R}_{+0}=\{x\in\mathbb{R}|\ x\geq 0\}$.

By the fundamental theory of functional differential equations, system (1.1) has a unique solution $(S(t),I(t),R(t))$ satisfying the initial conditions (1.2). It is easy to show that all solutions of system (1.1) with the initial conditions (1.2) are defined on $[0,+\infty)$ and remain positive for all $t>0$. We see that system (1.1) always has a disease-free equilibrium $E_0=(b/d,0,0)$. The basic reproduction number of system (1.1) is

$$R_0=\frac{bk\exp(-d\tau)}{d(d+\mu)}. \tag{1.3}$$

If $R_0>1$, then system (1.1) has a unique endemic equilibrium $E_*=(S^*,I^*,R^*)$ (see Huo and Ma [4]), where

$$S^*=\frac{1}{d}\left\{b-\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)I^*\right\}, \ I^*=\frac{-k\exp(-d\tau)\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)+\sqrt{\triangle}}{2\alpha d(d+\mu)}, \ R^*=\frac{\mu}{d+\gamma}I^*, \tag{1.4}$$

and

$$\triangle=k^2\exp(-2d\tau)\left(d+\mu-\frac{\gamma\mu}{d+\gamma}\right)^2-4\alpha d^2(d+\mu)^2(1-R_0). \tag{1.5}$$

More recently, Yang and Xiao [9] extended the result of Xiao and Ruan [6] to the incident rate of a specific form, namely, $\frac{\beta I(t-\tau)S(t)}{1+\alpha I^p(t-\tau)}$, where $p\geq 1$. Stability of the disease-free equilibrium and existence, uniqueness and stability of an endemic equilibrium for the model, were investigated. It was shown that if $R_0\leq 1$, then the disease-free equilibrium is globally asymptotic stable, whereas if $R_0>1$, then the unique endemic equilibrium is globally asymptotically stable in the interior of the feasible region for the model for no latency, and periodic solutions can arise at a critical latency. Some numerical simulations were provided to support their analytical conclusions. However, they showed no sufficient conditions for the global stability of the endemic equilibrium of the model with latency, and when $1<p\leq 2$ and $R_0>1$, they also proposed a conjecture that the endemic equilibrium of the model is globally stable for all $\tau>0$.

Therefore, it is important to study more carefully the global asymptotic stability of endemic equilibrium of system (1.1), which corresponds to the case $p=2$ and $\beta=k\exp(-d\tau)$. In this paper, for the basic reproduction number $R_0>1$ of system (1.1), applying both of new monotone iterative techniques which are improved versions of Xu and Ma [8] (see also Xu and Ma [7]), and Lyapunov functional techniques in McCluskey [5], we obtain two types of sufficient conditions for the global asymptotic stability of endemic equilibrium E_* of system (1.1) which is a partial answer to the open problem in Huo and Ma [4] and also Yang and Xiao [9].

The first main theorem is established by applying new monotone iterative techniques (see Lemma 3.4 in Section 3) such that the each lower and upper bounds for each $S(t)$, $I(t)$ and $R(t)$ of system (1.1) for a sufficiently large $t\geq 0$, approach to an endemic steady state by simple conditions of contractive convergence for suitable monotone iterations (see (3.9)).

Theorem 1.1. *Assume $R_0>1$. Then, the endemic equilibrium $E_*=(S^*,I^*,R^*)$ of system (1.1) exists. If*

$$\frac{k\exp(-d\tau)}{(d+\mu)\sqrt{\alpha}}+2>R_0 \quad \text{and} \ \frac{d+\gamma}{\mu}\left(1+\frac{d+\mu}{k\exp(-d\tau)}\alpha I^*\right)\geq 1, \tag{1.6}$$

then, the endemic equilibrium $E_*=(S^*,I^*,R^*)$ of system (1.1) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.

We solve the open question for an example offered by Huo and Ma [4, Example], because we can see that this example ($b=4$, $k=0.8$, $d=\alpha=\gamma=\mu=1$ and $1<R_0=1.6\exp(-0.8\tau)<2$) satisfied the condition (1.6) in Theorem 1.1.

Let us consider two functions $G(I)$ and $h(I)$ of I such that

$$\begin{cases} G(I) = \frac{I}{1+\alpha I^2}, & 0 \le I < +\infty, \quad \text{and} \\ h(I) = \frac{I}{G(I)} = 1 + \alpha I^2, & 0 < I < +\infty, \end{cases} \tag{1.7}$$

and $\hat{I} = \frac{1}{\sqrt{\alpha}}$ be the local maximal point of $G(I)$, where $G'(\hat{I}) = 0$ and $G(I) \le G(\hat{I}) = \frac{1}{2\sqrt{\alpha}}$ for any $0 \le I < +\infty$. For any $0 \le \underline{I} \le \bar{I}$, put

$$\bar{G}(\underline{I}, \bar{I}) = \max_{\underline{I} \le I \le \bar{I}} G(I) = \begin{cases} G(\bar{I}), & \text{if } \bar{I} < \hat{I}, \\ G(\hat{I}), & \text{if } \underline{I} \le \hat{I} \le \bar{I}, \\ G(\underline{I}), & \text{if } \hat{I} < \underline{I}, \end{cases} \tag{1.8}$$

and

$$\begin{cases} \bar{K}(\underline{I}, \bar{I}) = \bar{I} + \frac{d+\mu}{k\exp(-d\tau)} \bar{h}(\underline{I}, \bar{I}), \quad \bar{h}(\underline{I}, \bar{I}) = \frac{\bar{I}}{\bar{G}(\underline{I}, \bar{I})}, \\ K(\underline{I}) = \underline{I} + \frac{d+\mu}{k\exp(-d\tau)} h(\underline{I}), \end{cases} \tag{1.9}$$

and consider three constants $\underline{I}_0$, $\bar{I}_1$ and $\underline{I}_1$ such that

$$\begin{cases} 0 \le \underline{I}_0 \le \liminf_{t \to +\infty} I(t), \\ \bar{K}(\underline{I}_0, \bar{I}_1) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma} \underline{I}_0, K(\underline{I}_1) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma} \bar{I}_1, \end{cases} \tag{1.10}$$

and three constants a, σ and c such that

$$a = \frac{d+\gamma}{\mu} \frac{d+\mu}{k\exp(-d\tau)} \alpha, \ \sigma = \frac{d+\gamma}{\mu} \left(1 + \frac{d+\mu}{k\exp(-d\tau)} \alpha 2I^*\right), \ c = \frac{\sigma - 1 + \sqrt{(\sigma-1)(\sigma+3)}}{2}. \tag{1.11}$$

For (1.11), we note that

$$\sigma = \frac{d+\gamma}{\mu} + 2aI^*, \ c^2 = (\sigma-1)(c+1), \ c > \sigma - 1 > 0, \quad \text{if } \sigma > 1. \tag{1.12}$$

Then, the second main theorem is obtained by applying the best possible lower bound $\underline{I}^*$ and upper bounds $\bar{I}^*$ of $I(t)$ for a sufficiently large t obtained by new monotone iterative techniques (see (3.25) in Theorem 3.2) to Lyapunov functional techniques to the SIR model by perturbation (see Lemma 3.4 and Section 4).

Theorem 1.2. *Assume that $R_0 > 1$ and $I^* \le \hat{I}$, and for three constants $\underline{I}_0$, $\bar{I}_1$ and $\underline{I}_1$ defined by (1.10), suppose that*

$$\underline{I}_0 < \underline{I}_1 < \bar{I}_1, \tag{1.13}$$

and consider three constants a, σ and c defined by (1.11).
i) If

$$\begin{cases} \sigma \le 1, \\ \text{or} \\ \sigma > 1, \quad \text{and} \quad c > a(I^* - \underline{I}_0) \quad \text{or} \quad c \ge (\sigma - 1) + a(\hat{I} - I^*), \end{cases} \tag{1.14}$$

then the endemic equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.1) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.
ii) For the constants $\underline{I}^* \ge 0$, $\bar{I}^* \le \frac{b}{d}$ and $\underline{S}^*$ such that

$$\underline{I}^* \le \liminf_{t \to +\infty} I(t) \le \limsup_{t \to +\infty} I(t) \le \bar{I}^*, \ \underline{S}^* = \frac{\sigma(d+\gamma)/d - \gamma \bar{I}^*}{(d+\gamma) + k\exp(-d\tau)G(\bar{I}^*)}, \tag{1.15}$$

if

$$\gamma^2 < 4(d+\mu)(d+\gamma)\underline{S}^* \frac{\alpha(\underline{I}^* + I^*)(1 - \alpha \bar{I}^* I^*)}{(1 + \alpha(\bar{I}^*)^2)(1 + \alpha(I^*)^2)}, \tag{1.16}$$

then the endemic equilibrium E_* of system (1.1) exists and is globally asymptotically stable.
In particular, if

$$\sigma > 1, \quad c \le a(I^* - \underline{I}_0) \quad \text{and} \quad c < (\sigma - 1) + a(\hat{I} - I^*), \tag{1.17}$$

then the following $\underline{I}^*$ and $\bar{I}^*$ such that

$$\underline{I}^* = I^* - \frac{c}{a}, \quad \text{and} \quad \bar{I}^* = I^* + \frac{c - (\sigma - 1)}{a}, \tag{1.18}$$

satisfy (1.15).

In the case (1.17), we say that $I^* - \frac{c}{a}$, and $I^* + \frac{c-(\sigma-1)}{a}$ are the best possible lower bound $\underline{I}^*$ and upper bounds $\bar{I}^*$ of $I(t)$ for a sufficiently large t, respectively, in the meaning of (1.15) obtained by new monotone iterative techniques in this paper.

The organization of this paper is as follows. In Section 2, we give known results for system (1.1) by Huo and Ma [4]. In Section 3, we offer new monotone iterative techniques to SIRS model (1.1). In Section 4, applying Lyapunov functional techniques of McCluskey [5], we establish another type of conditions of the global stability of endemic equilibrium E_* of system (1.1) for $R_0 > 1$. In Section 5, we investigate two numerical examples. Finally, a brief discussion is offered in Section 6.

2 Preliminaries

In this section, we give the following known results obtained by Huo and Ma [4].

Theorem 2.1. *If $R_0 \le 1$ (i.e. $\tau \ge \tau^*$), then system (1.1) only has the disease-free equilibrium E_0 and if $R_0 > 1$ (i.e. $0 \le \tau < \tau^*$), then system (1.1) has a unique endemic equilibrium E_*.*

Theorem 2.2. *If $R_0 < 1$ (i.e. $\tau > \tau^*$), then the disease-free equilibrium E_0 of system (1.1) is globally asymptotically stable. If $R_0 > 1$ (i.e. $0 \le \tau < \tau^*$), E_0 becomes unstable.*

Theorem 2.3. *If $R_0 = 1$ (i.e. $\tau = \tau^*$), then the disease-free equilibrium E_0 of system (1.1) is globally attractive.*

Theorem 2.4. *If $R_0 > 1$ (i.e. $0 \le \tau < \tau^*$), then the endemic equilibrium E_* of system (1.1) is locally stable.*

Theorem 2.5. *If $R_0 > 1$ (i.e. $0 \le \tau < \tau^*$), then the disease of system (1.1) is permanent.*

3 Monotone iterative techniques for $R_0 > 1$

In this section, for $R_0 > 1$, we improve the monotone iterative techniques offered by Xu and Ma [8, Theorem 3.1] for system (1.1). For any solutions $(S(t), I(t), R(t))$ of (1.1) with initial conditions (1.2), by Theorem 2.5, there exist positive constants v_i $(i = 1, 2, 3)$ such that

$$\begin{cases} \liminf\limits_{t\to+\infty} S(t) = \hat{\underline{S}} \ge v_1,\ \liminf\limits_{t\to+\infty} I(t) = \hat{\underline{I}} \ge v_2,\ \liminf\limits_{t\to+\infty} R(t) = \hat{\underline{R}} \ge v_3, \\ \limsup\limits_{t\to+\infty} S(t) = \hat{\hat{S}} \le \dfrac{b}{d},\ \limsup\limits_{t\to+\infty} I(t) = \hat{\hat{I}} \le \dfrac{b}{d},\ \limsup\limits_{t\to+\infty} R(t) = \hat{\hat{R}} \le \dfrac{b}{d}. \end{cases} \tag{3.1}$$

From (1.1), it follows that $N'(t) = b - dN(t)$ for $N(t) \equiv S(t) + I(t) + R(t)$, and we obtain $\lim_{t\to+\infty} N(t) = \frac{b}{d}$. Therefore, hereafter, we may restrict our attention to the case that

$$S(t) + I(t) + R(t) = \frac{b}{d}, \quad \text{for any} \quad t \ge 0. \tag{3.2}$$

Then, we have the following lemma.

Lemma 3.1.

$$\frac{b}{d} - \hat{\underline{I}} - \hat{\hat{R}} > 0, \quad \textit{and} \quad \frac{b}{d} - \hat{\hat{I}} - \hat{\underline{R}} > 0. \tag{3.3}$$

Proof. Suppose that $\frac{b}{d} - \hat{\hat{I}} - \hat{\underline{R}} \le 0$. Then, by (3.1), there is a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim\limits_{n\to+\infty} I(t_n) = \hat{\hat{I}}$. Since $\liminf\limits_{n\to+\infty} R(t_n) \ge \hat{\underline{R}}$, by (3.2), we have that

$$0 < \limsup_{n\to+\infty} S(t_n) \le \frac{b}{d} - \liminf_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \le \frac{b}{d} - \hat{\hat{I}} - \hat{\underline{R}} \le 0,$$

which is a contradiction. Thus, we have $\frac{b}{d} - \hat{\hat{I}} - \hat{\underline{R}} > 0$. Similarly, we can prove that $\frac{b}{d} - \hat{\underline{I}} - \hat{\hat{R}} > 0$. □

Lemma 3.2.

$$\begin{cases} 0 \ge b - d\hat{\underline{S}} - k\exp(-d\tau)\hat{\underline{S}}\bar{G}(\hat{\underline{I}}, \hat{\hat{I}}) + \gamma(\frac{b}{d} - \hat{\underline{S}} - \hat{\hat{I}}), \\ 0 \ge k\exp(-d\tau)(\frac{b}{d} - \hat{\underline{I}} - \hat{\hat{R}})G(\hat{\underline{I}}) - (d+\mu)\hat{\underline{I}}, \\ 0 \ge \mu\hat{\underline{I}} - (d+\gamma)\hat{\underline{R}}, \end{cases} \tag{3.4}$$

and

$$\begin{cases} 0 \le b - d\hat{\bar{S}} - k\exp(-d\tau)\hat{\bar{S}}G(\hat{\underline{I}}) + \gamma(\frac{b}{d} - \hat{\bar{S}} - \hat{\underline{I}}), \\ 0 \le k\exp(-d\tau)(\frac{b}{d} - \hat{\bar{I}} - \hat{\underline{R}})\bar{G}(\hat{\underline{I}}, \hat{\bar{I}}) - (d+\mu)\hat{\bar{I}}, \\ 0 \le \mu\hat{\bar{I}} - (d+\gamma)\hat{\bar{R}}. \end{cases} \tag{3.5}$$

Proof. First, we assume that $I(t)$ is eventually monotone decreasing for $t \ge 0$. Then, by Theorem 2.5, there exists $\lim_{t\to+\infty} I(t) = \hat{\bar{I}} = \hat{\underline{I}} > 0$. Then, by the third equation of (1.1), we obtain that there exists $\lim_{t\to+\infty} R(t) = \hat{\bar{R}} = \hat{\underline{R}} > 0$. Then, by the first equation of (1.1), we obtain that there exists $\lim_{t\to+\infty} S(t) = \hat{\bar{S}} = \hat{\underline{S}} > 0$. Since the endemic equilibrium $E_* = (S^*, I^*, R^*)$ is unique, we have that $\hat{S}^* = \hat{\bar{S}} = \hat{\underline{S}}$, $\hat{I}^* = \hat{\bar{I}} = \hat{\underline{I}}$ and $\hat{R}^* = \bar{\bar{R}} = \hat{\underline{R}}$. Thus, by (1.4), we obtain (3.5).

Second, we assume that $I(t)$ is not eventually monotone decreasing for $t \ge 0$. Then, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to+\infty} I'(t_n) \ge 0$ and $\lim_{n\to+\infty} I(t_n) = \hat{\bar{I}}$. Then, by (3.2), it follows that

$$\limsup_{n\to+\infty} S(t_n) \le \frac{b}{d} - \lim_{n\to+\infty} I(t_n) - \liminf_{n\to+\infty} R(t_n) \le \frac{b}{d} - \hat{\bar{I}} - \hat{\underline{R}},$$

from which we can obtain (3.5). Similar to the above discussion, we can obtain (3.4). □

Hereafter, for simplicity, we assume that $I^* \le \hat{I}$. Then, we easily obtain the following lemma.

Lemma 3.3.

$$\hat{\underline{S}} \ge \frac{b(1+\frac{\gamma}{d}) - \gamma\hat{\bar{I}}}{(d+\gamma) + k\exp(-d\tau)\bar{G}(\hat{\underline{I}}, \hat{\bar{I}})}, \ \hat{\underline{I}} + \frac{d+\mu}{k\exp(-d\tau)}h(\hat{\underline{I}}) \ge \frac{b}{d} - \frac{\mu}{d+\gamma}\hat{\bar{I}}, \ \hat{\underline{R}} \ge \frac{\mu}{d+\gamma}\hat{\underline{I}}, \tag{3.6}$$

and

$$\hat{\bar{S}} \le \frac{b(1+\frac{\gamma}{d}) - \gamma\hat{\underline{I}}}{(d+\gamma) + k\exp(-d\tau)G(\hat{\underline{I}})}, \ \hat{\bar{I}} + \frac{d+\mu}{k\exp(-d\tau)}\bar{h}(\hat{\underline{I}}, \hat{\bar{I}}) \le \frac{b}{d} - \frac{\mu}{d+\gamma}\hat{\underline{I}}, \ \hat{\bar{R}} \le \frac{\mu}{d+\gamma}\hat{\bar{I}}. \tag{3.7}$$

We now first take a initial value $\underline{I}_0 = 0$ or more appropriate $\underline{I}_0$ such that

$$0 \le \underline{I}_0 \le \liminf_{t\to+\infty} I(t), \tag{3.8}$$

and consider the following six sequences $\bar{S}_n$, $\bar{I}_n$, $\bar{R}_n$, $\underline{S}_n$, $\underline{I}_n$ and $\underline{R}_n$ $(n = 1, 2, \cdots)$ as follows (cf. Xu and Ma [8, (3.3)]).

$$\begin{cases} \bar{K}(\underline{I}_{n-1}, \bar{I}_n) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\underline{I}_{n-1}, \\ K(\underline{I}_n) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\bar{I}_n, \quad n = 1, 2, 3\cdots, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \underline{S}_n = \dfrac{b(d+\gamma)/d - \gamma\bar{I}_n}{(d+\gamma) + k\exp(-d\tau)\bar{G}(\underline{I}_n, \bar{I}_n)}, \quad \underline{R}_n = \dfrac{\mu}{d+\gamma}\underline{I}_n, \\ \bar{S}_n = \dfrac{b(d+\gamma)/d - \gamma\underline{I}_{n-1}}{(d+\gamma) + k\exp(-d\tau)G(\underline{I}_{n-1})}, \quad \bar{R}_n = \dfrac{\mu}{d+\gamma}\bar{I}_n. \end{cases} \tag{3.10}$$

Then, by Lemma 3.3, (3.7) and (3.9), we have that

$$\underline{I}_0 \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \bar{I}_1. \tag{3.11}$$

Lemma 3.4. *For the sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$ defined by (3.9) and (3.10), assume $\underline{I}_0 < \bar{I}_1$. Then, (1.13) holds true, if and only if,*

$$\frac{\mu}{d+\gamma} < 1 + \frac{d+\mu}{k\exp(-d\tau)}\frac{\bar{h}(\underline{I}_0, \bar{I}_1) - h(\underline{I}_1)}{\bar{I}_1 - \underline{I}_1}. \tag{3.12}$$

In this case, the three sequences $\{\underline{I}_n\}_{n=1}^{\infty}$, $\{\underline{S}_n\}_{n=1}^{\infty}$ and $\{\underline{R}_n\}_{n=1}^{\infty}$ are strictly monotone increasing sequences and converge to $\underline{I}^*$, $\underline{S}^*$ and $\underline{R}^*$, respectively and the three sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\bar{S}_n\}_{n=1}^{\infty}$ and $\{\bar{R}_n\}_{n=1}^{\infty}$ are strictly monotone decreasing sequences and converge to $\bar{I}^*$, $\bar{S}^*$ and $\bar{R}^*$, respectively, as n tends to $+\infty$ satisfying

$$\begin{cases} \lim_{n\to+\infty} \underline{I}_n = \underline{I}^* \le \liminf_{t\to+\infty} I(t) \le \limsup_{t\to+\infty} I(t) \le \lim_{n\to+\infty} \bar{I}_n = \bar{I}^*, \\ \lim_{n\to+\infty} \underline{S}_n = \underline{S}^* \le \liminf_{t\to+\infty} S(t) \le \limsup_{t\to+\infty} S(t) \le \lim_{n\to+\infty} \bar{S}_n = \bar{S}^*, \\ \lim_{n\to+\infty} \underline{R}_n = \underline{R}^* \le \liminf_{t\to+\infty} R(t) \le \limsup_{t\to+\infty} R(t) \le \lim_{n\to+\infty} \bar{R}_n = \bar{R}^*, \end{cases} \tag{3.13}$$

and

$$\begin{cases} \bar{I}^* + \frac{\mu}{d+\gamma}\underline{I}^* + \frac{d+\mu}{k\exp(-d\tau)}\bar{h}(\underline{I}^*, \bar{I}^*) = \frac{b}{d}, \\ \underline{I}^* + \frac{\mu}{d+\gamma}\bar{I}^* + \frac{d+\mu}{k\exp(-d\tau)}h(\underline{I}^*) = \frac{b}{d}, \\ \text{and} \\ 1 + \frac{d+\mu}{k\exp(-d\tau)}\frac{\bar{h}(\underline{I}^*,\bar{I}^*) - h(\underline{I}^*)}{\bar{I}^* - \underline{I}^*} = \frac{\mu}{d+\gamma}, \quad \text{if } \underline{I}^* < \bar{I}^*. \end{cases} \tag{3.14}$$

Moreover, assume that there exist two constants $\underline{I} < \bar{I}$ such that

$$\begin{cases} \underline{I} \le \liminf_{t\to+\infty} I(t) \le I^* \le \limsup_{t\to+\infty} I(t) \le \bar{I}, \\ \dfrac{\mu}{d+\gamma} < 1 + \dfrac{d+\mu}{k\exp(-d\tau)}\dfrac{\bar{h}(\underline{I}, \bar{I}) - h(\underline{I})}{\bar{I} - \underline{I}}, \\ \text{and} \\ \left\{ \begin{array}{l} \underline{I} \le \underline{I}^* \le I^* \le \bar{I}^* \le \bar{I}, \\ \bar{I}^* + \dfrac{d+\mu}{k\exp(-d\tau)}\bar{h}(\underline{I}^*, \bar{I}^*) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\underline{I}^*, \\ \text{and} \\ \underline{I}^* + \dfrac{d+\mu}{k\exp(-d\tau)}h(\underline{I}^*) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\bar{I}^*, \end{array} \right\} \quad \text{imply} \quad \underline{I}^* = \bar{I}^* = I^*. \end{cases} \tag{3.15}$$

Then, the endemic equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.1) is globally asymptotically stable in the interior of $\mathbb{R}_+^3$. In particular, if (1.6) holds, then both conditions (3.12) and (3.15) are satisfied.

Proof. By (1.9) and (3.9),

$$\begin{cases} \bar{I}_n + \dfrac{d+\mu}{k\exp(-d\tau)}\bar{h}(\underline{I}_{n-1}, \bar{I}_n) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\underline{I}_{n-1}, \\ \underline{I}_n + \dfrac{d+\mu}{k\exp(-d\tau)}h(\underline{I}_n) = \dfrac{b}{d} - \dfrac{\mu}{d+\gamma}\bar{I}_n, \quad n = 1, 2, 3, \cdots, \end{cases} \tag{3.16}$$

from which we have that for $\underline{I}_n < \bar{I}_n$ and $n = 1, 2, 3, \cdots$,

$$\left(1 + \frac{d+\mu}{k\exp(-d\tau)}\frac{\bar{h}(\underline{I}_{n-1}, \bar{I}_n) - h(\underline{I}_n)}{\bar{I}_n - \underline{I}_n}\right)(\bar{I}_n - \underline{I}_n) = \frac{\mu}{d+\gamma}(\bar{I}_n - \underline{I}_{n-1}).$$

Hence, we obtain that for $\underline{I}_n < \bar{I}_n$,

$$\bar{I}_n - \underline{I}_n = \frac{\frac{\mu}{d+\gamma}}{1 + \frac{d+\mu}{k\exp(-d\tau)}\frac{\bar{h}(\underline{I}_{n-1}, \bar{I}_n) - h(\underline{I}_n)}{\bar{I}_n - \underline{I}_n}}(\bar{I}_n - \underline{I}_{n-1}), \quad n = 1, 2, 3, \cdots, \tag{3.17}$$

from which one can see that (1.13) holds, if and only if, (3.12) holds. Then, by the monotonicity and inductions in (3.16), we can prove that $\underline{I}_{n-1} < \underline{I}_n < \bar{I}_n < \bar{I}_{n-1}$, $n = 2, 3, \cdots$, (3.13) and (3.14) hold. Moreover, suppose that (3.15) holds. Then, by $I^* \le \hat{I}$ and (3.17), we obtain $\underline{I}^* = \bar{I}^* = I^*$, from which we have $\underline{S}^* = \bar{S}^* = S^*$ and $\underline{R}^* = \bar{R}^* = R^*$. Hence, from (3.6) and (3.7), E_* is globally asymptotically stable in the interior of $\mathbb{R}_+^3$.

In particular, assume that (1.6) holds. By the first equation of (3.14), we have that

$$\frac{k\exp(-d\tau)}{d+\mu}\bar{I}^* + \bar{h}(\underline{I}^*, \bar{I}^*) = R_0 - \frac{k\exp(-d\tau)}{d+\mu}\frac{\mu}{d+\gamma}\underline{I}^* \le R_0.$$

If $\bar{I}^* \ge \hat{I}$, then, by (1.8), (1.9) and the first equation of (1.6), we have that

$$\frac{k\exp(-d\tau)}{d+\mu}\bar{I}^* + \bar{h}(\underline{I}^*, \bar{I}^*) = \frac{k\exp(-d\tau)}{d+\mu}\bar{I}^* + \frac{\bar{I}^*}{G(\hat{I})} \ge \frac{k\exp(-d\tau)}{d+\mu}\bar{I}^* + \frac{\hat{I}}{G(\hat{I})} > R_0.$$

which is a contradiction. Therefore, the first equation of (1.6) implies $\bar{I}^* < \hat{I}$. Then, $\bar{h}(\underline{I}, \bar{I}) = h(\bar{I}) = 1 + \alpha \bar{I}^2$ for any $0 < \underline{I} < \bar{I} < \hat{I}$, and

$$\frac{\bar{h}(\underline{I}, \bar{I}) - h(\underline{I})}{\bar{I} - \underline{I}} = \frac{(1 + \alpha \bar{I}^2) - (1 + \alpha \underline{I}^2)}{\bar{I} - \underline{I}} = \alpha(\bar{I} + \underline{I}) > \alpha I^*, \quad \text{for } \bar{I} \geq I^*.$$

Then, by the second equation of (1.6), both conditions (3.12) and (3.15) are satisfied. Hence, we obtain the conclusion of this lemma. □

Proof of Theorem 1.1 By Lemma 3.4, we can immediately obtain the conclusion of Theorem 1.1. □

Now, assume that $R_0 > 1$ and consider the solutions $\underline{I}^*$ and $\bar{I}^*$ of (3.14) in Lemma 3.4. Put $\underline{I}^* = I^* - \epsilon$ and $\bar{I}^* = I^* + \kappa$. Then, we need the following restriction:

$$0 \leq \epsilon \leq I^* - \underline{I}_0 \quad \text{and} \quad 0 \leq \kappa < \bar{I}_1 - I^*. \tag{3.18}$$

Lemma 3.5. *Assume $R_0 > 1$. Then, under (3.18), there is a unique solution $(\underline{I}^*, \bar{I}^*)$ of (3.14) in Lemma 3.4 such that if*

$$\sigma \leq 1, \tag{3.19}$$

or

$$\sigma > 1, \ c \leq a(I^* - \underline{I}_0) \ \textit{or} \ c \leq (\sigma - 1) + a(\bar{I}_1 - I^*), \tag{3.20}$$

then

$$\underline{I}^* = \bar{I}^* = I^*, \tag{3.21}$$

otherwise,

$$\underline{I}^* = I^* - \frac{c}{a} \ \textit{and} \ \bar{I}^* = I^* + \frac{c + (1 - \sigma)}{a}. \tag{3.22}$$

Proof. First, assume that $\bar{I}^* < \hat{I}$. By (3.14), (1.9) and $I^* + \kappa \leq \hat{I}$, we have that

$$\begin{cases} (I^* + \kappa) + \dfrac{\mu}{d + \gamma}(I^* - \epsilon) + \dfrac{d + \mu}{k \exp(-d\tau)}\{1 + \alpha(I^* + \kappa)^2\} = \dfrac{b}{d}, \\ (I^* - \epsilon) + \dfrac{\mu}{d + \gamma}(I^* + \kappa) + \dfrac{d + \mu}{k \exp(-d\tau)}\{1 + \alpha(I^* - \epsilon)^2\} = \dfrac{b}{d}, \\ \text{and} \\ 1 + \frac{d+\mu}{k \exp(-d\tau)}\alpha\{2I^* + (\kappa - \epsilon)\} = \frac{\mu}{d+\gamma}, \quad \text{if } \ \epsilon + \kappa > 0. \end{cases} \tag{3.23}$$

Then, by (1.4) and (1.11), (3.23) becomes

$$\frac{\mu}{d + \gamma}(-\epsilon + \sigma\kappa + a\kappa^2) = 0, \ \frac{\mu}{d + \gamma}(\kappa - \sigma\epsilon + a\epsilon^2) = 0.$$

Thus,

$$\begin{cases} \kappa + \epsilon = 0, \quad \text{or} \\ \sigma + a(\kappa - \epsilon) = 1, \quad \text{if } \ \kappa + \epsilon > 0, \end{cases}$$

from which we have $\epsilon = \kappa = 0$, or $\kappa = \epsilon + \frac{1-\sigma}{a}$ if $\epsilon > 0$, respectively. Suppose that $\epsilon > 0$. Then, by $\kappa - \sigma\epsilon + a\epsilon^2 = 0$, it follows that $a^2\epsilon^2 + a(1 - \sigma)\epsilon + (1 - \sigma) = 0$. Hence, we obtain that $\sigma > 1$, $\epsilon = \frac{c}{a}$ and $\kappa = \frac{c+(1-\sigma)}{a} > 0$.

Second, we suppose that $\bar{I}^* \geq \hat{I}$. Then, we have that $\frac{1}{G_2(\hat{I})} - \frac{1}{G_2(\bar{I}^*)} \leq 0$ and by (1.9) and (3.14),

$$\begin{cases} (I^* + \kappa) + \frac{\mu}{d+\gamma}(I^* - \epsilon) + \frac{d+\mu}{k \exp(-d\tau)}[\{1 + \alpha(I^* + \kappa)^2\} + \alpha(\frac{1}{G_2(\hat{I})} - \frac{1}{G_2(\bar{I}^*)})\bar{I}^*] = \frac{b}{d}, \\ (I^* - \epsilon) + \frac{\mu}{d+\gamma}(I^* + \kappa) + \frac{d+\mu}{k \exp(-d\tau)}\{1 + \alpha(I^* - \epsilon)^2\} = \frac{b}{d}, \\ \text{and} \quad \text{if } \ \epsilon + \kappa > 0, \ \text{then} \\ 1 + \frac{d+\mu}{k \exp(-d\tau)}\alpha\{2I^* + (\kappa - \epsilon)\} + \frac{d+\mu}{k \exp(-d\tau)}\alpha(\frac{1}{G_2(\hat{I})} - \frac{1}{G_2(\bar{I}^*)})\bar{I}^* = \frac{\mu}{d+\gamma}. \end{cases} \tag{3.24}$$

Then, by $\frac{1}{G_2(\hat{I})} - \frac{1}{G_2(\bar{I}^*)} \leq 0$, similar to the above discussion, we can derive that for (3.24), $\underline{I}^* \geq I^* - \frac{c}{a}$ and $\bar{I}^* \leq I^* + \frac{c-(\sigma-1)}{a}$, from which we can conclude that there exist the same two solutions that $\epsilon = \kappa = 0$, or $\sigma > 1$, $\epsilon = \frac{c}{a}$ and $\kappa = \frac{c+(1-\sigma)}{a}$.

Next, we investigate the restriction (3.18). If $\sigma \leq 1$ then c is not a real number. Thus, we only have $\epsilon = \kappa = 0$ and we only have the solution $\underline{I}^* = \bar{I}^* = I^*$. If (3.20) holds, then

$$\underline{I}^* = I^* - \frac{c}{a} < \underline{I}_0, \quad \text{or} \quad \bar{I}^* = I^* + \frac{c + (1 - \sigma)}{a} > \bar{I}_1.$$

Since by the monotone convergence of the sequences $\{\bar{I}_n\}_{n=1}^{\infty}$, $\{\underline{I}_n\}_{n=1}^{\infty}$ to $\underline{I}^*$ and $\bar{I}^*$, respectively, in Lemma 3.4, this is a contradiction if $\epsilon > 0$. Thus, we conclude that under the restriction (3.18), there is only solution $\underline{I}^* = \bar{I}^* = I^*$. Suppose that $\sigma > 1$ and (3.20) is not satisfied. Then, by Lemma 3.4, we conclude that (3.22) holds. Hence, we complete the proof of this lemma. □

By Lemmas 3.4 and 3.5, we can obtain the following two theorems.

Theorem 3.1. *Assume that $R_0 > 1$, $I^* \leq \hat{I}$ and for (1.10) and suppose that (1.13) and (1.14) hold. Then, the endemic equilibrium $E_* = (S^*, I^*, R^*)$ of system (1.1) is globally asymptotically stable in the interior of $\mathbb{R}^3_+$.*

Theorem 3.2. *Assume that $R_0 > 1$ and $I^* \leq \hat{I}$ and for (1.10), suppose that (1.13) and (1.17) hold. Then, for (3.14), it holds that*

$$\begin{cases} \underline{I}^* = I^* - \frac{c}{a}, \quad \bar{I}^* = I^* + \frac{c-(\sigma-1)}{a}, \\ \underline{S}^* = \frac{\sigma(d+\gamma)/d - \gamma\bar{I}^*}{(d+\gamma)+k\exp(-d\tau)G(\bar{I}^*)}, \quad \bar{S}^* = \frac{\sigma(d+\gamma)/d - \gamma\underline{I}^*}{(d+\gamma)+k\exp(-d\tau)G(\underline{I}^*)}, \\ \underline{R}^* = \frac{\mu}{d+\gamma}\underline{I}^*, \quad \bar{R}^* = \frac{\mu}{d+\gamma}\bar{I}^*. \end{cases} \tag{3.25}$$

4 Lyapunov functional techniques for $R_0 > 1$

In this section, we establish another conditions of the global stability of the endemic equilibrium E_* of system (1.1) by Lyapunov functional techniques. We here note that the first and second equations of system (1.1) do not depend on the third equation, and the limit set of (1.1) is on the plane $S + I + R = b/d$. Hence, the dynamics of system (1.1) in Ω_1 is equivalent to the following reduced system.

$$\begin{cases} \dfrac{dS(t)}{dt} = \tilde{b} - \tilde{d}S(t) - \tilde{k}S(t)G(I(t-\tau)) - \gamma I(t), \\ \dfrac{dI(t)}{dt} = \tilde{k}S(t)G(I(t-\tau)) - (d+\mu)I(t), \end{cases} \tag{4.1}$$

with the initial conditions

$$S(\theta) = \phi_1(\theta),\ \ I(\theta) = \phi_2(\theta),\ \ \phi_i(\theta) \geq 0,\ \ \theta \in [-h, 0],\ \ \phi_i(0) > 0,\ \ \phi_i \in C([-h,0], \mathbb{R}_{+0}),\ \ i = 1, 2, \tag{4.2}$$

where $\tilde{b} = b + \gamma$, $\tilde{d} = d + \gamma$, $\tilde{k} = k\exp(-d\tau)$.

In this section, for the case $R_0 > 1$, we offer sufficient conditions for the endemic equilibrium $Q_* = (S^*, I^*)$ of the reduced system (4.1) to be globally asymptotically stable. We recall that by Theorem 2.1, the existence of the endemic equilibrium Q_* of this system is guaranteed for the case $R_0 > 1$.

For preparations, we need some lemmas. Put

$$x_t = \frac{S(t)}{S^*}, \quad y_t = \frac{I(t)}{I^*}, \quad y_{t,\tau} = \frac{I(t-\tau)}{I^*}, \ z_t = \frac{G(I(t))}{G(I^*)}, \quad z_{t,\tau} = \frac{G(I(t-\tau))}{G(I^*)} \tag{4.3}$$

and $g(x) = x - 1 - \ln x \geq g(1) = 0$.

Lemma 4.1. *For all $t \geq 0$, it holds that*

$$g(y_t) - g(z_t) \geq \frac{1}{I^*}\{G(I(t)) - G(I^*)\}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right). \tag{4.4}$$

Proof. First, by (4.3), we have that

$$z_t - 1 = \frac{1}{G(I^*)}\{G(I(t)) - G(I^*)\} \tag{4.5}$$

and

$$y_t - z_t = \frac{I(t)}{I^*} - \frac{G(I(t))}{G(I^*)} = \frac{G(I(t))}{I^*}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right). \tag{4.6}$$

Then, we immediately obtain

$$(z_t - 1)(y_t - z_t) = \frac{G(I(t))}{I^*G(I^*)}\{G(I(t)) - G(I^*)\}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right). \tag{4.7}$$

Moreover, by $g''(x) = \frac{1}{x^2} > 0$ for $x > 0$ and (4.3), one can obtain that

$$\begin{aligned} g(y_t) - g(z_t) &\geq g'(z_t)(y_t - z_t) \\ &= \left(1 - \frac{1}{z_t}\right)(y_t - z_t) \\ &= \frac{1}{z_t}(z_t - 1)(y_t - z_t) \\ &= \frac{G(I^*)}{G(I(t))}\frac{G(I(t))}{I^*G(I^*)}\{G(I(t)) - G(I^*)\}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right) \\ &= \frac{1}{I^*}\{G(I(t)) - G(I^*)\}\left(\frac{I(t)}{G(I(t))} - \frac{I^*}{G(I^*)}\right). \end{aligned} \tag{4.8}$$

Hence, we get the conclusion of Lemma 4.1. □

Lemma 4.2. *$G(I) = \frac{I}{1+\alpha I^2}$ is a unimodal function of $I > 0$ and it holds that*

$$\left\{\frac{I}{G(I)} - \frac{I^*}{G(I^*)}\right\}(I - I^*) = \alpha(I + I^*)(I - I^*)^2 \geq 0, \tag{4.9}$$

and

$$\{G(I) - G(I^*)\}(I - I^*) = \frac{1 - \alpha I I^*}{(1 + \alpha I^2)(1 + \alpha(I^*)^2)}(I - I^*)^2. \tag{4.10}$$

Proof. Since $G'(I) = \frac{1-\alpha I^2}{(1+\alpha I^2)^2}$, $G(I)$ is a unimodal function of $I > 0$. The remained part of this lemma is proved by simple calculations. □

We now claim the following lemma which plays a crucial role to obtain Theorem 1.2.

Lemma 4.3. *It holds that*

$$g(y_t) - g(z_t) \geq \alpha(I(t) + I^*)I^*\frac{1 - \alpha I(t)I^*}{(1 + \alpha I^2(t))(1 + \alpha(I^*)^2)}(y_t - 1)^2. \tag{4.11}$$

Assume that for (3.9), it holds (3.12). Moreover, suppose that (1.17) holds for (1.16), and $\underline{I}^* < \bar{I}^*$ for (3.14). If

$$0 < \alpha < \frac{1}{(I^* + \frac{c-(\sigma-1)}{a})I^*}, \tag{4.12}$$

then for any sufficiently large $t > 0$,

$$1 - \alpha I(t)I^* > 0. \tag{4.13}$$

Proof. By Lemmas 4.1 and 4.2, we obtain (4.11). If (1.17) and $\underline{I}^* < \bar{I}^*$ hold, then for (4.12), it holds that

$$1 - \alpha(I(t)I^*) > 1 - \frac{1}{(I^* + \frac{c-(\sigma-1)}{a})I^*}\left(I^* + \frac{c - (\sigma - 1)}{a}\right)I^* = 0,$$

for any sufficiently large $t > 0$. Hence, we complete the proof of this lemma. □

Now, by applying the best possible lower bound $\underline{I}^*$ and upper bounds $\bar{I}^*$ of $I(t)$ for a sufficiently large t obtained by new monotone techniques (see (3.25) in Theorem 3.2 which may be not convergent to I^*) to Lyapunov techniques established by McCluskey [5, Proof of Theorem 4.1], we are in a position to prove the global asymptotic stability of the endemic equilibrium E_* for $R_0 > 1$.

Proof of Theorem 1.2.
i) By Theorem 3.1, it is evident.
ii) Now, assume (1.16). Let us consider the following Lyapunov functional (cf. McCluskey [5, Theorem 4.1]).

$$U(t) = \frac{1}{\tilde{k}G(I^*)}U_S(t) + \frac{I^*}{\tilde{k}S^*G(I^*)}U_I(t) + U_+(t), \tag{4.14}$$

where

$$U_S(t) = g\left(\frac{S(t)}{S^*}\right),\ U_I(t) = g\left(\frac{I(t)}{I^*}\right),\ U_+(t) = \int_{t-\tau}^{t} g\left(\frac{G(I(u))}{G(I^*)}\right)du.$$

We now show that $\frac{dU(t)}{dt} \le 0$. First, we calculate $\frac{dU_S(t)}{dt}$.

$$\frac{dU_S(t)}{dt} = \frac{1}{S^*}\left(1 - \frac{S^*}{S(t)}\right)\{\tilde{b} - \tilde{d}S(t) - \tilde{k}S(t)G(I(t-\tau)) - \gamma I(t)\}.$$

Substituting $\tilde{b} = \tilde{d}S^* + \tilde{k}S^*G(I^*) + \gamma I^*$ gives

$$\begin{aligned}
\frac{dU_S(t)}{dt} &= \frac{S(t)-S^*}{S^*S(t)}\left\{\tilde{d}(S^* - S(t)) + \tilde{k}\{S^*G(I^*) - S(t)G(I(t-\tau))\} + \gamma(I^* - I(t))\right\} \\
&= -\frac{\tilde{d}S(t)}{S^*}\left(1 - \frac{S^*}{S(t)}\right)^2 + \tilde{k}G(I^*)\left(1 - \frac{S^*}{S(t)}\right)\left\{1 - \frac{S(t)G(I(t-\tau))}{S^*G(I^*)}\right\} + \frac{\gamma I^*}{S^*}\left(1 - \frac{S^*}{S(t)}\right)\left(1 - \frac{I(t)}{I^*}\right) \\
&= -\tilde{d}x_t\left(1 - \frac{1}{x_t}\right)^2 + \tilde{k}G(I^*)\left(1 - \frac{1}{x_t}\right)(1 - x_t z_{t,\tau}) + \frac{\gamma I^*}{S^*}\left(1 - \frac{1}{x_t}\right)(1 - y_t). \qquad (4.15)
\end{aligned}$$

We secondly calculate $\frac{dU_I(t)}{dt}$.

$$\frac{dU_I(t)}{dt} = \frac{I(t) - I^*}{I^*I(t)}\{\tilde{k}S(t)G(I(t-\tau)) - (d+\mu)I(t)\}.$$

Substituting $(d+\mu)I^* = \tilde{k}S^*G(I^*)$ gives

$$\begin{aligned}
\frac{dU_I(t)}{dt} &= \frac{I(t) - I^*}{I^*I(t)}\left\{\tilde{k}S(t)G(I(t-\tau)) - \frac{\tilde{k}S^*G(I^*)}{I^*}I(t)\right\} \\
&= \frac{\tilde{k}S^*G(I^*)}{I^*}\left(1 - \frac{I^*}{I(t)}\right)\left(\frac{S(t)G(I(t-\tau)}{S^*G(I^*)} - \frac{I(t)}{I^*}\right) \\
&= \frac{\tilde{k}S^*G(I^*)}{I^*}\left(1 - \frac{1}{y_t}\right)(x_t z_{t,\tau} - y_t). \qquad (4.16)
\end{aligned}$$

Next, calculating $\frac{dU_+(t)}{dt}$ gives as follows.

$$\frac{dU_+(t)}{dt} = g\left(\frac{G(I(t))}{G(I^*)}\right) - g\left(\frac{G(I(t-\tau))}{G(I^*)}\right) = g(z_t) - g(z_{t,\tau}). \qquad (4.17)$$

Since it follows that

$$\begin{aligned}
&\left(1 - \frac{1}{x_t}\right)(1 - x_t z_{t,\tau}) + \left(1 - \frac{1}{y_t}\right)(x_t z_{t,\tau} - y_t) + \{g(z_t) - g(z_{t,\tau})\} \\
=\ & \left(1 - \frac{1}{x_t} - x_t z_{t,\tau} + z_{t,\tau}\right) + \left(x_t z_{t,\tau} - \frac{x_t z_{t,\tau}}{y_t} - y_t + 1\right) + \{z_t - z_{t,\tau} - \ln z_t + \ln z_{t,\tau}\} \\
=\ & -\left(\frac{1}{x_t} - 1 - \ln x_t\right) - \left(\frac{x_t z_{t,\tau}}{y_t} - 1 - \ln\frac{x_t z_{t,\tau}}{y_t}\right) - (y_t - 1 - \ln y_t) + (z_t - 1 - \ln z_t) \\
=\ & -\left\{g\left(\frac{1}{x_t}\right) + g\left(\frac{x_t z_{t,\tau}}{y_t}\right)\right\} - \{g(y_t) - g(z_t)\},
\end{aligned}$$

by combining (4.15)-(4.17) and Lemma 4.3, we obtain

$$\begin{aligned}
\frac{dU(t)}{dt} =& -\frac{\tilde{d}x_t}{\tilde{k}G(I^*)}\left(1 - \frac{1}{x_t}\right)^2 - \frac{\gamma I^*}{\tilde{k}S^*G(I^*)}\left(1 - \frac{1}{x_t}\right)(1 - y_t) \\
&+ \left(1 - \frac{1}{x_t}\right)(1 - x_t z_{t,\tau}) + \left(1 - \frac{1}{y_t}\right)(x_t z_{t,\tau} - y_t) + \{g(z_t) - g(z_{t,\tau})\} \\
=& -\frac{\tilde{d}x_t}{\tilde{k}G(I^*)}\left(1 - \frac{1}{x_t}\right)^2 - \frac{\gamma I^*}{\tilde{k}S^*G(I^*)}\left(1 - \frac{1}{x_t}\right)(1 - y_t) - \left\{g\left(\frac{1}{x_t}\right) + g\left(\frac{x_t z_{t,\tau}}{y_t}\right)\right\} - \{g(y_t) - g(z_t)\} \\
\le& -\frac{\tilde{d}x_t}{\tilde{k}G(I^*)}\left(1 - \frac{1}{x_t}\right)^2 + \frac{\gamma I^*}{\tilde{k}S^*G(I^*)}\left(1 - \frac{1}{x_t}\right)(y_t - 1) - \frac{\alpha(I(t) + I^*)I^*\{1 - \alpha I(t)I^*\}}{(1 + \alpha I^2(t))(1 + \alpha(I^*)^2)}(y_t - 1)^2 - g\left(\frac{x_t z_{t,\tau}}{y_t}\right).
\end{aligned}$$

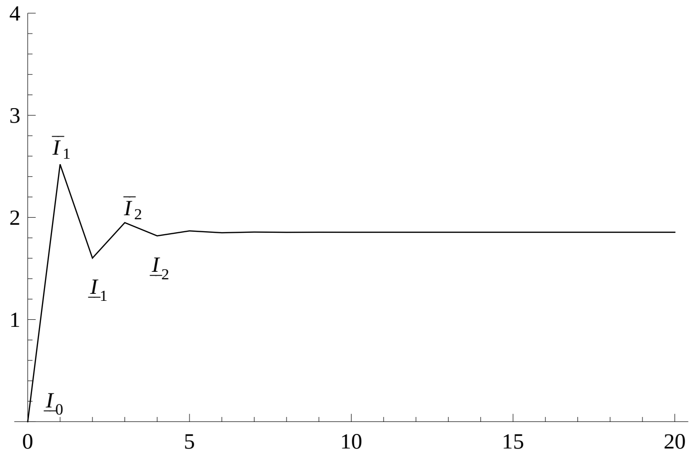

Figure 1: A graph trajectory of $\underline{I}_{n-1}$ and $\overline{I}_n$ for $n \geq 1$ ($0 = \underline{I}_0 \to \overline{I}_1 \to \underline{I}_1 \to \cdots$) of (3.16) for the case (5.3).

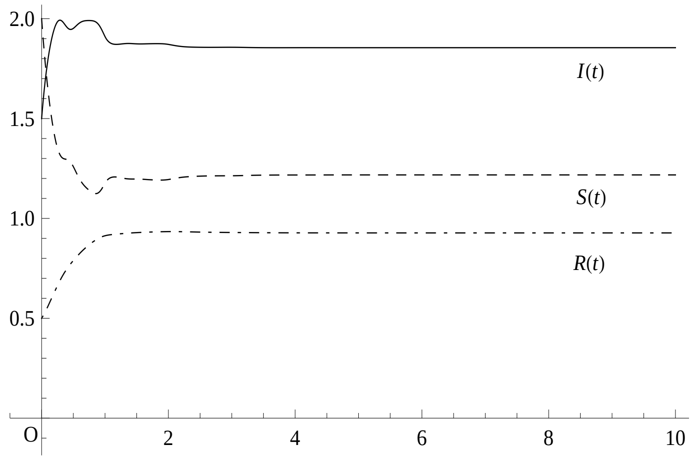

Figure 2: A graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of (1.1) for the case (5.3).

Since it follows from (1.16) and the relation $\tilde{k}S^*G(I^*) = (d+\mu)I^*$ that

$$\begin{aligned} &\left(\frac{\gamma I^*}{\tilde{k}S^*G(I^*)}\right)^2 - 4\frac{\tilde{d}x_t}{\tilde{k}G(I^*)} \cdot \frac{\alpha(I(t)+I^*)I^*\{1-\alpha I(t)I^*\}}{(1+\alpha I^2(t))(1+\alpha(I^*)^2)} \\ =& \left(\frac{I^*}{\tilde{k}S^*G(I^*)}\right)^2 \left\{\gamma^2 - 4\tilde{d}S^*x_t\frac{\tilde{k}S^*G(I^*)}{I^*} \cdot \frac{\alpha(I(t)+I^*)\{1-\alpha I(t)I^*\}}{(1+\alpha I^2(t))(1+\alpha(I^*)^2)}\right\} \\ =& \left(\frac{1}{d+\mu}\right)^2 \left\{\gamma^2 - 4(d+\mu)(d+\gamma)S(t) \cdot \frac{\alpha(I(t)+I^*)\{1-\alpha I(t)I^*\}}{(1+\alpha I^2(t))(1+\alpha(I^*)^2)}\right\} < 0, \end{aligned} \tag{4.18}$$

for any sufficiently large $t > 0$, we obtain $\frac{dU(t)}{dt} \leq 0$. By Theorem 2.4, solutions of system (4.1) limit to M, the largest invariant subset of $\{\frac{dU(t)}{dt} = 0\}$. Since $\frac{dU(t)}{dt} = 0$ holds if $x_t = 1$, $y_t = 1$ and $x_t z_{t,\tau}/y_t = 1$ or equivalently, if $S(t) = S^*$ and $I(t) = I^*$. Therefore, M consists only the endemic equilibrium Q_* of (4.1). It follows from the permanence result in Theorem 2.5 and LaSalle's invariance principle that Q_* is globally asymptotically stable. This corresponds to the proof of the main part of *ii*) in Theorem 1.2, and the last part of Theorem 1.2 is evident by Theorem 3.2. □

5 Numerical Examples

In this section, we provide two numerical examples which is applicable to one of Theorems 1.1 and 1.2. Set

$$f_1 = \frac{k\exp(-d\tau)}{(d+\mu)\sqrt{\alpha}} + 2 - R_0,\ f_2 = \frac{d+\gamma}{\mu}\left(1 + \frac{d+\mu}{k\exp(-d\tau)}I^*\right) - 1, \tag{5.1}$$

and

$$f_3 = 4(d+\mu)(d+\gamma)\underline{S}^*\frac{\alpha(\underline{I}^* + I^*)(1-\alpha\overline{I}^*I^*)}{(1+\alpha(\overline{I}^*)^2)(1+\alpha(I^*)^2)} - \gamma^2. \tag{5.2}$$

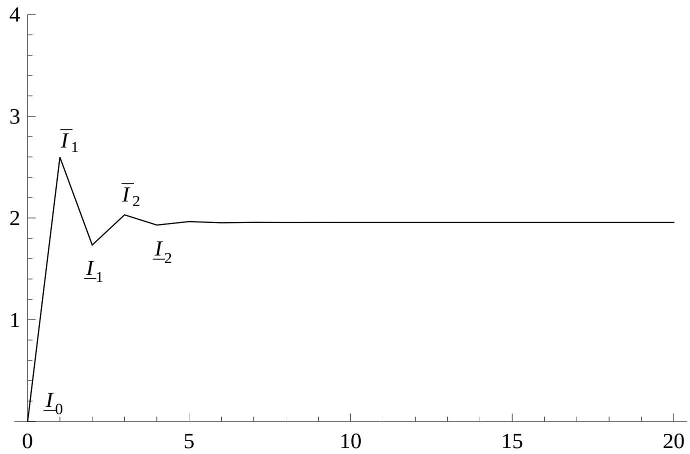

Figure 3: A graph trajectory of $\underline{I}_{n-1}$ and $\overline{I}_n$ for $n \geq 1$ $(0 = \underline{I}_0 \to \overline{I}_1 \to \underline{I}_1 \to \cdots)$ of (3.16) for the case (5.4).

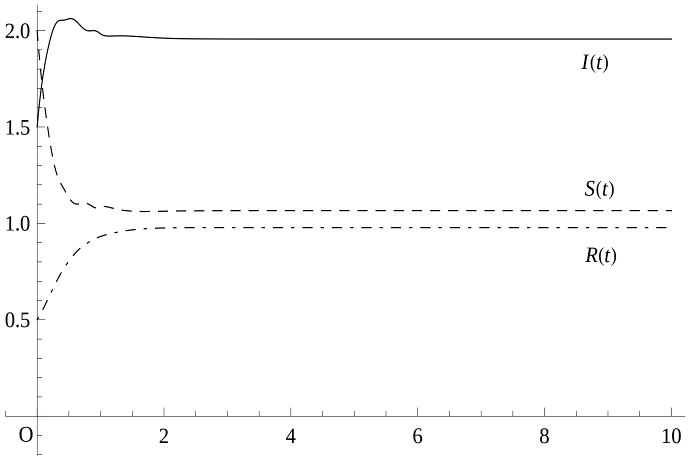

Figure 4: A graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of (1.1) for the case (5.4).

If $f_1 > 0$ and $f_2 > 0$, then the condition (1.6) in Theorems 1.1 holds true and if $I^* \leq \hat{I}$ and $f_3 > 0$, then the condition (1.16) in Theorem 1.2 holds true.

First, consider an example of (1.1) with

$$b = 4,\ k = 6,\ \alpha = 0.1,\ d = \gamma = \mu = 1,\ \text{and}\ \tau = 1, \tag{5.3}$$

and consider (1.10) with $\underline{I}_0 = 0$. Then, $\bar{I}_1 \leq b/d = 4$. For this case, we obtain that $R_0 = 4.4145\cdots > 1$ and $E_* = (1.2178\cdots, 1.8547\cdots, 0.9273\cdots)$. Then, it holds that $f_1 = 1.0754\cdots > 0$, $f_2 = 4.3612\cdots > 0$. By Theorem 1.2, we obtain that the endemic equilibrium E_* of system (1.1) is globally asymptotically stable. Figure 1 indicates that the both sequences $\{\bar{I}_n\}_{n=1}^{+\infty}$ and $\{\underline{I}_n\}_{n=1}^{+\infty}$ defined by (3.9), converge to I^* for the case (5.3). Figures 2 shows us a graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of (1.1) for the case (5.3) which indicates that the endemic equilibrium E_* of system (1.1) is globally asymptotically stable for the case (5.3).

Second, we give an another example of (1.1) with

$$b = 4,\ k = 9,\ \alpha = 0.2,\ d = \gamma = \mu = 1,\ \text{and}\ \tau = 1. \tag{5.4}$$

For this case, $R_0 = 6.6218\cdots > 1$ and $E_* = (1.0662\cdots, 1.9558\cdots, 0.9779\cdots)$ with $I^* < \hat{I} = 2.2360$, but $f_1 = -0.9201\cdots < 0$, $f_2 = 3.3629\cdots > 0$. If we chose that $\underline{I}^* = \underline{I}_0 = 0$ and $\bar{I}^* = b/d = 4$, then $1 - a\bar{I}_0 I^* \leq -0.5646\cdots < 0$. Thus, the conditions in Theorems 1.1 and (1.16) in Theorem 1.2 are not satisfied. On the other hand, by monotone iterations (3.9), we use that $\underline{I}^* = \bar{I}_4 = 1.9552\cdots$ and $\bar{I}^* = \bar{I}_4 = 1.9568\cdots$, from which we obtain $1 - a\bar{I}_4 I^* = 0.2345\cdots > 0$ and $f_3 \geq 0.0039\cdots > 0$. Therefore, by applying Theorem 1.2, we obtain that the endemic equilibrium E_* of system (1.1) is globally asymptotically stable. Note that for the case (5.4), Figure 3 indicates that the both sequences $\{\bar{I}_n\}_{n=1}^{+\infty}$ and $\{\underline{I}_n\}_{n=1}^{+\infty}$ defined by (3.9), seems to converge numerically to I^*, as $n \to \infty$. Figures 4 shows us a graph trajectory of $S(t)$, $I(t)$ and $R(t)$ of (1.1) for the case (5.4) which indicates that the endemic equilibrium E_* of system (1.1) is globally asymptotically stable for the case (5.4).

6 Conclusion

In this paper, for the basic reproduction number $R_0 > 1$ of system (1.1), applying both of new monotone iteration techniques and Lyapunov functional techniques in McCluskey [5], we establish two types of sufficient conditions for the global asymptotic stability of endemic equilibrium E_* of system (1.1), one (see Theorem 1.1) is obtained by a simple conditions of contractive convergence for suitable monotone iterations (see (3.9)), and the other (Theorem 1.2) is derived by applying the best possible lower bound $\underline{I}^*$ and upper bounds $\bar{I}^*$ of $I(t)$ for a sufficiently large t which are obtained by new monotone iterative techniques (see (3.25) in Theorem 3.2) to Lyapunov functional techniques to an SIR model by perturbation (see Lemma 3.4 and Section 4). Note that by the sake of Lemma 3.3, our monotone iterative techniques become much improved one than that in Xu and Ma [8] (see also Xu and Ma [7]) which was applied to the saturated incidence rate $G(I) = \frac{I}{1+\alpha I}$.

Then, we have solved the conjecture to the example in Huo and Ma [4] that the endemic equilibrium of system (1.1) is globally asymptotically stable if $R_0 > 1$, and also give partial answers to the open problem in Huo and Ma [4] and also Yang and Xiao [9].

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