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Global stability for a discrete epidemic model for disease with immunity and latency spreading in a heterogeneous host population

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Abstract. In this paper, we propose a discrete epidemic model for disease with immunity and latency spreading in a heterogeneous host population which is derived from the continuous case by using the well-known backward Euler method and applying a Lyapunov function technique which is a discrete version of that in the paper [Prüss, Pujo-Menjouet, Webb, Zacher, Analysis of a model for the dynamics of prions, *Discrete and Continuous Dynamical Systems-Series B* **6** (2006), 225-235]. It is shown that the global dynamics of this discrete epidemic model with latency is fully determined by a single threshold parameter.

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1 Introduction

The application of theories of functional differential/difference equations in mathematical biology has been developed rapidly. Various mathematical models have been proposed in the literature of population dynamics, ecology and epidemiology. Many authors have studied the dynamical behavior of several epidemic models (see [1–15] and the references therein).

Consider the following integro-differential system:

$$\begin{cases} S'(t) = B - \delta S(t) - \beta S(t)I(t) + \sigma I(t), \\ I'(t) = \beta \int_0^t S(u)I(u)\hat{g}(t-u)\mathrm{e}^{-\delta(t-u)}du - (\delta + \epsilon + \gamma + \sigma)I(t), \\ R'(t) = \gamma I(t) - \delta R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$, and $R(t)$ denote the numbers of susceptible, infectious, and recovered of individuals at time t, respectively. The non-negative constant β is the transmission rate due to the contact of susceptible individuals with infectious individuals. The non-negative constants δ, ϵ, γ and σ are natural death rates, disease-caused death rates, recovery rates and immigration rates of infectives, respectively. The function $g(t)$ is the probability density function for the time (a random variable) it takes for an infected individual to become infectious defined by the following gamma distribution:

$$\hat{g}(u) \equiv \frac{u^{n-1}}{(n-1)!b^n}\mathrm{e}^{-u/b}, \tag{1.2}$$

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where $b > 0$ is a real number and $n > 1$ is an integer.

By using "linear chain trick" to transfer from this model (see Yuan and Zou [15]), we can derive the following system of ordinary differential equations for a disease with an immigration of infectives:

$$\begin{cases} S'(t) = B - \delta S(t) - \beta S(t)I(t) + \sigma I(t), \\ y_1'(t) = c(S(t))y_{n+1}(t) - dy_1(t), \\ y_j'(t) = dy_1(t) - dy_2(t), \quad j = 2, 3, \ldots, n, \\ y_{n+1}'(t) = dy_n(t) - (e+\sigma)y_{n+1}(t), \end{cases} \tag{1.3}$$

where

$$c(S) = \frac{\beta S}{(1+\delta b)^n}, \quad d = \frac{1}{\hat{b}}, \quad e = \delta + \epsilon + \gamma, \quad \hat{b} = \frac{b}{1+\delta b}, \quad y_{n+1}(t) = I(t). \tag{1.4}$$

System (1.3) always has a disease-free equilibrium $E^0 = (S^0, 0, \ldots, 0) \in \mathbb{R}^{n+2}$, where

$$S^0 = \frac{B}{\mu}. \tag{1.5}$$

The reproduction number of system (1.3) becomes

$$R_0 = \frac{c(S^0)}{e+\sigma} = \frac{\beta S^0}{(1+\delta b)^n(\delta+\epsilon+\gamma+\sigma)}. \tag{1.6}$$

Apart from the disease-free equilibrium E^0, system (1.3) allows a unique endemic equilibrium $E^* = (S^*, y_1^*, \ldots, y_n^*, I^*) \in Int(\mathbb{R}^{n+2})$ under the conditions $R_0 > 1$, with $y_l^* = \hat{b}(\delta+\epsilon+\gamma+\sigma)I^* > 0$, $l = 1, 2, \ldots, n$. For the case $\sigma = 0$, Yuan and Zou [15] established a complete analysis of the global asymptotic stability of system (1.3) with a single threshold parameter R_0.

On the other hand, if $n = 1$, then (1.3) is corresponding to the following continuous-time SEIS epidemic model:

$$\begin{cases} S'(t) = B - \mu_1 S(t) - \beta S(t)I(t) + \sigma I(t), \\ E'(t) = \beta S(t)I(t) - (\mu_2+\lambda)E(t), \\ I'(t) = \lambda E(t) - (\mu_3+\sigma)I(t). \end{cases} \tag{1.7}$$

System (1.7) always has a disease-free equilibrium $P_0 = (B/\mu_1, 0, 0)$. Furthermore, if $\hat{R}_0 := \frac{\lambda\beta B}{\mu_1(\mu_2+\lambda)(\mu_3+\sigma)} > 1$, then system (1.7) has a unique endemic equilibrium $P_* = (S_*, E_*, I_*)$ (see Prüss *et al.* [11, Theorem 2.2]), where

$$\begin{cases} 0 < S_* = \frac{(\mu_2+\lambda)(\mu_3+\sigma)}{\lambda\beta} < \frac{B}{\mu_1}, \ 0 < E_* = \frac{(\mu_3+\sigma)\{\lambda\beta B - \mu_1(\mu_2+\lambda)(\mu_3+\sigma)\}}{\lambda\beta(\lambda\mu_3+\sigma\mu_2+\mu_2\mu_3)} < \frac{B}{\mu_1}, \\ 0 < I_* = \frac{\lambda\beta B - \mu_1(\mu_2+\lambda)(\mu_3+\sigma)}{\beta(\lambda\mu_3+\sigma\mu_2+\mu_2\mu_3)} < \frac{B}{\mu_1}. \end{cases} \tag{1.8}$$

By using a geometric approach developed in [2,8] have first proved Theorem A for the case $\mu_1 = \mu_3 \le \mu_2$ below, and later, by using appropriate Lyapunov functions, Prüss *et al.* [11] established complete analysis of a mathematical model for the dynamics of prion proliferation whose result is also applicable to system (1.7) for $\hat{R}_0 \le 1$ and $\hat{R}_0 > 1$ as follows (see Prüss *et al.* [11, Theorem 2.2]).

Theorem A. *The disease-free equilibrium P_0 of system (1.7) is globally asymptotically stable, if and only if, $\hat{R}_0 \le 1$. Moreover, the endemic equilibrium P_* is globally asymptotically stable in $\mathbb{R}^3_{+0} \setminus [\mathbb{R}_+ \times \{0\} \times \{0\}]$, if and only if, $\hat{R}_0 > 1$.*

In those cases, how to choose the discrete schemes which preserve the global asymptotic stability of the endemic equilibrium of the models remained an open problem. For a delayed SIR epidemic model, a complete solution of this problem has been elusive until recent paper Enatsu *et al.* [1].

Motivated by the above results, in this paper, we propose the following discrete epidemic model which is derived from system (1.3) by applying the well-known backward Euler method (cf. Izzo and Vecchio [5]).

$$\begin{cases} s(p+1) = s(p) + B - \delta s(p+1) - \beta s(p+1)y_{n+1}(p+1) + \sigma y_{n+1}(p+1), \\ y_1(p+1) = y_1(p) + c(s(p+1))y_{n+1}(p+1) - dy_1(p+1), \\ y_j(p+1) = y_j(p) + dy_{j-1}(p+1) - dy_j(p+1), \quad j = 2, 3, \ldots, n, \\ y_{n+1}(p+1) = y_{n+1}(p) + dy_n(p+1) - (e+\sigma)y_{n+1}(p+1), \\ i(p+1) = y_{n+1}(p+1) > 0, \quad p = 0, 1, 2, \ldots, \end{cases} \tag{1.9}$$

with the initial conditions of system (1.9)

$$s(0) > 0, \ y_j(p) > 0, \quad \text{and} \quad j = 1, \ldots, n+1. \tag{1.10}$$

Remark 1.1. To prove the positivity of $s(p)$, $y_j(p)$, $1 \leq j \leq n$ and $i(p)$ for any $p \geq 0$ and to apply both Lyapunov functional techniques in Enatsu *et al.* [1] and a discrete time analogue to Prüss *et al.* [11], we need to use the backward Euler discretization (see, e.g., Izzo and Vecchio [5] and Izzo *et al.* [6]) which is a different scheme from that of Jang and Elaydi [7] and Sekiguchi [12]. Moreover, in order to consider only the positive solution $(s(p+1), y_1(p+1), y_2(p+1), \cdots, i(p+1))$ for (1.9) for any obtained positive solution $(s(p), y_1(p), y_2(p), \cdots, y_n(p), i(p))$, we need the restriction $i(p+1) > 0$ in (1.9), because without the condition $i(p+1) > 0$ in (1.9), there exist just two solutions $(s(p+1), y_1(p+1), y_2(p+1), \cdots, y_n(p+1), i(p+1))$ of (1.9), one is $i(p+1) < 0$ and the other is $i(p+1) > 0$ for any obtained positive solution $(s(p), y_1(p), y_2(p), \cdots, y_n(p), i(p))$ (see Proof of Lemma 2.1).

Using the threshold R_0, one can see that system (1.9) always has a disease-free equilibrium $E^0 = (S^0, 0, 0, \cdots, 0, 0)$ and if $R_0 > 1$, then system (1.9) has a unique endemic equilibrium $E^* = (S^*, y_1^*, y_2^*, \cdots, y_n^*, I^*)$ (see Lemma 2.3). Applying Lyapunov function techniques in Prüss *et al.* [11] to both cases for $R_0 \leq 1$ and $R_0 > 1$, we establish a complete analysis of the global asymptotic stability for the model (1.9). In particular, we also apply techniques of Lyapunov functionals in McCluskey [9] (see Lemma 5.2) to prove the global asymptotic stability of the endemic equilibrium of (1.9) for the case $R_0 > 1$ which no longer needs any of the theory of non-negative matrices and graph theory (cf. Guo *et al.* [4]). Moreover, by applying results in Enatsu *et al.* [1, Lemma 4.1], we give a proof of the permanence of (1.9) for $R_0 > 1$, which is more simplified than that in Sekiguchi [12] and Sekiguchi and Ishiwata [13].

Our main result in this paper, is as follows.

Theorem 1.1. *The disease-free equilibrium E^0 of system (1.9) is globally asymptotically stable if and only if $R_0 \leq 1$. Moreover, the endemic equilibrium E^* is globally asymptotically stable, if and only if, $R_0 > 1$.*

Remark 1.2. For the case $\sigma = 0$, Theorem 1.1 is just a discrete analogue of the result in Yuan and Zou [15] for system (1.3).

The organization of this paper is as follows. In Section 2, we offer some basic results for system (1.9). In Section 3, we give a proof of the first part of Theorem 1.1 for $R_0 \leq 1$. In Section 4, by applying Lemmas 4.1-4.4, we offer a new proof to obtain lower positive bounds for the permanence of system (1.9) for $R_0 > 1$ (see Enatsu *et al.* [1] and cf. Thieme [14]). In Section 5, we prove the second part of Theorem 1.1 for $R_0 > 1$ by extending a discrete time analogue of the Lyapunov function proposed by Prüss *et al.* [11] to system (1.9). Finally, short conclusions is offered in Section 6.

2 Basic properties

The following lemma is a basic result in this paper (cf. Izzo and Vecchio [5] and Izzo *et al.* [6]).

Lemma 2.1. *Let $s(p)$, $y_j(p)$, $j = 1, \ldots, n$, and $y_{n+1}(p) = i(p)$ be the solutions of system (1.9) with the initial conditions (1.10). Then $s(p) > 0$ and $y_j(p) > 0$, $j = 1, \ldots, n+1$ for any $p \geq 0$, and (1.9) is equivalent to the following iteration system.*

$$\begin{cases} s(p+1) = \dfrac{B + s(p) + \sigma i(p+1)}{1 + \delta + \beta i(p+1)}, \\ y_1(p+1) = \dfrac{c(s(p+1))i(p+1) + y_1(p)}{1+d}, \\ y_j(p+1) = \dfrac{d y_{j-1}(p+1) + y_j(p)}{1+d}, \quad j = 2, 3, \ldots, n, \\ i(p+1) = \dfrac{d y_n(p+1) + i(p)}{1 + e + \sigma}, \quad \text{and} \quad i(p+1) > 0, \quad p = 0, 1, 2, \ldots, \end{cases} \tag{2.1}$$

which is equivalent to

$$i(p+1) = \frac{-\tilde{B}_p + \sqrt{\tilde{B}_p^2 + 4\tilde{A}\tilde{C}_p}}{2\tilde{A}} = \frac{2\tilde{C}_p}{\tilde{B}_p + \sqrt{\tilde{B}_p^2 + 4\tilde{A}\tilde{C}_p}}, \tag{2.2}$$

where

$$\begin{aligned} \tilde{A} &= \beta\{(1+e+\sigma)(1+\delta b)^n(1+d)^{n+1} - \delta d^n\}, \\ \tilde{B}_p &= \Bigg[(1+\delta)(1+e+\sigma)(1+\delta b)^n(1+d)^{n+1} \\ &\quad - \beta\bigg\{d^n(B + s(p)) + (1+\delta b)^n(1+d)^{n+1}\bigg(\frac{d^n y_1(p)}{(1+d)^n} + \frac{d^{n-1} y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{d y_n(p)}{1+d} + i(p)\bigg)\bigg\}\Bigg], \\ \tilde{C}_p &= (1+\delta)(1+\delta b)^n(1+d)^{n+1}\bigg(\frac{d^n y_1(p)}{(1+d)^n} + \frac{d^{n-1} y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{d y_n(p)}{1+d} + i(p)\bigg). \end{aligned}$$

Proof. It is evident that the first $(n+1)$ equations of (1.9) are equivalent to the second-$(n+2)$-th equations of (2.1). The $(n+2)$-th equation with the first $(n+1)$ equations of (1.9) is equivalent to the first $(n+1)$ equations of (2.1) and

$$\begin{aligned}
(1+e+\sigma)i(p+1) &= dy_n(p+1) + i(p)\\
&= d\frac{dy_{n-1}(p+1)+y_n(p)}{1+d} + i(p)\\
&= \frac{d^2y_{n-1}(p+1)}{1+d} + \frac{dy_n(p)}{1+d} + i(p)\\
&\cdots\\
&= \frac{d^ny_1(p+1)}{(1+d)^{n-1}} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)\\
&= \frac{d^nc(s(p+1))i(p+1)}{(1+d)^{n+1}} + \frac{d^ny_1(p)}{(1+d)^n} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)\\
&= \frac{\beta d^n}{(1+\delta b)^n(1+d)^{n+1}}\frac{B+s(p)+\sigma i(p+1)}{1+\delta+\beta i(p+1)}i(p+1)\\
&\quad + \frac{d^ny_1(p)}{(1+d)^n} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)
\end{aligned}$$

and $i(p+1)>0$ for $p=0,1,2,\ldots$, which is equivalent to the following quadratic equation $P(x)=0$ with $x=i(p+1)>0$ such that

$$\begin{aligned}
P(x) &= (1+e+\sigma)(1+\delta b)^n(1+d)^{n+1}(1+\delta+\beta x)x - \beta d^n(B+s(p)+\sigma x)x\\
&\quad - (1+\delta b)^n(1+d)^{n+1}\left(\frac{d^ny_1(p)}{(1+d)^n} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)\right)(1+\delta+\beta x)\\
&= \beta\{(1+e+\sigma)(1+\delta b)^n(1+d)^{n+1} - \sigma d^n\}x^2 + \Bigg[(1+\delta)(1+e+\sigma)(1+\delta b)^n(1+d)^{n+1}\\
&\quad - \beta\left\{d^n(B+s(p)) + (1+\delta b)^n(1+d)^{n+1}\left(\frac{d^ny_1(p)}{(1+d)^n} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)\right)\right\}\Bigg]x\\
&\quad - (1+\delta)(1+\delta b)^n(1+d)^{n+1}\left(\frac{d^ny_1(p)}{(1+d)^n} + \frac{d^{n-1}y_2(p)}{(1+d)^{n-1}} + \cdots + \frac{dy_n(p)}{1+d} + i(p)\right).
\end{aligned}$$

For $s(p)>0$ and $i(p)>0$, it is evident that $i(p+1)$ defined by the first equation of (2.1) is a unique positive solution of the quadratic equation $P(x)=0$.

Assume that $s(p)>0$ and $y_j(p)>0$ $j=1,\ldots,n+1$ for some $p\ge 0$. Suppose that $s(p+1)<S^0$. Then, we have $B-\delta s(p+1)>0$. Then, system (1.9) becomes

$$\begin{cases}
(1+\beta y_{n+1}(p+1))s(p+1) = s(p) + \{B-\delta s(p+1)\} + \sigma y_{n+1}(p+1) > 0,\\
(1+d)y_1(p+1) = y_1(p) + c(s(p+1))y_{n+1}(p+1) > 0.\\
(1+d)y_j(p+1) = y_j(p) + dy_{j-1}(p+1) > 0, \quad j=2,3,\ldots,n,\\
(1+e+\sigma)y_{n+1}(p+1) = y_{n+1}(p) + dy_n(p+1) > 0.
\end{cases} \tag{2.3}$$

Then, from the first equation of (2.3), $s(p+1)>0$. For the other case $s(p+1)\ge S^0$, it is evident that $s(p+1)>0$. Then, from the second equation of (2.3), we have $y_1(p+1)>0$, and similarly we obtain $y_2(p+1)$, $y_3(p+1)$, $\cdots$, $y_{n+1}(p+1)=i(p+1)>0$. Hence by induction of $p\ge 0$, we complete the proof of this lemma. □

Hereafter, in order to simplify the proofs of remaining sections, let us set $y_0(p)=y_{n+2}(p)=s(p)$ and

$$\begin{cases}
\underline{s} = \underline{y}_0 = \underline{y}_{n+2} = \liminf_{p\to+\infty} s(p), \quad \overline{s} = \overline{y}_0 = \overline{y}_{n+2} = \limsup_{p\to+\infty} s(p),\\
\underline{y}_j = \liminf_{p\to+\infty} y_j(p), \quad \overline{y}_j = \limsup_{p\to+\infty} y_j(p), \quad j=1,2,\ldots,n+1,
\end{cases} \tag{2.4}$$

and put

$$\overline{\kappa} = (1+\delta b)^n, \quad \kappa = \frac{1}{1+\delta b}, \quad V(p) = s(p) + \overline{\kappa}\{y_1(p) + \kappa y_2(p) + \cdots + \kappa^n y_{n+1}(p)\}. \tag{2.5}$$

Then, by (1.4),

$$\overline{\kappa} > 1 > \kappa > 0, \quad \overline{\kappa}\kappa^n = 1, \quad (1-\kappa)d = \delta, \quad \text{and} \quad e > \delta. \tag{2.6}$$

By (1.9) and $i(p+1)=y_{n+1}(p+1)$, one can see that

$$\begin{aligned} V(p+1)-V(p) &= B-\delta s(p+1)+\sigma y_{n+1}(p+1)\\ &\quad -\overline{\kappa}\{(1-\kappa)dy_1(p+1)-\kappa(1-\kappa)dy_2(p+1)+\cdots+\kappa^{n-1}(1-\kappa)dy_n(p+1)\}-\overline{\kappa}\kappa^n(e+\sigma)y_{n+1}(p+1)\\ &\le B-\delta V(p+1). \end{aligned} \tag{2.7}$$

Then, we obtain the following basic lemma of the boundedness of $s(p+1)$ and $y_j(p+1)$, $j=1,2,\ldots,n+1$.

Lemma 2.2. *Let $s(p)$ and $y_j(p)$, $j=1,\ldots,n+1$ be the solutions of system (1.9) with the initial conditions (1.10). Then, it holds that*

$$\limsup_{p\to+\infty} V(p)\le S^0=\frac{B}{\delta}, \tag{2.8}$$

and

$$\overline{s}\le S^0=\frac{B}{\delta},\quad \text{and}\quad \overline{y}_j\le \frac{S^0}{\overline{\kappa}\kappa^{j-1}},\quad j=1,2,\ldots,n+1. \tag{2.9}$$

Proof. Let $\overline{V}=\limsup_{p\to+\infty}V(p)$. First, we suppose that $\overline{V}=+\infty$. Then, there exists a sequence $\{p_l\}_{l=1}^{\infty}$ such that $p_l<p_{l+1}$, $l=1,2,\ldots$, $\lim\limits_{l\to+\infty}p_l=+\infty$ and

$$V(p)<V(p_l),\quad \text{for any } p<p_l,\quad \text{and}\quad \lim_{l\to+\infty}V(p_l)=+\infty. \tag{2.10}$$

By (2.7), we have

$$0<V(p_l)-V(p_l-1)\le B-\delta V(p_l),$$

from which it holds $V(p_l)<\frac{B}{\delta}=S^0$ for any $l\ge 1$. This is a contradiction. Thus, we have $\overline{V}<+\infty$. If there exists a sequence $\{q_l\}_{l=1}^{\infty}$ such that $q_l<q_{l+1}$, $l=1,2,\ldots$, $\lim\limits_{l\to+\infty}q_l=+\infty$ and

$$V(q_l-1)\le V(q_l),\quad \text{for any } l=1,2,\ldots,\quad \text{and}\quad \lim_{l\to+\infty}V(q_l)=\overline{V}. \tag{2.11}$$

Then, similarly, we obtain that

$$B-\delta V(q_l)\ge V(q_l)-V(q_l-1)\ge 0,$$

from which we obtain $\overline{V}\le S^0$. For the other case that $V(p)\ge 0$ is eventually monotone decreasing, there exists a $\lim_{p\to+\infty}V(p)=\overline{V}\ge 0$, and hence, $\lim_{p\to+\infty}(V(p+1)-V(p))=0$. By (2.7), we obtain $0\le B-\delta\overline{V}$ and $\overline{V}\le S^0$, from which we get (2.9). Hence, the proof of this lemma is complete. $\Box$

Lemma 2.3. *System (1.9) has an equilibrium $E^0=(S^0,0,0,\cdots,0)$, and if $R_0\le 1$, then E^0 is a unique equilibrium, but if $R_0>1$, then there exists an another equilibrium $E^*=(S^*,y_1^*,y_2^*,\cdots,y_{n+1}^*)$, where $\beta S^*-\sigma>0$ and*

$$\begin{cases} 0<S^*=\frac{(1+\delta b)^n(e+\sigma)}{\beta}<S^0, \quad c(S^*)=e+\sigma,\\ y_1^*=y_2^*=\cdots=y_n^*=\frac{e+\sigma}{d}y_{n+1}^*,\ y_{n+1}^*=\frac{B-\mu S^*}{\beta S^*-\sigma}. \end{cases} \tag{2.12}$$

Proof. By Lemma 2.1, positivity of the sequences $\{s(p)\}_{p=1}^{\infty}$ and $\{y_j(p)\}_{p=1}^{\infty}$, $j=1,2,\ldots,n+1$ is assured. Then, the equilibrium $\hat{E}=(\hat{S},\hat{y}_1,\hat{y}_2,\cdots,\hat{y}_{n+1})$ of (1.9) satisfies the following equations.

$$\begin{cases} B-\delta\hat{S}=(\beta\hat{S}-\sigma)\hat{y}_{n+1},\quad c(\hat{S})\hat{y}_{n+1}=d\hat{y}_1,\\ \hat{y}_1=\hat{y}_2=\cdots=\hat{y}_n,\quad \text{and}\quad d\hat{y}_n=(e+\sigma)\hat{y}_{n+1}, \end{cases} \tag{2.13}$$

that is,

$$B-\delta\hat{S}=(\beta\hat{S}-\sigma)\hat{y}_{n+1},\quad \text{and}\quad (c(\hat{S})-e-\sigma)\hat{y}_{n+1}=0. \tag{2.14}$$

Then,

$$\hat{y}_{n+1}=0,\quad \text{or}\quad c(\hat{S})=e+\sigma. \tag{2.15}$$

If $\hat{y}_{n+1}=0$, then by (2.13), we have that

$$\hat{S}=S^0,\quad \text{and}\quad \hat{y}_1=\hat{y}_2=\cdots=\hat{y}_{n+1}=0. \tag{2.16}$$

If $R_0=\frac{c(S^0)}{e+\sigma}<1$, then $\hat{y}_{n+1}=0$, and by (2.13), we have (2.16). If $R_0=1$, then $c(S^0)=e+\sigma$. Then, by (2.15), $\hat{S}=S^0$ and by (2.13), we also have (2.16). If $R_0>1$, then $c(S^0)>e+\sigma$. Then, by (1.4), there exists a $0<\frac{\sigma}{\beta}<\hat{S}=S^*=\frac{(1+\delta b)^n(e+\sigma)}{\beta}<S^0$ such that $c(S^*)=e+\sigma$, and by (2.13), it holds that $\hat{y}_{n+1}=y_{n+1}^*=\frac{B-\delta S^*}{\beta S^*-\sigma}$ and $\hat{y}_j=y_j^*=\frac{(e+\sigma)(B-\delta S^*)}{d(\beta S^*-\sigma)}$, $j=1,2,\cdot,n$. Therefore, system (1.9) has an equilibrium E^0. If $R_0\le 1$, then E^0 is a unique equilibrium, but if $R_0>1$, then there exists an another equilibrium E^*. This completes the proof of this lemma. $\Box$

3 Global stability of the disease-free equilibrium E^0 for $R_0 \leq 1$

In this section we assume $R_0 \leq 1$. By applying the similar Lyapunov function techniques in Prüss *et al.* [11], we prove the first part of Theorem 1.1.

Proof of the first part of Theorem 1.1. By means of a Lyapunov functional, we show that in this case the disease-free equilibrium $E^0 = (S^0, 0, 0, \cdots, 0)$ is globally asymptotically stable in $\mathbb{R}_+^{n+2}$. First, we consider the case $S^0 - \frac{\sigma}{\beta} \geq 0$. We define

$$W(p) = \frac{(s(p) - S^0)^2}{2} + \frac{\beta S^0}{e+\sigma}\left(S^0 - \frac{\sigma}{\beta}\right)\sum_{j=1}^{n+1} y_j(p). \tag{3.1}$$

Then, we obtain

$$\begin{aligned}
W(p+1) - W(p) =& \frac{(s(p+1) - S^0)^2}{2} - \frac{(s(p) - S^0)^2}{2}\\
&+ \frac{\beta S^0}{e+\sigma}\left(S^0 - \frac{\delta}{\beta}\right)\Big\{c(s(p+1))y_{n+1}(p+1) - dy_1(p+1)\\
&+ d\sum_{j=2}^{n}(y_{j-1}(p+1) - y_j(p+1)) + dy_n(p+1) - (e+\sigma)y_{n+1}(p+1)\Big\}\\
=& \frac{(s(p+1) - S^0)^2}{2} - \frac{(s(p) - S^0)^2}{2} + \frac{\beta S^0}{e+\sigma}\left(S^0 - \frac{\delta}{\beta}\right)\{c(s(p+1))y_{n+1}(p+1) - (e+\sigma)y_{n+1}(p+1)\}.
\end{aligned}$$

By the relations $R_0 = \frac{\beta S^0}{(1+\delta b)^n (e+\sigma)} \leq 1$ and

$$c(s(p+1)) = \frac{\beta s(p+1)}{(1+\delta b)^n} = \frac{(e+\sigma) R_0 s(p+1)}{S^0},$$

we have

$$\begin{aligned}
W(p+1) - W(p) =& \frac{\{(s(p+1) - S^0) + (s(p) - S^0)\}(s(p+1) - s(p))}{2} + \beta\left(S^0 - \frac{\delta}{\beta}\right)(R_0 s(p+1) - S^0)y_{n+1}(p+1)\\
\leq& -\frac{(s(p+1) - s(p))^2}{2} + (s(p+1) - S^0)(s(p+1) - s(p)) + \beta\left(S^0 - \frac{\sigma}{\beta}\right)(s(p+1) - S^0)i(p+1)\\
=& -\frac{(s(p+1) - s(p))^2}{2} + (s(p+1) - S^0)(B - \beta s(p+1)i(p+1) - \delta s(p+1) + \sigma i(p+1))\\
&+ \beta\left(S^0 - \frac{\sigma}{\beta}\right)(s(p+1) - S^*)i(p+1)\\
=& -\frac{(s(p+1) - s(p))^2}{2} - \delta(s(p+1) - S^0)^2 - \beta(s(p+1) - S^0)\left(s(p+1) - \frac{\sigma}{\beta}\right)i(p+1)\\
&+ \beta\left(S^0 - \frac{\sigma}{\beta}\right)(s(p+1) - S^0)i(p+1)\\
=& -\frac{(s(p+1) - s(p))^2}{2} - \delta(s(p+1) - S^0)^2\\
&- \beta(s(p+1) - S^0)\left\{(s(p+1) - S^0) + \left(S^0 - \frac{\sigma}{\beta}\right)\right\}i(p+1)\\
&+ \beta\left(S^0 - \frac{\sigma}{\beta}\right)(s(p+1) - S^0)i(p+1)\\
=& -\frac{(s(p+1) - s(p))^2}{2} - (\delta + \beta i(p+1))(s(p+1) - S^0)^2 \leq 0
\end{aligned}$$

for all $p \geq 0$. Thus, $W(p+1) \leq W(p) \leq W(0)$ for all $p \geq 0$ and $\lim_{p \to +\infty} W(p) = 0$.

If $S^0 - \frac{\sigma}{\beta} > 0$, then $\lim_{p \to +\infty} W(p) = 0$, if and only if $\lim_{p \to +\infty} s(p) = S^0 = B/\delta$ and $y_j(p+1) = 0$, $j = 1, 2, \ldots, n+1$. If $S^0 - \frac{\sigma}{\beta} = 0$, then $\lim_{p \to +\infty} W(p) = 0$, if and only if $\lim_{p \to +\infty} s(p) = S^0 = B/\delta$. By (2.5) and (2.7), we obtain that $V(p+1) - S^0 \leq \frac{1}{1+\delta}(V(0) - S^0)$, $V(p+1) \leq (\frac{1}{1+\delta})^p (V(0) - S^0)$ and

$$\sum_{j=1}^{n+1} \kappa^{j-1} y_j(p) \leq \left(\frac{1}{1+\delta}\right)^p (V(0) - S^0) - \frac{1}{\kappa}(s(p) - S^0).$$

Therefore, for the case $S^0 - \frac{\sigma}{\beta} \geq 0$, the disease-free equilibrium $E^0 = (S^0, 0, 0, \cdots, 0)$ is uniformly stable, and hence, it is globally asymptotically stable in $\mathbb{R}_+^{n+2}$.

Second, we consider the other case $S^0 - \frac{\sigma}{\beta} < 0$. By Lemma 2.2, $\overline{s} \leq S^0$, and hence, there exists a sufficiently large $p_0 > 0$ such that $s(p) - \frac{\sigma}{\beta} < 0$ for any $p \geq p_0$. For $\liminf_{p\to+\infty} s(p) = \underline{s}$, we first suppose that $\underline{s} = 0$. Then, there exists a sequence $\{p_l\}_{l=1}^{\infty}$ such that $p_0 \leq p_l < p_{l+1}$, $l = 1, 2, \ldots$, $\lim_{l\to+\infty} p_l = 0$ and

$$s(p) > s(p_l), \quad \text{for any } p < p_l, \quad \text{and} \quad \lim_{l\to+\infty} s(p_l) = 0, \tag{3.2}$$

and by (1.9), we have

$$0 > s(p_l) - s(p_l - 1) \geq B - \delta s(p_l) - (\beta s(p_l) - \sigma) i(p_l) \geq B - \delta s(p_l),$$

from which it holds $s(p_l) > \frac{B}{\delta} = S^0$ for any $l \geq 1$. This is a contradiction. Thus, we have $\underline{s} > 0$. If there exists a sequence $\{q_l\}_{l=1}^{\infty}$ such that $q_l < q_{l+1}$, $l = 1, 2, \ldots$, $\lim_{l\to+\infty} q_l = +\infty$ and

$$s(q_l - 1) \geq s(q_l), \quad \text{for any } l = 1, 2, \ldots, \text{and} \quad \lim_{l\to+\infty} s(q_l) = \underline{s}. \tag{3.3}$$

Then, similarly, we obtain that

$$B - \delta s(q_l) \leq s(q_l) - s(q_l - 1) \leq 0,$$

from which we obtain $\underline{s} \geq S^0 \geq \overline{s}$. Thus, $\lim_{p\to+\infty} s(p) = S^0$. Next, consider the other case that $s(p) > 0$ is eventually monotone increasing. Then, by Lemma 2.2, there exists a $\lim_{p\to+\infty} s(p) = \underline{s} > 0$, and hence, $\lim_{p\to+\infty}(s(p+1) - s(p)) = 0$. By (1.9), we obtain $0 \geq B - \delta\underline{s}$ and $\underline{s} \geq S^0 \geq \overline{s}$, from which we also get $\lim_{p\to+\infty} s(p) = S^0$. By (2.8), we obtain $\lim_{p\to+\infty} y_j(p) = 0$, $j = 1, 2, \ldots, n+1$, which implies that the disease-free equilibrium $E^0 = (S^0, 0, 0, \cdots, 0)$ is globally asymptotically stable in $\mathbb{R}_+^{n+2}$. Hence, the proof of the first part of Theorem 1.1 is complete. □

4 Permanence for $R_0 > 1$

In this section, we assume that $R_0 > 1$ and prove the permanence of system (1.9) for $R_0 > 1$. By Lemma 2.3, the endemic equilibrium $E^* = (S^*, y_1^*, y_2^*, \cdots, y_{n+1}^*)$ exists. We have basic lemmas as follows.

Lemma 4.1. *For $E^* = (S^*, y_1^*, y_2^*, \cdots, y_{n+1}^*)$, it holds that*

$$\frac{c(S^*) y_{n+1}^*}{d y_1^*} = 1, \ \frac{y_{j-1}^*}{y_j^*} = 1, \ j = 2, 3, \ldots, n, \quad \text{and} \quad \frac{d y_n^*}{(e+\sigma) y_{n+1}^*} = 1, \tag{4.1}$$

and

$$\frac{c(S^*)}{e+\sigma} = 1. \tag{4.2}$$

Proof. By (2.12), we can easily prove this lemma. □

By Lemma 4.1, we put

$$\tilde{y}_j(p) = \frac{y_j(p)}{y_j^*}, \quad j = 1, 2, \ldots, n+1. \tag{4.3}$$

Then, it follows from (4.1) that (1.9) is equivalent to

$$\begin{cases} s(p+1) - s(p) = B - \delta s(p+1) - y_{n+1}^*(\beta s(p+1) - \sigma)\tilde{y}_{n+1}(p+1), \\ \tilde{y}_1(p+1) - \tilde{y}_1(p) = d\left(\dfrac{c(s(p+1))}{c(S^*)}\tilde{y}_{n+1}(p+1) - \tilde{y}_1(p+1)\right), \\ \tilde{y}_j(p+1) - \tilde{y}_j(p) = d(\tilde{y}_{j-1}(p+1) - \tilde{y}_j(p+1)), \ j = 2, 3, \ldots, n, \\ \tilde{y}_{n+1}(p+1) - \tilde{y}_{n+1}(p) = (e+\sigma)(\tilde{y}_n(p+1) - \tilde{y}_{n+1}(p+1)), \quad p \geq 0. \end{cases} \tag{4.4}$$

Lemma 4.2. *If*

$$\min_{1\leq j\leq n+1} \tilde{y}_j(p+1) < \min_{1\leq j\leq n+1} \tilde{y}_j(p), \tag{4.5}$$

then

$$\tilde{y}_1(p+1) = \min_{1\leq j\leq n+1} \tilde{y}_j(p+1) < \min_{1\leq j\leq n+1} \tilde{y}_j(p) \quad \text{and} \quad s(p+1) < S^*. \tag{4.6}$$

Inversely,

$$\tilde{y}_1(p+1) > \min_{1\leq j\leq n+1} \tilde{y}_j(p+1) \quad \text{or} \quad s(p+1) \geq S^*, \tag{4.7}$$

then

$$\min_{1\leq j\leq n+1} \tilde{y}_j(p+1) \geq \min_{1\leq j\leq n+1} \tilde{y}_j(p). \tag{4.8}$$

Proof. We choose j_0 such that $\tilde{y}_{j_0}(p+1) = \min_{1\le j\le n+1} \tilde{y}_j(p)$. It then follows from (4.5) that $\tilde{y}_{j_0}(p+1) < \min_{1\le j\le n+1} \tilde{y}_j(p) \le \tilde{y}_{j_0}(p)$, which implies $\tilde{y}_{j_0}(p+1) - \tilde{y}_{j_0}(p) < 0$ and $\tilde{y}_{j_0-1}(p+1) - \tilde{y}_{j_0}(p+1) \ge 0$ hold. Suppose that $2 \le j_0 \le n+1$. Then, for the case $2 \le j_0 \le n$, by (4.4), we have

$$0 > \tilde{y}_{j_0}(p+1) - \tilde{y}_{j_0}(p) = d(\tilde{y}_{j_0-1}(p+1) - \tilde{y}_{j_0}(p+1)) \ge 0,$$

which is a contradiction. If $j_0 = n+1$, then by (4.4),

$$0 > \tilde{y}_{n+1}(p+1) - \tilde{y}_{n+1}(p) = (e+\sigma)(\tilde{y}_n(p+1) - \tilde{y}_{n+1}(p+1)) \ge 0,$$

which is also a contradiction. Thus, we have $j_0 = 1$. Then, by $\tilde{y}_1(p+1) < \tilde{y}_{n+1}(p)$ and the second equation of (4.4), we have

$$0 > \tilde{y}_1(p+1) - \tilde{y}_1(p) \ge d\left(\frac{c(s(p+1))}{c(S^*)} - 1\right)\tilde{y}_1(p+1),$$

from which we obtain $s(p+1) < S^*$. Hence, (4.5) implies (4.6). The remaining part of this lemma is evident. □

Lemma 4.3. *For (4.4), it holds that*

$$\begin{cases} \tilde{y}_j(p+1) \ge \frac{1}{1+d}\tilde{y}_j(p), \quad j = 1,2,\dots,n, \\ \tilde{y}_{n+1}(p+1) \ge \frac{1}{1+e+\sigma}\tilde{y}_{n+1}(p), \\ \tilde{y}_1(p+1) \ge \frac{d}{1+d}\frac{c(s(p+1))}{c(S^*)}\tilde{y}_{n+1}(p+1), \\ \tilde{y}_j(p+1) \ge \frac{d}{1+d}\tilde{y}_{j-1}(p+1), \quad j = 2,\dots,n, \\ \tilde{y}_{n+1}(p+1) \ge \frac{e+\sigma}{1+e+\sigma}\tilde{y}_n(p+1), \quad p \ge 0. \end{cases} \tag{4.9}$$

In particular, if (4.5) holds for some $p \ge 0$, then

$$\tilde{y}_1(p+1) = \min_{1\le j\le n+1} \tilde{y}_j(p+1) \ge \frac{c(s(p+1))}{c(S^*)}\tilde{y}_{n+1}(p). \tag{4.10}$$

Proof. The proof of this lemma is evident from (4.4) and (4.6). □

Hereafter, in order to simplify the proofs of remaining sections, let us set $y_0(p) = y_{n+2}(p) = s(p)$ and

$$\begin{cases} \underline{S} = \liminf_{p\to+\infty} s(p), \quad \overline{S} = \limsup_{p\to+\infty} s(p), \\ \underline{y}_j = \liminf_{p\to+\infty} y_j(p), \quad \overline{y}_j = \limsup_{p\to+\infty} y_j(p), \quad j = 1,2,\dots,n+1, \\ \underline{I} = \underline{y}_{n+1}, \quad \overline{I} = \overline{y}_{n+1}. \end{cases} \tag{4.11}$$

Lemma 4.4. *If $R_0 > 1$, then for any solution of system (1.9), it holds that*

$$\begin{cases} \underline{S} \ge v_0 \equiv \dfrac{B}{\delta + \beta B/\delta} > 0, \\ \dfrac{e}{d}\underline{y}_{n+1} \ge \underline{y}_n \ge \underline{y}_{n-1} \ge \cdots \ge \underline{y}_1 \ge \dfrac{c(v_0)}{d}\underline{y}_{n+1}, \\ \underline{I} = \underline{y}_{n+1} \ge v_{n+1}(q) \equiv \dfrac{d^n}{(1+d)^n}\left(\dfrac{1}{1+e+\sigma}\right)^{l_0(q)+1} I^* > 0, \end{cases} \tag{4.12}$$

where for any $0 < q < 1$, the integer $l_0(q) \ge 0$ is sufficiently large such that

$$S^* < \frac{B}{k_q}\left\{1 - \left(\frac{1}{1+k_q}\right)^{l_0(q)}\right\} \quad \textit{and} \quad k_q = \delta + \beta I^*. \tag{4.13}$$

Proof. By Lemmas 2.1 and 2.2, we obtain that every sequence $\{s(p)\}_{p=0}^{\infty}$, $\{y_j(p)\}_{p=0}^{\infty}$, $j = 1,2,\dots,n$ and $\{i(p)\}_{p=0}^{\infty}$ is positive and eventually bounded, that is, $\overline{S} \le S^0 = \frac{B}{\delta}$ and $\overline{I} = \overline{y}_{n+1} \le \frac{B}{\delta}$. Then, we have $0 \ge B - \delta\underline{S} - \beta\underline{S}\overline{I}$, from which we obtain $\underline{S} \ge \frac{B}{\delta+\beta\overline{I}} \ge \frac{B}{\delta+\beta B/\delta}$. Thus, we obtain the first equation of (4.12).

By (1.9), we also obtain the second equation of (4.12).

Now, we show the last equation of (4.12).

First, we prove the claim that any solution $(s(p), y_1(p), y_2(p), \cdots, y_n(p), i(p))$ of system (1.9) does not have the following property: for any $0 < q < 1$, there exists a non-negative integer p_0 such that $y_j(p) \le q y_j^*$, $j = 1,2,\dots,n$, and $i(p) \le qI^*$ for all $p \ge p_0$. Suppose on the contrary that there exist a solution $(s(p), y_1(p), y_2(p), \cdots, y_n(p), i(p))$ of system (1.9) and a non-negative integer p_0 such that $y_j(p) \le q y_j^*$, $j = 1,2,\dots,n$ and $i(p) \le qI^*$ for all $p \ge p_0$. Then, $\tilde{y}_j(p) \le q$, $j = 1,2,\dots,n+1$ for all $p \ge p_0$.

Consider the sequence $\{w(p)\}_{p=0}^{\infty}$ defined by

$$w(p)=\sum_{j=1}^{n}\frac{\tilde{y}_j(p)}{d}+\frac{\tilde{y}_{n+1}(p)}{e+\sigma}. \tag{4.14}$$

Then, by (4.4), we have

$$\begin{aligned} w(p+1)-w(p) &= \frac{c(s(p+1))}{c(S^*)}\tilde{y}_{n+1}(p+1)-\tilde{y}_1(p+1)+\sum_{j=2}^{n}(\tilde{y}_{j-1}(p+1)-\tilde{y}_j(p+1))+\tilde{y}_n(p+1)-\tilde{y}_{n+1}(p+1)\\ &= \left(\frac{c(s(p+1))}{c(S^*)}-1\right)\tilde{y}_{n+1}(p+1). \end{aligned} \tag{4.15}$$

i) Consider the case that $\{s(p)\}_{p=0}^{\infty}$ is eventually monotone increasing. Then, there is a limit of $\lim_{p\to\infty}s(p)=\hat{S}\le\frac{B}{\delta}$. We show that $\hat{S}=S^*$ holds true.

Suppose that $\beta\hat{S}-\sigma<0$, then $\hat{S}<\frac{\sigma}{\beta}$ and there exists an integer $p_1\ge p_0$ such that $\beta s(p+1)-\sigma<0$ for any $p\ge p_1$. By (4.19), we have

$$s(p+1)-s(p)=B-\delta s(p+1)-(\beta s(p+1)-\sigma)i(p+1)>B-\delta s(p+1),$$

and by $p\to+\infty$, we have that $\hat{S}-\hat{S}\ge B-\delta\hat{S}$, that is, $\hat{S}\ge\frac{B}{\delta}$, which implies that $\frac{\sigma}{\beta}>\hat{S}\ge\frac{B}{\delta}$. On the other hand, since by (1.6) and Lemma 2.3, $S^*=\frac{(1+\delta b)(e+\sigma)}{\beta}$ and $R_0=\frac{c(S^0)}{e+\sigma}=\frac{S^0}{S^*}>1$, we have that $S^0=\frac{B}{\delta}>S^*$. Moreover, by $S^*-\frac{\sigma}{\beta}=\frac{(1+\delta b)e+\delta b\sigma}{\beta}>0$, we have that $\frac{B}{\delta}>S^*>\frac{\sigma}{\beta}$, which is a contradiction. Thus, we prove that $\beta\hat{S}-\sigma\ge 0$.

Then, by the first equation of (1.9), we have that

$$\begin{aligned} B-\delta s(p+1) &> s(p+1)-s(p)=B-\delta s(p+1)-(\beta s(p+1)-\sigma)i(p+1)\\ &\ge B-\delta s(p+1)-(\beta s(p+1)-\sigma)qI^*, \end{aligned}$$

and by $p\to+\infty$, we have that $B-\delta\hat{S}\ge 0\ge B-\delta\hat{S}-(\beta\hat{S}-\sigma)qI^*$, that is, $\frac{B}{\delta}>\hat{S}\ge\frac{B+\sigma qI^*}{\delta+\beta qI^*}>\frac{\sigma}{\beta}$.

Consider the following $\tilde{S}>0$ such that

$$B-\delta\tilde{S}-\beta\tilde{S}qI^*+\sigma qI^*=0,\quad \text{that is}\quad \tilde{S}=\frac{B+\sigma qI^*}{\delta+\beta qI^*}. \tag{4.16}$$

Then, by $\frac{\sigma}{\beta}<S^*<S^0=\frac{B}{\delta}$, we have $\beta B-\sigma\delta>0$ and

$$\tilde{S}-S^*=\frac{B+\sigma qI^*}{\delta+\beta qI^*}-\frac{B+\sigma I^*}{\delta+\beta I^*}=\frac{(\beta B-\sigma\delta)(1-q)I^*}{(\delta+\beta qI^*)(\delta+\beta I^*)}>0. \tag{4.17}$$

Thus, we obtain that $\hat{S}\ge\tilde{S}>S^*$. Then, there exists an integer $p_1\ge 0$ such that $s(p+1)>S^*$ for any $p\ge p_1$. Therefore, by the second part of Lemma 4.2,

$$\min_{1\le j\le n+1}\tilde{y}_j(p+1)\ge\min_{1\le j\le n+1}\tilde{y}_j(p),\quad \text{for any } p\ge p_1, \tag{4.18}$$

and hence, there exists a positive constant $\underline{y}$ such that $\min_{1\le j\le n+1}\tilde{y}_j(p)\ge\underline{y}$ for any $p\ge p_1$. Thus, from (4.15), we have $\lim_{p\to\infty}w(p)=+\infty$. However, by (4.14) and Lemma 2.2, it holds that there is a positive constant $\overline{w}$ such that $w(p)\le\overline{w}$ for any $p\ge p_1$, which leads to a contradiction.

ii) Consider the case that $\{s(p)\}_{p=0}^{\infty}$ is not eventually monotone increasing. Then, there exists a sequence $\{p_l\}_{l=0}^{\infty}$ such that

$$s(p_l+1)\le s(p_l),\quad \text{and}\quad \lim_{l\to\infty}s(p_l+1)=\underline{S}\le\frac{B}{\delta}. \tag{4.19}$$

We show that $\beta\underline{S}-\sigma\ge 0$. If $\beta\underline{S}-\sigma<0$, then there exists an integer $l_1\ge 0$ such that $\beta s(p_l+1)-\sigma<0$ for any $l\ge l_1$. By the first equation of (1.9) and (4.19), we have

$$0\ge s(p_l+1)-s(p_l)=B-\delta s(p_l+1)-(\beta s(p_l+1)-\sigma)i(p_l+1)>B-\delta s(p_l+1),$$

and by $l\to+\infty$, we have that $0\ge B-\delta\underline{S}$, that is, $\underline{S}\ge\frac{B}{\delta}$, which implies that there is a limit of $\lim_{p\to\infty}s(p)=\frac{B}{\delta}>S^*$ and by the above discussion on (4.15), we conclude that $\lim_{p\to\infty}(s(p),y_1(p),y_2(p),\cdots,y_n(p),i(p))=(S^*,y_1^*,y_2^*,\cdots,y_n^*,I^*)$. This is a contradiction. Thus, we prove that $\beta\underline{S}-\sigma\ge 0$.

By the first equation of (1.9) and (4.19), we have

$$\begin{aligned} B-\delta s(p_l+1) \geq 0 \geq s(p_l+1)-s(p_l) &= B-\delta s(p_l+1)-(\beta s(p_l+1)-\sigma)i(p_l+1) \\ &\geq B-\delta s(p_l+1)-(\beta s(p_l+1)-\sigma)qI^*, \end{aligned}$$

and by $l \to +\infty$, we have that $B-\delta \underline{S} \geq 0 \geq B-\delta\underline{S}-(\beta\underline{S}-\sigma)qI^*$, that is, $\frac{B}{\delta} > \underline{S} \geq \tilde{S} = \frac{B+\delta qI^*}{\delta+\beta qI^*} > \frac{\sigma}{\beta}$. Hence, by $\beta B - \sigma\delta > 0$ and (4.17), we obtain that $\underline{S} \geq \tilde{S} > S^*$, which similarly leads to a contradiction by the above discussion on (4.15). Hence, the claim is proved.

Put $\underline{\tilde{y}}(p) = \min_{1\leq j\leq n+1}\{y_j(p)\}$. Then, by the claim, we are left to consider the two possibilities. First, $\underline{\tilde{y}}(p) \geq q$ for all p sufficiently large. Second, we consider the case that $\underline{\tilde{y}}(p)$ oscillates about q for all sufficiently large p. If the first condition that $\underline{\tilde{y}}(p) \geq q$ holds for all sufficiently large p, then we get the conclusion of the proof. For the second case that $\underline{\tilde{y}}(p)$ oscillates about q for all sufficiently large p, let $p_3 < p_4$ be sufficiently large such that

$$\underline{\tilde{y}}(p_3-1), \underline{\tilde{y}}(p_4+1) > q, \quad \text{and} \quad \underline{\tilde{y}}(p) \leq q \quad \text{for any } p_3 \leq p \leq p_4.$$

We first estimate the lower bound of $\tilde{y}_{n+1}(p)$ for $p_3 \leq p \leq p_3 + l_0(q)$. By the last equation of (4.4), we have that for $p_3 \leq p \leq p_3 + l_0(q)$,

$$\tilde{y}_{n+1}(p) \geq \frac{1}{1+e+\sigma}\tilde{y}_{n+1}(p-1) \geq \cdots \geq \left(\frac{1}{1+e+\sigma}\right)^{p+1-p_3}\tilde{y}_{n+1}(p_3-1) > \left(\frac{1}{1+e+\sigma}\right)^{l_0(q)+1} q.$$

Second, by (4.13), since one can obtain that for $p_3 \leq p \leq p_4$,

$$s(p+1) \geq s(p) + B - \delta s(p+1) - \beta s(p+1)qI^* = s(p) + B - (\delta + \beta qI^*)s(p+1) = s(p) + B - k_q s(p+1),$$

we get

$$s(p+1) \geq \frac{s(p)}{1+k_q} + \frac{B}{1+k_q}, \quad \text{for } p_3 \leq p \leq p_4,$$

which yields

$$\begin{aligned} s(p+1) &\geq \left(\frac{1}{1+k_q}\right)^{p+1-p_3} s(p_3) + \frac{B}{1+k_q}\sum_{l=0}^{p-p_3}\left(\frac{1}{1+k_q}\right)^l \\ &\geq \frac{B}{1+k_q}\frac{1-(\frac{1}{1+k_q})^{p+1-p_3}}{1-\frac{1}{1+k_q}} \\ &\geq \frac{B}{k_q}\left\{1-\left(\frac{1}{1+k_q}\right)^{p+1-p_3}\right\}, \quad \text{for any } p_3 \leq p \leq p_4. \end{aligned}$$

Therefore, if $p_4 - p_3 \geq l_0(q) - 1$, then by (4.13) we have that for any $p_3 + l_0(q) \leq p+1 \leq p_4$,

$$s(p+1) \geq s^{\triangle} \equiv \frac{B}{k_q}\left\{1-\left(\frac{1}{1+k_q}\right)^{l_0(q)}\right\} > S^*, \tag{4.20}$$

and by the second part of Lemma 4.2, we obtain

$$\min_{1\leq j\leq n+1}\{y_j(p+1)\} \geq \min_{1\leq j\leq n+1}\{y_j(p)\} \quad \text{for any } p_3 + l_0(q) \leq p \leq p_4, \tag{4.21}$$

which implies that $\min_{1\leq j\leq n+1}\{y_j(p)\} \geq \min_{1\leq j\leq n+1}\{y_j(p_3+l_0(q))\}$ for any $p_3 + l_0(q) \leq p \leq p_4$. Thus, $s(p_3 + l_0(q)) \geq S^*$ and by (4.9) in Lemma 4.3, we have

$$\begin{aligned} \min_{1\leq j\leq n+1}\tilde{y}_j(p_3+l_0(q)) &\geq \min\left\{\min\left(1, \frac{d}{1+d}, \frac{d^2}{(1+d)^2}, \cdots, \frac{d^{n-1}}{(1+d)^{n-1}}\right)\frac{d}{1+d}\frac{c(s(p+1))}{c(s^*)}, 1\right\}\tilde{y}_{n+1}(p_3+l_0(q)), \\ &\geq \frac{d^n}{(1+d)^n}\tilde{y}_{n+1}(p_3+l_0(q)) \\ &\geq \frac{d^n}{(1+d)^n}\left(\frac{1}{1+e+\sigma}\right)^{l_0(q)+1} q. \end{aligned}$$

Hence, we prove that

$$\underline{y}_{n+1} \geq \frac{d^n}{(1+d)^n}\left(\frac{1}{1+e+\sigma}\right)^{l_0(q)+1} qI^*.$$

Since q $(0 < q < 1)$ is arbitrarily chosen, we may conclude that

$$\underline{y}_{n+1} \geq \frac{d^n}{(1+d)^n} \left(\frac{1}{1+e+\sigma} \right)^{l_0(q)+1} I^*.$$

Hence, we prove the last equation of (4.12). This completes the proof. $\square$

By Lemmas 2.1 and 4.4, we obtain the permanence of system (1.9).

5 Global stability of the endemic equilibrium E^* for $R_0 > 1$

Assume $R_0 > 1$. Then, by Lemma 4.4, system (1.9) is permanent and by Lemma 2.3, system (1.9) has a unique endemic equilibrium $E^* = (S^*, y_1^*, y_2^*, \cdots, y_n^*, I^*)$. Moreover, (1.9) is equivalent to (4.4), which has a unique endemic equilibrium $\tilde{E}^* = (S^*, \tilde{y}_1^*, \tilde{y}_2^*, \cdots, \tilde{y}_n^*, \tilde{y}_{n+1}^*)$ with $\tilde{y}_1^* = \tilde{y}_2^* = \cdots = \tilde{y}_n^* = \tilde{y}_{n+1}^* = 1$. In the rest of this paper, we prove that the endemic equilibrium $\tilde{E}^*$ of (4.4) is globally asymptotically stable.

By $R_0 = \frac{c(S^0)}{e+\sigma} = \frac{\beta S^0}{(1+\delta b)^n(\delta+\epsilon+\gamma+\sigma)} > 1$, we have

$$S^0 = \frac{B}{\delta} > \frac{(1+\delta b)^n(\delta+\epsilon+\gamma+\sigma)}{\beta} = S^*$$

and $\beta S^* - \sigma = (1+\delta b)^n(\delta+\epsilon+\gamma+\sigma) - \sigma = (1+\delta b)^n(\delta+\epsilon+\gamma) + \{(1+\delta b)^n - 1\}\sigma > 0$. We define

$$U_s(p) = g\left(\frac{s(p)}{S^*}\right),\ U_{y_j}(p) = g\left(\frac{y_j(p)}{y_j^*}\right), \quad j = 1, 2, \ldots, n,\ U_i(p) = g\left(\frac{i(p)}{I^*}\right),$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$. For simplicity, we put

$$x(p+1) = \frac{s(p+1)}{S^*}, \quad z(p+1) = \frac{i(p+1)}{I^*} = \tilde{y}_{n+1}(p+1). \tag{5.1}$$

The following lemma is a key result which is a discrete version to that in Prüss *et al.* [11].

Lemma 5.1.

$$\begin{aligned} \left(1 + \frac{\sigma}{\beta S^* - \sigma}\right) S^*(U_s(p+1) - U_s(p)) \quad \leq \quad & -\frac{\beta S^*(\sigma I^* z(p+1) + \delta S^*)}{\beta S^* - \sigma} \frac{(x(p+1)-1)^2}{x(p+1)} \\ & + \beta S^* I^* \left(1 - \frac{1}{x(p+1)}\right)(1 - x(p+1) \cdot z(p+1)). \end{aligned} \tag{5.2}$$

Proof. By (1.9), we have that

$$\begin{aligned} U_s(p+1) - U_s(p) \quad &= \quad \frac{s(p+1) - s(p)}{S^*} - \ln \frac{s(p+1)}{s(p)} \\ &\leq \quad \frac{s(p+1) - s(p)}{S^*} - \frac{s(p+1) - s(p)}{s(p+1)} \\ &= \quad \frac{s(p+1) - S^*}{S^* s(p+1)}(s(p+1) - s(p)) \\ &= \quad \frac{s(p+1) - S^*}{S^* s(p+1)}(B - \beta s(p+1)i(p+1) - \delta s(p+1) + \sigma i(p+1)), \end{aligned} \tag{5.3}$$

because $\ln(1-x) \leq -x$ holds for all $x < 1$, one can obtain that

$$-\ln \frac{s(p+1)}{s(p)} = \ln\left\{1 - \left(1 - \frac{s(p)}{s(p+1)}\right)\right\} \leq -\left(1 - \frac{s(p)}{s(p+1)}\right) = -\frac{s(p+1) - s(p)}{s(p+1)}.$$

Substituting $B = \beta S^* I^* + \delta S^* - \sigma I^*$ into (5.3), we see that

$$\begin{aligned}
U_s(p+1) - U_s(p) &\leq \frac{s(p+1) - S^*}{S^* s(p+1)}(\beta S^* I^* + \delta S^* - \sigma I^* - \beta s(p+1)i(p+1) - \delta s(p+1) + \sigma i(p+1)) \\
&= -\frac{\delta(s(p+1) - S^*)^2}{S^* s(p+1)} + \beta I^* \left(1 - \frac{S^*}{s(p+1)}\right)\left(1 - \frac{s(p+1)}{S^*} \cdot \frac{i(p+1)}{I^*}\right) \\
&\quad + \frac{\sigma I^*}{S^*}\left(1 - \frac{S^*}{s(p+1)}\right)\left(\frac{i(p+1)}{I^*} - 1\right) \\
&= -\delta \frac{(x(p+1) - 1)^2}{x(p+1)} + \beta I^* \left(1 - \frac{1}{x(p+1)}\right)(1 - x(p+1) \cdot z(p+1)) \\
&\quad + \frac{\sigma I^*}{S^*}\left(1 - \frac{1}{x(p+1)}\right)(z(p+1) - 1).
\end{aligned}$$

On the other hand, by $B = \beta S^* I^* + \delta S^* - \sigma I^*$, we have

$$\begin{aligned}
s(p+1) - s(p) &= B - \beta s(p+1)i(p+1) - \delta s(p+1) + \sigma i(p+1) \\
&= \beta S^* I^* + \delta S^* - \delta I^* - \beta s(p+1)i(p+1) - \delta s(p+1) + \sigma i(p+1) \\
&= -(\beta i(p+1) + \delta)(s(p+1) - S^*) - (\beta S^* - \sigma)(i(p+1) - I^*) \\
&\quad + \beta S^* I^* + \delta S^* - \sigma I^* - (\beta i(p+1) + \delta)S^* + \beta S^*(i(p+1) - I^*) + \sigma I^* \\
&= -(\beta i(p+1) + \delta)(s(p+1) - S^*) - (\beta S^* - \delta)(i(p+1) - I^*),
\end{aligned}$$

and hence,

$$\begin{aligned}
U_s(p+1) - U_s(p) &= \frac{s(p+1) - s(p)}{S^*} - \ln \frac{s(p+1)}{s(p)} \\
&\leq \frac{s(p+1) - s(p)}{S^*} - \frac{s(p+1) - s(p)}{s(p+1)} \\
&= (s(p+1) - s(p))\left(\frac{1}{S^*} - \frac{1}{s(p+1)}\right) \\
&= -\{(\beta i(p+1) + \delta)(s(p+1) - S^*) + (\beta S^* - \sigma)(i(p+1) - I^*)\}\frac{s(p+1) - S^*}{s(p+1)S^*} \\
&= -(\beta I^* z(p+1) + \delta)\frac{(x(p+1) - 1)^2}{x(p+1)} - \frac{(\beta S^* - \sigma)I^*}{S^*}(z(p+1) - 1)\left(1 - \frac{1}{x(p+1)}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left(1 + \frac{\sigma}{\beta S^* - \sigma}\right)S^*(U(p+1) - U(p)) \\
\leq\ & -\delta S^* \frac{(x(p+1) - 1)^2}{x(p+1)} + \beta S^* I^* \left(1 - \frac{1}{x(p+1)}\right)(1 - x(p+1) \cdot z(p+1)) + \sigma I^* \left(1 - \frac{1}{x(p+1)}\right)(z(p+1) - 1) \\
& - \frac{\sigma S^*}{\beta S^* - \sigma}(\beta I^* z(p+1) + \delta)\frac{(x(p+1) - 1)^2}{x(p+1)} - \sigma I^*(z(p+1) - 1)\left(1 - \frac{1}{x(p+1)}\right) \\
=\ & -S^* \left\{\delta + \frac{\sigma}{\beta S^* - \sigma}(\beta I^* z(p+1) + \delta)\right\}\frac{(x(p+1) - 1)^2}{x(p+1)} + \beta S^* I^* \left(1 - \frac{1}{x(p+1)}\right)(1 - x(p+1)z(p+1)) \\
=\ & -\frac{\beta S^*(\sigma I^* z(p+1) + \delta S^*)}{\beta S^* - \sigma}\frac{(x(p+1) - 1)^2}{x(p+1)} + \beta S^* I^* \left(1 - \frac{1}{x(p+1)}\right)(1 - x(p+1)z(p+1)).
\end{aligned}$$

Hence, the proof of this lemma is complete. □

The following lemma plays an important role to apply techniques of equation deformation in McCluskey [9, Proof of Theorem 4.1] to the global stability analysis of endemic equilibrium of system (1.9).

Lemma 5.2. *If $R_0 > 1$, then it holds that*

$$\begin{aligned}
&\left(1-\frac{1}{x(p+1)}\right)(1-x(p+1)\cdot z(p+1))\\
&+\left(1-\frac{1}{\tilde{y}_1(p+1)}\right)(x(p+1)\cdot\tilde{y}_{n+1}(p+1)-\tilde{y}_1(p+1))+\sum_{j=2}^{n+1}\left(1-\frac{1}{\tilde{y}_j(p+1)}\right)(\tilde{y}_{j-1}(p+1)-\tilde{y}_j(p+1))\\
&=-\left\{g\left(\frac{1}{x(p+1)}\right)+g\left(\frac{x(p+1)\cdot\tilde{y}_{n+1}(p+1)}{\tilde{y}_1(p+1)}\right)+\sum_{j=2}^{n+1}g\left(\frac{\tilde{y}_{j-1}(p+1)}{\tilde{y}_j(p+1)}\right)\right\}\leq 0. \qquad (5.4)
\end{aligned}$$

Proof.

$$\begin{aligned}
&\left(1-\frac{1}{x(p+1)}\right)(1-x(p+1)\cdot z(p+1))\\
&+\left(1-\frac{1}{\tilde{y}_1(p+1)}\right)(x(p+1)\cdot\tilde{y}_{n+1}(p+1)-\tilde{y}_1(p+1))+\sum_{j=2}^{n+1}\left(1-\frac{1}{\tilde{y}_j(p+1)}\right)(\tilde{y}_{j-1}(p+1)-\tilde{y}_j(p+1))\\
=\;&1-\frac{1}{x(p+1)}-x(p+1)\cdot z(p+1)+z(p+1)\\
&+x(p+1)\cdot\tilde{y}_{n+1}(p+1)-\frac{x(p+1)\cdot\tilde{y}_{n+1}(p+1)}{\tilde{y}_1(p+1)}-\tilde{y}_1(p+1)+1+\sum_{j=2}^{n+1}\left(\tilde{y}_{j-1}(p+1)-\frac{\tilde{y}_{j-1}(p+1)}{\tilde{y}_j(p+1)}-\tilde{y}_j(p+1)+1\right)\\
=\;&n+2-\frac{1}{x(p+1)}-\frac{x(p+1)\cdot\tilde{y}_{n+1}(p+1)}{\tilde{y}_1(p+1)}-\sum_{j=2}^{n+1}\frac{\tilde{y}_{j-1}(p+1)}{\tilde{y}_j(p+1)}\\
=\;&-\left\{g\left(\frac{1}{x(p+1)}\right)+g\left(\frac{x(p+1)\cdot\tilde{y}_{n+1}(p+1)}{\tilde{y}_1(p+1)}\right)+\sum_{j=2}^{n+1}g\left(\frac{\tilde{y}_{j-1}(p+1)}{\tilde{y}_j(p+1)}\right)\right\}\leq 0.
\end{aligned}$$

Hence this completes the proof. □

Proof of the second part of Theorem 1.1. Consider the following Lyapunov function (see Prüss *et al.* [11]).

$$U(p)=\frac{1}{\beta S^*I^*}\left(1+\frac{\sigma}{\beta S^*-\sigma}\right)S^*U_s(p)+\frac{1}{d}\sum_{j=1}^{n}U_{y_j}(p)+\frac{1}{e+\sigma}U_i(p), \qquad (5.5)$$

where

$$U_s(p)=g\left(\frac{s(p)}{S^*}\right),\; U_{y_j}(p)=g(\tilde{y}_j(p)),\quad j=1,2,\dots,n,\; U_i(p)=g(\tilde{y}_{n+1}(p)).$$

First, by the relation $\frac{c(s(p+1))}{c(S^*)}=\frac{s(p+1)}{S^*}=x(p+1)$, calculating $U_{y_j}(p+1)-U_{y_j}(p)$ for $j=1,2,\dots,n+1$ gives as follows.

$$\begin{aligned}
U_{y_j}(p+1)-U_{y_j}(p)&=\tilde{y}_j(p+1)-\tilde{y}_j(p)-\ln\frac{\tilde{y}_j(p+1)}{\tilde{y}_j(p)}\\
&\leq\tilde{y}_j(p+1)-\tilde{y}_j(p)-\frac{\tilde{y}_j(p+1)-\tilde{y}_j(p)}{\tilde{y}_j(p+1)}\\
&=\frac{\tilde{y}_j(p+1)-1}{\tilde{y}_j(p+1)}(\tilde{y}_j(p+1)-\tilde{y}_j(p))\\
&=\begin{cases}
d\left(1-\dfrac{1}{\tilde{y}_1(p+1)}\right)(x(p+1)\cdot\tilde{y}_{n+1}(p+1)-\tilde{y}_1(p+1)), & \text{if } j=1,\\
d\left(1-\dfrac{1}{\tilde{y}_j(p+1)}\right)(\tilde{y}_{j-1}(p+1)-\tilde{y}_j(p+1)), & \text{if } j=2,3,\dots,n,\\
(e+\sigma)\left(1-\dfrac{1}{\tilde{y}_{n+1}(p+1)}\right)(\tilde{y}_n(p+1)-\tilde{y}_{n+1}(p+1)), & \text{if } j=n+1.
\end{cases}
\end{aligned}$$

Therefore, by Lemmas 5.1 and 5.2, we have that

$$\begin{aligned}
U(p+1)-U(p) &\leq -\frac{(\sigma I^* z(p+1)+\mu S^*)}{I^*(\beta S^*-\sigma)}\frac{(x(p+1)-1)^2}{x(p+1)} \\
&\quad +\left(1-\frac{1}{x(p+1)}\right)(1-x(p+1)\cdot z(p+1)) \\
&\quad +\left(1-\frac{1}{\tilde{y}_1(p+1)}\right)(x(p+1)\cdot \tilde{y}_{n+1}(p+1)-\tilde{y}_1(p+1)) \\
&\quad +\sum_{j=2}^{n}\left(1-\frac{1}{\tilde{y}_j(p+1)}\right)\Big(\tilde{y}_{j-1}(p+1)-\tilde{y}_j(p+1)\Big) \\
&\quad +\left(1-\frac{1}{\tilde{y}_{n+1}(p+1)}\right)(\tilde{y}_n(p+1)-\tilde{y}_{n+1}(p+1)) \\
&= -\frac{\sigma I^* z(p+1)+\delta S^*}{I^*(\beta S^*-\sigma)}\frac{(x(p+1)-1)^2}{x(p+1)} \\
&\quad -\left\{g\left(\frac{1}{x(p+1)}\right)+g\left(\frac{x(p+1)\cdot \tilde{y}_{n+1}(p+1)}{\tilde{y}_1(p+1)}\right)+\sum_{j=2}^{n+1} g\left(\frac{\tilde{y}_{j-1}(p+1)}{\tilde{y}_j(p+1)}\right)\right\} \\
&\leq 0.
\end{aligned}$$

Hence, $U(p+1)-U(p)\leq 0$ for any $p\geq 0$. Since $U(p)\geq 0$ is a monotone decreasing sequence, there is a limit $\lim_{p\to+\infty}U(p)\geq 0$. Then, $\lim_{p\to+\infty}(U(p+1)-U(p))=0$, from which we obtain

$$\lim_{p\to+\infty}s(p+1)=S^*,\quad \lim_{p\to+\infty}\tilde{y}_j(p+1)=\tilde{y}_j^*,\quad j=1,2,\ldots,n+1,$$

that is, $\lim_{p\to+\infty}(s(p),y_1(p),y_2(p),\cdots,y_{n+1}(p))=(S^*,y_1^*,y_2^*,\cdots,y_{n+1}^*)$. Since $U(p)\leq U(0)$ for all $p\geq 0$ and $g(x)\geq 0$ with equality if and only if $x=1$, E^* is uniformly stable. Hence, the proof is complete. □

6 Conclusions

In this paper, we propose a discrete epidemic model for disease with immunity and latency spreading in a heterogeneous host population which is derived from the continuous case of model by using the well-known backward Euler method, and applying a Lyapunov functional technique which is a discrete version to that in Prüss *et al.* [11], it is shown that the global dynamics of this discrete epidemic model with latency are fully determined by a single threshold parameter.

Despite the proofs of main results in Yuan and Zou [15] make use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov functions guided by graph theory, we apply the techniques of Lyapunov functions in McCluskey [9] and Prüss *et al.* [11] to prove the global asymptotic stability for the endemic equilibrium of system (1.9) for the case $R_0>1$ which is simpler and no longer needs using any of the theory of non-negative matrices and graph theory (cf. Guo *et al.* [4]). Moreover, we offer new techniques (cf. Muroya *et al.* [10]) for obtaining lower bounds for the permanence of group epidemic models which are derived by the backward Euler method from continuous group epidemic models and will be useful in applications (cf. persistence theory in dynamical systems, for example, Thieme [14], Freedman *et al* [3] and Guo *et al.* [4]). The extension of these techniques to the other types of discrete and continuous group epidemic models will be considered in future work.

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