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Global stability for a class of discrete SIR epidemic models

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Abstract. In this paper, we propose a class of discrete SIR epidemic models which are derived from SIR epidemic models with distributed delays by using a variation of the backward Euler method. Applying a Lyapunov functional technique, it is shown that the global dynamics of each discrete SIR epidemic model are fully determined by a single threshold parameter and the effect of discrete time delays are harmless for the global stability of the endemic equilibrium of the model.

Keywords: Backward Euler method; discrete SIR epidemic model; distributed delays; globally asymptotic stability; Lyapunov functional.
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1 Introduction

The application of theories of functional differential/difference equations in mathematical biology has been developed rapidly. Various mathematical models have been proposed in the literature of population dynamics, ecology and epidemiology. Many authors have studied the epidemic models, which displays the dynamical behavior of the transmission of infectious diseases (see also [1–18] and the references therein).

Cooke [5] formulated an SIR epidemic model with a bilinear incidence rate and a discrete time delay which takes the form βSI to investigate the spread of an infectious disease transmitted by a vector (e.g. mosquitoes, rats, etc.) after an incubation time denoting the time during which the infectious agents develop in the vector. This is called the phenomena of time delay effect which now has important biological meanings in epidemic models (see [1, 5]).

Based on their ideas, Ma *et al.* [13] considered a continuous-time SIR model with a discrete delay and investigated the global asymptotic stability of the equilibrium, while a continuous SIR model with distributed delays has been also studied in [1, 2, 12, 14, 17]. The distributed delays is more appropriate form than the discrete one because it is considered more realistic to assume that the time delay is not a fixed time but a distributed parameter which is upper bounded by some positive finite time. Beretta and Takeuchi [1] have studied the following continuous SIR model with distributed delays:

$$\begin{cases} \dfrac{ds(t)}{dt} = b - \beta s(t) \displaystyle\int_0^h f(\tau) i(t-\tau) d\tau - \mu_1 s(t), \\ \dfrac{di(t)}{dt} = \beta s(t) \displaystyle\int_0^h f(\tau) i(t-\tau) d\tau - (\mu_2 + \lambda) i(t), \\ \dfrac{dr(t)}{dt} = \lambda i(t) - \mu_3 r(t), \end{cases} \tag{1.1}$$

where $s(t)$, $i(t)$ and $r(t)$ denote the proportions of the population susceptible to the disease, of infective members and of members who have been removed from the possibility of infection at time t, respectively. It is assumed that all newborns

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are susceptible and all recruitment is into the susceptible class at a constant rate $b > 0$. The positive constants μ_1, μ_2 and μ_3 represent the death rates of susceptible, infectious and recovered classes, respectively. Infectiousness is assumed to vary over time from the initial time of infection until a duration h has passed and the function $f(\tau)$ denotes the fraction of vector population in which the time taken to become infectious is τ. The mass action coefficient is β and $f(\tau)$ is chosen so that it is nonnegative and continuous on $[0, h]$ and assume that, for simplicity, $\int_0^h f(\tau)d\tau = 1$.

System (1.1) always has a disease-free equilibrium $E_0 = (b/\mu_1, 0, 0)$. Furthermore, if $R_0 > 1$, then system (1.1) has a unique endemic equilibrium $E_* = (s^*, i^*, r^*)$, where

$$0 < s^* = \frac{\mu_2 + \lambda}{\beta} < \frac{b}{\mu_1},\ 0 < i^* = \frac{b - \mu_1 s^*}{\beta s^*} < \frac{b}{\mu_1},\ 0 < r^* = \frac{\lambda i^*}{\mu_3} < \frac{b}{\mu_1}, \tag{1.2}$$

and

$$R_0 = \frac{b\beta}{\mu_1(\mu_2 + \lambda)}. \tag{1.3}$$

For the case $R_0 \le 1$, it is known that disease dies out for (1.1) (see also [1, 2, 17]). For the case $R_0 > 1$, it is shown that system (1.1) is permanent by Ma *et al.* [12]. Later, by using a Lyapunov functional for $R_0 > 1$, complete analysis of system (1.1) has been established by McCluskey [14]. Thereafter, using the similar techniques, McCluskey [15] also established a complete analysis of an SIR epidemic models with a saturated incidence rate and a discrete delay.

Summarizing the above discussion, the following result holds (see [1, 2, 12, 17] and [14, Theorem 4.1]).

Theorem A. *The following statement holds.*

(i) *Let us assume $R_0 \le 1$. Then the disease-free equilibrium E_0 of (1.1) is globally attractive, and locally asymptotically stable if the inequality is strict.*

(ii) *Let us assume $R_0 > 1$. Then the endemic equilibrium E_* of (1.1) is globally asymptotically stable.*

On the other hand, there occur situations such that constructing discrete epidemic models is more appropriate approach to understand disease transmission dynamics and to evaluate eradication policies because they permit arbitrary time-step units, preserving the basic features of corresponding continuous-time models. Furthermore, this allows better use of statistical data for numerical simulations due to the reason that the infection data are compiled at discrete given time intervals. For a discrete epidemic model with immigration of infectives, Jang and Elaydi [10] showed the global asymptotic stability of the disease-free equilibrium, the local asymptotic stability of the endemic equilibrium and the strong persistence of susceptible class by means of the nonstandard discretization method. Recently, using a discretization called "mixed type" formula in Izzo and Vecchio [8] and Izzo *et al.* [9], Sekiguchi [16] obtained the permanence of a class of SIR discrete epidemic models with one delay and SEIRS discrete epidemic models with two delays if an endemic equilibrium of each model exists. For the detailed property for a class of discrete epidemic models, we refer to [3, 4, 8–11, 16, 18].

However, in those cases, how to choose the discrete schemes which guarantee the global asymptotic stability for the endemic equilibrium of the models, was still an open problem.

In this paper, motivated by the above facts, we propose the following discrete SIR epidemic model which is derived from system (1.1) by applying a variation of backward Euler method.

$$\begin{cases} s(p+1) - s(p) = b - \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) - \mu_1 s(p+1), \\ i(p+1) - i(p) = \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) - (\mu_2 + \lambda) i(p+1), \\ r(p+1) - r(p) = \lambda i(p+1) - \mu_3 r(p+1), \end{cases} \tag{1.4}$$

where b, β, μ_i $(i = 1, 2, 3)$, λ and m are positive constants and $f(j) \ge 0,\ j = 0, 1, \cdots, m$. For simplicity, we may assume that $\sum_{j=0}^{m} f(j) = 1$.

Similar to the case of continuous system (1.1), system (1.4) always has a disease-free equilibrium $E_0 = (b/\mu_1, 0, 0)$. Furthermore, if $R_0 > 1$, then system (1.4) has a unique endemic equilibrium $E_* = (s^*, i^*, r^*)$ defined by (1.2).

The initial conditions of system (1.4) is

$$\begin{cases} s(p) = \phi(p) \ge 0,\ i(p) = \psi(p) \ge 0,\ r(p) = \sigma(p) \ge 0,\ p = -m, -(m-1), \cdots, -1, \\ s(0) > 0,\ i(0) > 0,\ r(0) > 0. \end{cases} \tag{1.5}$$

Remark 1.1. To prove the positivity of $s(p)$, $i(p)$ and $r(p)$ for any $p \geq 0$, we need to use the backward Euler discretization instead of the forward Euler discretization (see, e.g., [8, 9]). Moreover, to apply a discrete time analogue of the Lyapunov function proposed by McCluskey [14, 15], we adopt a variation of the backward Euler method which is different from that of Sekiguchi [16].

Using the same threshold $R_0 = \frac{b\beta}{\mu_1(\mu_2+\gamma)}$ and applying the both techniques in Izzo and Vecchio [8] and Izzo *et al.* [9] for the case $R_0 \leq 1$ and McCluskey [14, Theorem 4.1] for the case $R_0 > 1$ to system (1.4), we establish a complete analysis of the global asymptotic stability for system (1.4). In particular, by applying Lemma 4.1, we offer a simplified proof for the permanence of system (1.4) for $R_0 > 1$ (cf. Sekiguchi [16]).

Our main result in this paper is as follows.

Theorem 1.1. *The disease-free equilibrium $E_0 = (b/\mu_1, 0, 0)$ of (1.4) is globally asymptotically stable if and only if $R_0 \leq 1$, and the endemic equilibrium $E_* = (s^*, i^*, r^*)$ of (1.4) exists uniquely and is globally asymptotically stable, if and only if, $R_0 > 1$.*

Remark 1.2. Theorem 1.1 for system (1.4) is just a discrete analogue of Theorem A for system (1.1).

The organization of this paper is as follows. In Section 2, we offer some basic results for system (1.4) and some asymptotic properties are investigated. The first part of Theorem 1.1 concerning the global asymptotic stability for $R_0 \leq 1$ is given in Section 3, and the second part of Theorem 1.1 concerning the permanence and the global stability for $R_0 > 1$ is given in Section 4. Finally, we offer concluding remarks in Section 5.

2 Basic properties

For system (1.4), since the variable r does not appear in the first and the second equations, it is sufficient to consider the following 2-dimensional system:

$$\begin{cases} s(p+1) - s(p) = b - \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) - \mu_1 s(p+1), \\ i(p+1) - i(p) = \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) - (\mu_2 + \lambda) i(p+1), \end{cases} \tag{2.1}$$

with the initial conditions

$$s(p) = \phi(p) \geq 0, \ i(p) = \psi(p) \geq 0, \ p = -m, -(m-1), \cdots, -1, \ s(0) > 0, \ i(0) > 0. \tag{2.2}$$

The following results in Sections 2 and 3 are obtained by applying techniques in Izzo and Vecchio [8] and Izzo *et al.* [9] to system (2.1).

Lemma 2.1. *Let $(s(p), i(p))$ be the solutions of system (2.1) with the initial conditions (2.2). Then $s(p) > 0$, $i(p) > 0$ for any $p > 0$.*

Proof. Assume that $s(p-j), i(p-j) > 0, \ j = 0, 1, \cdots, m$. Then, system (2.1) becomes

$$\begin{cases} s(p+1)\left\{1 + \mu_1 + \beta \sum_{j=0}^{m} f(j) i(p-j)\right\} = b + s(p) > 0, \\ i(p+1)\{1 + (\mu_2 + \lambda)\} = i(p) + \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) > 0. \end{cases} \tag{2.3}$$

By the first equation of (2.3), we have $s(p+1) > 0$, and by the second equation of (2.3), we have $i(p+1) > 0$. Hence, by induction, we prove this lemma. $\square$

Lemma 2.2. *Any solution $(s(p), i(p))$ of system (2.1) with the initial conditions (2.2) satisfies $\limsup_{p\to+\infty}(s(p) + i(p)) \leq b/\underline{\mu}$, where $\underline{\mu} = \min\{\mu_1, \mu_2 + \lambda\}$.*

Proof. Let $V(p) = s(p) + i(p)$. From system (2.1), we have that

$$V(p+1) - V(p) = b - \mu_1 s(p+1) - (\mu_2 + \lambda) i(p+1) \leq b - \underline{\mu} V(p+1),$$

from which we have $\limsup_{p\to+\infty} V(p) \leq \frac{b}{\underline{\mu}}$. Hence, the proof is complete. $\square$

We now put

$$\underline{s} = \liminf_{p\to+\infty} s(p), \ \ \overline{s} = \limsup_{p\to+\infty} s(p), \ \ \underline{i} = \liminf_{p\to+\infty} i(p), \ \ \overline{i} = \limsup_{p\to+\infty} i(p).$$

Then by $\sum_{j=0}^{m} f(j) = 1$, similar to the Izzo *et al.* [9, Proof of Lemma 3.3], we obtain the following basic lemma:

Lemma 2.3. *For any solution $(s(p), i(p))$ of system (2.1) with the initial conditions (2.2), we have that*

$$\begin{cases} 0 < \dfrac{b}{\mu_1 + \beta \bar{i}} \le \underline{s} \le \bar{s} \le \dfrac{b}{\mu_1 + \beta \underline{i}} \le \dfrac{b}{\mu_1}, \quad 0 \le \underline{i} \le \bar{i}, \\ \dfrac{\beta \underline{s}}{\mu_2 + \lambda} \le 1, \quad \text{if} \quad \underline{i} > 0, \\ \dfrac{\beta \bar{s}}{\mu_2 + \lambda} \ge 1, \quad \text{if} \quad \bar{i} > 0. \end{cases} \tag{2.4}$$

Furthermore, we put

$$P(s) \equiv \frac{\beta s}{\mu_2 + \lambda}, \tag{2.5}$$

then we easily obtain the following lemma:

Lemma 2.4. *$P(s)$ is strictly monotone increasing function on $[0, +\infty)$ satisfying*

$$P(0) = 0, \quad \text{and} \quad R_0 = P\left(\frac{b}{\mu_1}\right), \tag{2.6}$$

and

$$\begin{cases} P(\underline{s}) \le 1, \quad \text{if} \quad \underline{i} > 0, \\ P(\bar{s}) \ge 1, \quad \text{if} \quad \bar{i} > 0. \end{cases} \tag{2.7}$$

If $R_0 > 1$, then there exists a unique solution $s = s^*$ of $P(s) = 1$ such that

$$0 < s^* = \frac{\mu_2 + \lambda}{\beta} < \frac{b}{\mu_1}. \tag{2.8}$$

Proof. By the definition of $P(s)$,it is clear that $P(s)$ is strictly monotone increasing on $[0, +\infty)$. By (2.4), we have (2.6) and (2.7). If $R_0 > 1$, then $P(0) = 0 < 1 < R_0 = P(\frac{b}{\mu_1})$. Hence, there exists a unique solution $0 < s = s^* = \frac{\mu_2+\lambda}{\beta} < \frac{b}{\mu_1}$ of $P(s) = 1$. □

3 Global stability of the disease-free equilibrium $E_0 = (b/\mu_1, 0)$ for $R_0 \le 1$

In this section, we prove the first part of Theorem 1.1 for system (1.4). By applying techniques in Izzo *et al.* [9], we obtain the global asymptotic stability of the disease-free equilibrium E_0 of system (2.1) for $R_0 \le 1$.

Lemma 3.1. *If $R_0 \le 1$, then*

$$\lim_{p \to +\infty} s(p) = \frac{b}{\mu_1}, \quad \text{and} \quad \lim_{p \to +\infty} i(p) = 0, \tag{3.1}$$

and $E_0 = (\frac{b}{\mu_1}, 0)$ is globally asymptotically stable.

Proof. From (2.4) in Lemma 2.3, for any $\epsilon > 0$, there is an integer $p_0 \ge 0$ such that

$$s(p + 1) \le \frac{b}{\mu_1} + \epsilon \quad \text{for } p \ge p_0.$$

Consider the following sequence $\{w(p)\}_{p=p_0}^{+\infty}$ defined by

$$w(p) = i(p) + u(p), \quad p \ge p_0, \tag{3.2}$$

where

$$u(p) = \beta \sum_{j=0}^{m} f(j) \sum_{k=p-j}^{p} s(j + k + 1) i(k), \quad p \ge p_0.$$

Since

$$\begin{aligned} u(p+1) - u(p) &= \beta \sum_{j=0}^{m} f(j) \left\{ \sum_{k=p+1-j}^{p+1} s(j + k + 1) i(k) - \sum_{k=p-j}^{p} s(j + k + 1) i(k) \right\} \\ &= \beta \sum_{j=0}^{m} f(j) \left\{ s(p + j + 2) i(p + 1) - s(p + 1) i(p - j) \right\}, \end{aligned}$$

we have that

$$\begin{aligned} w(p+1)-w(p) &= \beta s(p+1)\sum_{j=0}^{m} f(j)i(p-j)-(\mu_2+\gamma)i(p+1)\\ &\quad+\beta\sum_{j=0}^{m} f(j)\Big\{s(p+j+2)i(p+1)-s(p+1)i(p-j)\Big\}\\ &=\beta\sum_{j=0}^{m} f(j)s(p+j+2)i(p+1)-(\mu_2+\gamma)i(p+1)\\ &\le\beta\sum_{j=0}^{m} f(j)\Big(\frac{b}{\mu_1}+\epsilon\Big)i(p+1)-(\mu_2+\gamma)i(p+1)\\ &=\Big\{\frac{b\beta}{\mu_1}-(\mu_2+\gamma)\Big\}i(p+1)+\beta\epsilon i(p+1). \end{aligned}$$

Since $\epsilon>0$ is arbitrary, we obtain that if $R_0\le 1$, then

$$w(p+1)-w(p)\le\Big\{\frac{b\beta}{\mu_1}-(\mu_2+\gamma)\Big\}i(p+1)\le 0, \tag{3.3}$$

and the nonnegative sequence $\{w(p)\}_{p=p_0}^{+\infty}$ is monotone decreasing. Therefore, there exists a nonnegative constant $\hat{w}$ such that $\lim_{p\to+\infty} w(p)=\hat{w}$. We will prove that $\hat{w}=0$ and (3.1) hold for $R_0\le 1$. If $R_0<1$, then $\frac{b\beta}{\mu_1}-(\mu_2+\gamma)<0$ and by (3.3), we conclude that $\lim_{p\to+\infty} i(p)=0$. Then, $\hat{w}=0$ and by Lemma 2.3, we obtain (3.1). Suppose that $R_0=1$. From (1.4), we have that

$$\begin{cases} s(p+1)=\tilde{b}+\tilde{c}s(p)-\tilde{\beta}_1 s(p+1)\displaystyle\sum_{j=0}^{m} f(j)i(p-j),\\ i(p+1)=\tilde{d}i(p)+\tilde{\beta}_2 s(p+1)\displaystyle\sum_{j=0}^{m} f(j)i(p-j), \end{cases} \tag{3.4}$$

where

$$\tilde{b}=\frac{b}{1+\mu_1},\ \tilde{c}=\frac{1}{1+\mu_1},\ \tilde{\beta}_1=\frac{\beta}{1+\mu_1},\ \tilde{d}=\frac{1}{1+\mu_2+\gamma},\ \tilde{\beta}_2=\frac{\beta}{1+\mu_2+\gamma}.$$

Then, by applying the similar techniques in Izzo *et al.* [9, Proofs of Lemmas 4.4 and 4.5] to (3.4) with Lemmas 2.3 and 2.4, we prove the claim that there is a sequence $\{p_k\}_{k=0}^{+\infty}$ such that each of $i(p_k-j),\ j=0,1,\cdots,m$ converges to 0 as $k\to+\infty$. If $\overline{i}=0$, then the claim is evident. Now, suppose that $\overline{i}>0$. By (2.7) in Lemma 2.4, we have that $P(\overline{s})\ge 1=R_0=P(\frac{b}{\mu_1})$, which implies $\overline{s}=\frac{b}{\mu_1}$ from (2.4) in Lemma 2.3. Therefore, there exists a sequence $\{p_k\}_{k=1}^{+\infty}$ such that $\lim_{k\to+\infty} s(p_k+1)=\overline{s}$. Then, $\frac{\tilde{b}}{1-\tilde{c}}=\frac{b}{\mu_1}=\overline{s}$ and

$$s(p_k+1)=\frac{\tilde{b}-\tilde{c}\{s(p_k+1)-s(p_k)\}-\tilde{\beta}_1 s(p_k+1)\sum_{j=0}^{m} f(j)i(p_k-j)}{1-\tilde{c}}\to\overline{s},$$

as $k\to+\infty$. We then obtain

$$\lim_{k\to+\infty}\Big[\tilde{c}\{s(p_k+1)-s(p_k)\}+\tilde{\beta}_1 s(p_k+1)\sum_{j=0}^{m} f(j)i(p_k-j)\Big]=0.$$

Since $\limsup_{k\to+\infty}\{s(p_k+1)-s(p_k)\}\ge 0$, we obtain that

$$\limsup_{k\to+\infty}\{s(p_k+1)-s(p_k)\}=0,\ \text{and}\ \limsup_{k\to+\infty}\sum_{j=0}^{m} f(j)i(p_k-j)=0.$$

Thus, it holds that

$$\lim_{k\to+\infty}\{s(p_k+1)-s(p_k)\}=0,\ \text{and}\ \lim_{k\to+\infty}\sum_{j=0}^{m} f(j)i(p_k-j)=0.$$

Hence, it follows that $\lim_{k\to+\infty} s(p_k) = \lim_{k\to+\infty} s(p_k+1) = \overline{s}$ and

$$\lim_{k\to+\infty} \sum_{j=0}^{m} f(j) i(p_k - j) = 0.$$

In the same way, we can prove that

$$\lim_{k\to+\infty} s(p_k - l) = \overline{s}, \ \lim_{k\to+\infty} \sum_{j=0}^{m} f(j) i(p_k - l - j) = 0, \ \text{for } l = 0, 1, \cdots, m.$$

Since by $\sum_{j=0}^{m} f(j) = 1$, there exists at least one $0 \le j_0 \le m$ such that $f(j_0) > 0$, we can obtain that $i(p_k - l - j_0) = 0$, $l = 0, 1, \cdots, m$. Hence, the claim is proved. Then, by the definition of $w(p)$, we obtain $\lim_{k\to+\infty} w(p_k+1) = 0$, which implies that $\hat{w} = 0$. Hence, by $\lim_{p\to+\infty} w(p) = 0$, we can conclude that (3.1) still holds for $R_0 = 1$.

Finally, we will prove that if $R_0 \le 1$, then $E_0 = (\frac{b}{\mu_1}, 0)$ is uniformly stable. First, consider the case that there exists a nonnegative integer p_1 such that $s(p+1) > \overline{s} = \frac{b}{\mu_1}$ for any $p \ge p_1$. From the first equation of (2.1), we have that for any $p > 0$,

$$s(p+1) - s(p) = (b - \mu_1 s(p+1)) - \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) \le 0. \tag{3.5}$$

Then, we obtain that $\frac{b}{\mu_1} \le s(p+1) \le s(p) \le s(p_1)$ for any $p \ge p_1$. Next, consider the case that there exists a nonnegative integer p_2 such that $s(p_2) \le \overline{s} = \frac{b}{\mu_1}$. Then, by the first equation of (2.1), we have that

$$s(p_2+1) \le \frac{b + s(p_2)}{1+\mu_1} \le \frac{b + b/\mu_1}{1+\mu_1} = \frac{b}{\mu_1}.$$

Thus, we obtain that $s(p) \le \overline{s} = \frac{b}{\mu_1}$, for any $p \ge p_2$. Then, by the second equation of (2.1) and $\frac{b\beta}{\mu_1} \le \mu_2 + \lambda$, we have that for any $p \ge p_2$,

$$i(p+1) \le \frac{i(p) + \beta(b/\mu_1) \sum_{j=0}^{m} f(j) i(p-j)}{1 + (\mu_2 + \lambda)} \le \frac{1 + (\mu_2+\lambda)\sum_{j=0}^{m} f(j)}{1 + (\mu_2+\lambda)} \max_{0\le j\le m} i(p-j) = \max_{0\le j\le m} i(p-j). \tag{3.6}$$

Moreover, by the first equation of (2.1), we have that

$$s(p+1) - \frac{b}{\mu_1} = \frac{1}{1+\mu_1}\left(s(p) - \frac{b}{\mu_1}\right) - \frac{\beta s(p+1)}{1+\mu_1} \sum_{j=0}^{m} f(j) i(p-j),$$

and hence, for any $p \ge p_2$, we obtain that

$$\begin{aligned}\left|s(p+1) - \frac{b}{\mu_1}\right| &\le \left(\frac{1}{1+\mu_1}\right)^{p-p_2+1} \left|s(p_2) - \frac{b}{\mu_1}\right| + \frac{\beta(b/\mu_1)}{1+\mu_1} \frac{\max_{0\le j\le m} i(p_2 - j)}{1 - 1/(1+\mu_1)} \\ &= \left(\frac{1}{1+\mu_1}\right)^{p-p_2+1} \left|s(p_2) - \frac{b}{\mu_1}\right| + \frac{b\beta}{\mu_1^2} \max_{0\le j\le m} i(p_2 - j).\end{aligned}$$

Thus, by the above discussion, we can conclude that $E_0 = (\frac{b}{\mu_1}, 0)$ is uniformly stable. Hence, if $R_0 \le 1$, then $E_0 = (\frac{b}{\mu_1}, 0)$ is globally asymptotically stable. □

Lemma 3.2. *(3.1) implies $R_0 \le 1$.*

Proof. Suppose that $R_0 = P(\frac{b}{\mu_1}) > 1$. Then, by Lemma 2.4, there exists a positive constant solution $(s(p), i(p)) = (s^*, i^*)$ of system (2.1) defined by (1.2). This contradicts the fact that (3.1) holds. Hence, the proof is complete. □

Proof of the first part of Theorem 1.1. By Lemmas 3.1 and 3.2, we immediately obtain the conclusion of this theorem. □

4 Global stability of the endemic equilibrium $E_* = (s^*, i^*)$ for $R_0 > 1$

In this section, we assume $R_0 > 1$. Similar to the result of McCluskey [14, Theorem 4.1] for the continuous-time SIR model, we prove the second part of Theorem 1.1 for system (1.4). By Lemmas 4.1-4.3, we obtain the permanence of system (2.1) for $R_0 > 1$.

Lemma 4.1. *If $i(p+1) < \min_{0\le j\le m} i(p-j)$, then $s(p+1) < s^*$. Inversely, if $s(p+1) \ge s^*$, then $i(p+1) \ge \min_{0\le j\le m} i(p-j)$.*

Proof. By the second equation of (2.1), we have that

$$i(p+1) = \frac{i(p) - i(p+1)}{\mu_2 + \lambda} + \frac{s(p+1)}{s^*}\sum_{j=0}^{m} f(j)i(p-j).$$

Therefore, if $i(p+1) < \min_{0\le j\le m} i(p-j)$, then by $i(p) - i(p+1) > 0$ and $\sum_{j=0}^m f(j)i(p-j) > i(p+1)$, we have $i(p+1) > \frac{s(p+1)}{s^*} i(p+1)$, from which we obtain $s(p+1) < s^*$. The remaining part of this lemma is evident. □

Lemma 4.2. *If $R_0 > 1$, then for any solution of system (2.1) with the initial conditions (1.5),*

$$\liminf_{p\to+\infty} s(p) \ge v_1 := \frac{b}{1+\mu_1+\beta\frac{b}{\mu}} > 0, \tag{4.1}$$

$$\liminf_{p\to+\infty} i(p) \ge v_2 := \left(\frac{1}{1+\mu_2+\lambda}\right)^{m+l_0} i^* > 0, \tag{4.2}$$

where for $k = \mu_1 + \beta q i^*$, the constant $l_0 \ge 1$ is sufficiently large such that $s^* < s^{\triangle} := \frac{b}{k}\{1 - (\frac{1}{1+k})^{l_0}\}$.

Proof. By the first equation of (2.3), it is straightforward to obtain (4.1). We now show that (4.2) holds. For any $0 < q < 1$, by (1.2), one can see that $s^* = \frac{b}{\mu_1+\beta i^*} < \frac{b}{\mu_1+\beta q i^*}$. We first prove the claim that any solution $(s(p), i(p))$ of system (2.1) does not have the following property: there exists a nonnegative integer p_0 such that $i(p) \le q i^*$ for all $p \ge p_0$. Suppose on the contrary that there exist a solution $(s(p), i(p))$ of system (2.1) and a nonnegative integer p_0 such that $i(p) \le q i^*$ for all $p \ge p_0$. From system (2.1), one can obtain that

$$s(p+1) \ge \frac{s(p)}{1+k} + \frac{b}{1+k}, \quad \text{for } p \ge p_0 + m,$$

which yields that

$$\begin{aligned} s(p+1) &\ge \left(\frac{1}{1+k}\right)^{p+1-(p_0+m)} s(p_0+m) + \frac{b}{1+k}\sum_{l=0}^{p-(p_0+m)}\left(\frac{1}{1+k}\right)^{l} \\ &\ge \frac{b}{1+k}\frac{1-(\frac{1}{1+k})^{p+1-(p_0+m)}}{1-\frac{1}{1+k}} \\ &= \frac{b}{k}\left\{1 - \left(\frac{1}{1+k}\right)^{p+1-(p_0+m)}\right\}, \quad \text{for any } p \ge p_0 + m. \end{aligned}$$

Therefore, we have

$$s(p+1) \ge \frac{b}{k}\left\{1 - \left(\frac{1}{1+k}\right)^{l_0}\right\} = s^{\triangle} > s^*, \quad \text{for any } p \ge p_0 + m + l_0 - 1. \tag{4.3}$$

By the second part of Lemma 4.1, we obtain

$$i(p+1) \ge \min_{0\le j\le m} i(p-j), \quad \text{for any } p \ge p_0 + m + l_0 - 1. \tag{4.4}$$

Thus, there exists a positive constant $\hat{i}$ such that $i(p) \ge \hat{i}$ for any $p \ge p_0 + m + l_0 - 1$. Moreover, for the sequence $\{w(p)\}_{p=p_0}^{+\infty}$ defined by (3.2), we have

$$\begin{aligned} w(p+1) - w(p) &= \beta\sum_{j=0}^{m} f(j)s(p+j+2)i(p+1) - (\mu_2+\gamma)i(p+1) > \{\beta s^{\triangle} - (\mu_2+\gamma)\}i(p+1) \\ &> \{\beta s^{\triangle} - (\mu_2+\gamma)\}\hat{i}, \end{aligned}$$

for $p \ge p_0 + m + l_0 - 1$. By $\beta s^{\triangle} - (\mu_2+\gamma) = \beta(s^{\triangle} - s^*) > 0$, this yields $\lim_{p\to+\infty} w(p) = +\infty$. However, from Lemma 2.2, it holds that $\limsup_{p\to+\infty} w(p) \le \bar{w}$, which leads to a contradiction. Hence, the claim is proved.

By the claim, we are left to consider two possibilities. First, $i(p) \ge q i^*$ for all p sufficiently large. Second, we consider the case that $i(p)$ oscillates about $q i^*$ for all p sufficiently large. We now show that $i(p) \ge q v_2$ for all p sufficiently large.

If the first case holds, then we get the conclusion of the proof. If the second case holds, let $p_1 < p_2$ be sufficiently large such that

$$i(p_1), i(p_2) > qi^*, \quad \text{and} \quad i(p) \leq qi^* \quad \text{for any } p_1 < p < p_2.$$

By the second equation of system (2.1), we have $i(p+1) - i(p) \geq -(\mu_2 + \lambda)i(p+1)$, that is,

$$i(p+1) \geq \frac{1}{1+\mu_2+\lambda} i(p), \text{ for any } p \geq p_1.$$

This implies that

$$i(p+1) \geq \left(\frac{1}{1+\mu_2+\lambda}\right)^{p+1-p_1} i(p_1) \geq \left(\frac{1}{1+\mu_2+\lambda}\right)^{p+1-p_1} qi^*, \text{ for any } p \geq p_1.$$

Therefore, we obtain that

$$i(p+1) \geq \left(\frac{1}{1+\mu_2+\lambda}\right)^{m+l_0} qi^* = qv_2, \text{ for any } p_1 \leq p \leq p_1 + m + l_0 - 1. \tag{4.5}$$

If $p_2 \geq p_1 + m + l_0$, then by applying the similar discussion to (4.3) and (4.4) in place of p_0 by $p_1 + 1$, we obtain that $i(p+1) \geq v_2$ for $p_1 + m + l_0 \leq p \leq p_2 - 1$. Hence, we prove that $i(p+1) \geq qv_2$ for $p_1 \leq p \leq p_2$. Since the interval $p_1 \leq p \leq p_2$ is arbitrarily chosen, we conclude that $i(p+1) \geq qv_2$ for all sufficiently large. Since q $(0 < q < 1)$ is also arbitrarily chosen, we obtain $\liminf_{p\to+\infty} i(p) \geq v_2$. This completes the proof. □

Lemma 4.3. *There exists a unique endemic equilibrium $E_* = (s^*, i^*)$ of system (2.1) and*

$$0 < \frac{b}{\mu_1 + \bar{i}} \leq \underline{s} \leq s^* \leq \bar{s} < \frac{b}{\mu_1}, \quad \text{and} \quad 0 < \underline{i} \leq i^* \leq \bar{i} < \frac{b}{\underline{\mu}}, \tag{4.6}$$

if and only if, $R_0 > 1$.

Proof. By (3.1)-(3.2) and Lemma 4.2, we have that $R_0 > 1$, if and only if, there exists a unique endemic equilibrium $E_* = (s^*, i^*)$ of system (2.1) and $\liminf_{p\to+\infty} i(p) > 0$. Then, by (2.4) and Lemma 2.4, we obtain (4.6). This completes the proof. □

Proof of the second part of Theorem 1.1. Consider the following Lyapunov function (see McCluskey [14,15]).

$$U(p) = \frac{1}{\beta i^*} U_s(p) + \frac{1}{\beta s^*} U_i(p) + U_+(p), \tag{4.7}$$

where

$$U_s(p) = g\left(\frac{s(p)}{s^*}\right), \; U_i(p) = g\left(\frac{i(p)}{i^*}\right), \; U_+(p) = \sum_{j=0}^{m} f(j) \sum_{k=p-j}^{p} g\left(\frac{i(k)}{i^*}\right),$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$, $x > 0$.

We now show that $U(p+1) - U(p) \leq 0$. First, we calculate $U_s(p+1) - U_s(p)$.

$$\begin{aligned}
U_s(p+1) - U_s(p) &= \frac{s(p+1) - s(p)}{s^*} - \ln\frac{s(p+1)}{s(p)} \\
&\leq \frac{s(p+1) - s(p)}{s^*} - \frac{s(p+1) - s(p)}{s(p+1)} \\
&= \frac{1}{s^*} \frac{s(p+1) - s^*}{s(p+1)} (s(p+1) - s(p)) \\
&= \frac{1}{s^*} \frac{s(p+1) - s^*}{s(p+1)} \left\{ b - \beta s(p+1) \sum_{j=0}^{m} f(j) i(p-j) - \mu_1 s(p+1) \right\},
\end{aligned} \tag{4.8}$$

because $\ln(1-x) \leq -x$ holds for any $x < 1$, one can obtain

$$-\ln\frac{s(p+1)}{s(p)} = \ln\left(1 - \left(1 - \frac{s(p)}{s(p+1)}\right)\right) \leq -\left(1 - \frac{s(p)}{s(p+1)}\right) = -\frac{s(p+1) - s(p)}{s(p+1)}.$$

Substituting $b = \beta s^* i^* + \mu_1 s^*$ into (4.8), we see that

$$\begin{aligned} U_s(p+1) - U_s(p) &\leq \frac{1}{s^*}\frac{s(p+1)-s^*}{s(p+1)}\left\{\beta s^* i^* + \mu_1 s^* - \beta s(p+1)\sum_{j=0}^{m} f(j) i(p-j) - \mu_1 s(p+1)\right\} \\ &= -\frac{\mu_1}{s^*}\frac{(s(p+1)-s^*)^2}{s(p+1)} + \beta i^* \sum_{j=0}^{m} f(j)\left(1 - \frac{s^*}{s(p+1)}\right)\left(1 - \frac{s(p+1)}{s^*}\cdot\frac{i(p-j)}{i^*}\right). \end{aligned}$$

Second, similarly, we calculate $U_i(p+1) - U_i(p)$.

$$\begin{aligned} U_i(p+1) - U_i(p) &= \frac{i(p+1)-i(p)}{i^*} - \ln\frac{i(p+1)}{i(p)} \\ &\leq \frac{i(p+1)-i(p)}{i^*} - \frac{i(p+1)-i(p)}{i(p+1)} \\ &= \frac{1}{i^*}\frac{i(p+1)-i^*}{i(p+1)}(i(p+1)-i(p)) \\ &= \frac{1}{i^*}\frac{i(p+1)-i^*}{i(p+1)}\left\{\beta s(p+1)\sum_{j=0}^{m} f(j) i(p-j) - (\mu_2+\lambda) i(p+1)\right\}. \end{aligned}$$

Since $\mu_2 + \lambda = \beta s^*$ holds, we obtain that

$$\begin{aligned} U_i(p+1) - U_i(p) &\leq \frac{1}{i^*}\frac{i(p+1)-i^*}{i(p+1)}\left\{\beta s(p+1)\sum_{j=0}^{m} f(j) i(p-j) - \beta s^* i(p+1)\right\} \\ &= \beta s^* \sum_{j=0}^{m} f(j)\left(1 - \frac{i^*}{i(p+1)}\right)\left(\frac{s(p+1)}{s^*}\cdot\frac{i(p-j)}{i^*} - \frac{i(p+1)}{i^*}\right). \end{aligned}$$

Finally, calculating $U_+(p+1) - U_+(p)$, we get that

$$\begin{aligned} U_+(p+1) - U_+(p) &= \sum_{j=0}^{m} f(j)\left\{\sum_{k=p+1-j}^{p+1}\left\{g\left(\frac{i(k)}{i^*}\right) - \sum_{k=p-j}^{p} g\left(\frac{i(k)}{i^*}\right)\right\}\right. \\ &= \sum_{j=0}^{m} f(j)\left\{g\left(\frac{i(p+1)}{i^*}\right) - g\left(\frac{i(p-j)}{i^*}\right)\right\} \\ &= \sum_{j=0}^{m} f(j) g\left(\frac{i(p+1)}{i^*}\right) - \sum_{j=0}^{m} f(j) g\left(\frac{i(p-j)}{i^*}\right). \end{aligned}$$

Defining

$$x_{p+1} = \frac{s(p+1)}{s^*},\ \ y_{p+1} = \frac{i(p+1)}{i^*},\ \ z_{p,j} = \frac{i(p-j)}{i^*},\ j = 0, 1, \cdots, m,$$

similar to McCluskey [14, Proof of Theorem 4.1], we obtain that

$$\begin{aligned} U(p+1) - U(p) &\leq -\frac{\mu_1}{\beta s^* i^*}\frac{(s(p+1)-s^*)^2}{s(p+1)} - \sum_{j=0}^{m} f(j)\left(-2 + \frac{1}{x_{p+1}} + \frac{x_{p+1} z_{p,j}}{y_{p+1}} + \ln y_{p+1} - \ln z_{p,j}\right) \\ &= -\frac{\mu_1}{\beta s^* i^*}\frac{(s(p+1)-s^*)^2}{s(p+1)} - \sum_{j=0}^{m} f(j)\left[g\left(\frac{1}{x_{p+1}}\right) + g\left(\frac{x_{p+1} z_{p,j}}{y_{p+1}}\right)\right] \\ &\leq 0. \end{aligned}$$

Hence, $U(p+1) - U(p) \leq 0$ for any $p \geq 0$. Since $U(p) \geq 0$ is monotone decreasing sequence, there is a limit $\lim_{p\to+\infty} U(p) \geq 0$. Then, $\lim_{p\to+\infty}(U(p+1) - U(p)) = 0$, from which we obtain $\lim\limits_{p\to+\infty} s(p+1) = s^*$ and

$$\lim_{p\to+\infty}\frac{i(p-j)}{i(p+1)} = \lim_{p\to+\infty}\frac{z_{p,j}}{y_{p+1}} = 1,$$

if $f(j) > 0$, $j = 0, 1, \cdots, m$. By the first equation of (2.1), we have that for $p \geq 0$,

$$\begin{aligned} s(p+1) - s(p) &= (b - \mu_1 s(p+1)) - \beta s(p+1)\sum_{j=0}^{m} f(j) i(p-j) \\ &= (b - \mu_1 s(p+1)) - \beta s(p+1)\frac{\sum_{j=0}^{m} f(j) i(p-j)}{i(p+1)} i(p+1), \end{aligned}$$

which implies

$$i(p+1) = \frac{b - (1+\mu_1)s(p+1) + s(p)}{\beta s(p+1) \frac{\sum_{j=0}^{m} f(j)i(p-j)}{i(p+1)}}.$$

Using the relations $\lim_{p\to+\infty}(b - (1+\mu_1)s(p+1) + s(p)) = b - \mu_1 s^* > 0$ and

$$\lim_{p\to+\infty} \beta s(p+1) \frac{\sum_{j=0}^{m} f(j)i(p-j)}{i(p+1)} = \beta s^* > 0,$$

we obtain that $\lim_{p\to+\infty} i(p+1) = i^*$, that is, $\lim_{p\to+\infty}(s(p), i(p)) = (s^*, i^*)$.

Since $U(p) \le U(0)$ for all $p \ge 0$, we conclude that E_* is uniformly stable. Hence, the proof is complete. □

5 Discussions

Recently, in order to investigate the transmission dynamics of infectious diseases, there have been many papers focusing on the analysis of the global stability of the disease-free equilibrium and the endemic equilibrium of a class of discrete and continuous SIR epidemic models. Motivated by the fact that the discrete models are more appropriate forms than the continuous ones in order to directly fit the statistical data concerning infectious diseases with a latent period such as malaria, in this paper, we show that the disease-free equilibrium E_0 is globally asymptotically stable if $R_0 \le 1$, and the unique endemic equilibrium E_* exists and is globally asymptotically stable if $R_0 > 1$ for the discrete SIR epidemic model (1.4) by applying a variation of the backward Euler discretization (cf. [8,9]) and Lyapunov functional techniques in McCluskey [14,15]. We note that its stability conditions no longer need any restriction of the size of time delays for the model (1.4) which is discretized by a variation of the backward Euler method. From a biological viewpoint, it is noteworthy that the global dynamics is determined by a single threshold parameter R_0 without imposing any restriction on the length of an incubation period of the diseases not only for the continuous SIR model in [1,2,12,14,17] but also for this discrete SIR model. Applying these techniques to the other types of discrete epidemic models is our future work.

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References

[1] E. Beretta and Y. Takeuchi, *Convergence results in SIR epidemic models with varying population size*, Nonlinear Anal., **28** (1997), 1909–1921.

[2] E. Beretta, T. Hara, W. Ma and Y. Takeuchi, *Global asymptotic stability of an SIR epidemic model with distributed time delay*, Nonlinear Anal., **47** (2001), 4107–4115.

[3] C. Castillo-Chavez and A-A. Yakubu, *Dispersal, disease and life-history evolution*, Math. Biosci., **173** (2001), 35–53.

[4] C. Castillo-Chavez and A-A. Yakubu, *Discrete-time S-I-S models with complex dynamics*, Nonlinear Anal., **47** (2001), 4753–4762.

[5] K.L. Cooke, *Stability analysis for a vector disease model*, Rocky Mountain J. Math., **9** (1979), 31–42.

[6] O. Diekmann, J.A.P. Heesterbeek and J.A.J. Metz, *On the definition and the computation of the basic reproduction R_0 in models for infectious diseases in heterogeneous populations*, J. Math. Biol., **28** (1990), 365–382.

[7] M.C.M. De Jong, O. Diekmann and J.A.P. Heesterbeek, *The computation of R_0 for discrete-time epidemic models with dynamic heterogeneity*, Math. Biosci., **119** (1994), 97–114.

[8] G. Izzo and A. Vecchio, *A discrete time version for models of population dynamics in the presence of an infection*, J. Comput. Appl. Math., **210** (2007), 210–221.

[9] G. Izzo, Y. Muroya and A. Vecchio, *A general discrete time model of population dynamics in the presence of an infection*, Discrete Dyn. Nat. Soc., **2009**, Art. ID 143019, 15 pages doi:10.1155/2009/143019.

[10] S. Jang and S.N. Elaydi, *Difference equations from discretization of a continuous epidemic model with immigration of infectives*, Can. Appl. Math. Q., **11** (2003), 93–105.

[11] J. Li, Z. Ma and F. Brauer, *Global analysis of discrete-time SI and SIS epidemic models*, Math. Biosci. Engi., **4** (2007), 699–710.

[12] W. Ma, Y. Takeuchi, T. Hara and E. Beretta, *Permanence of an SIR epidemic model with distributed time delays*, Tohoku Math. J., **54** (2002), 581–591.

[13] W. Ma, M. Song and Y. Takeuchi, *Global stability of an SIR epidemic model with time delay*, Appl. Math. Lett., **17** (2004), 1141–1145.

[14] C.C. McCluskey, *Complete global stability for an SIR epidemic model with delay-Distributed or discrete*, Nonlinear Anal. RWA., **11** (2010), 55–59.

[15] C.C. McCluskey, *Global stability for an SIR epidemic model with delay and nonlinear incidence*, Nonlinear Anal. RWA., **11** (2010), 3106–3109.

[16] M. Sekiguchi, *Permanence of some discrete epidemic models*, Int. J. Biomath., **2** (2009), 443–461.

[17] Y. Takeuchi, W. Ma and E. Beretta, *Global asymptotic properties of a delay SIR epidemic model with finite incubation times*, Nonlinear Anal., **42** (2000), 931–947.

[18] Y. Zhou, Z. Ma and F. Brauer, *A discrete epidemic model for SARS transmission and control in China*, Mathematical and Computer Modeling., **40** (2004), 1491–1506.

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