Published in Nonlinear Analysis: Real World Applications **14** (2013) 1693-1704
DOI: 10.1016/j.nonrwa.2012.11.005

Global stability for a multi-group SIRS epidemic model with varying population sizes

Yoshiaki Muroya*
Department of Mathematics, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan
E-mail: ymuroya@waseda.jp

Yoichi Enatsu
Graduate School of Fundamental Sciences and Engineering, Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo, 169-8555, Japan
E-mail: y.enatsu@kurenai.waseda.jp

Toshikazu Kuniya
Graduate School of Mathematical Sciences, University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan
E-mail: tkuniya@ms.u-tokyo.ac.jp

Abstract. In this paper, by extending well-known Lyapunov function techniques, we establish sufficient conditions for the global stability of an endemic equilibrium of a multi-group SIRS epidemic model with varying population sizes which has cross patch infection between different groups. Our proof no longer needs such a grouping technique by graph theory commonly used to analyze the multi-group SIR models.

Keywords: Multi-group SIRS epidemic model, varying population size, global stability, Lyapunov function.
2000 Mathematics Subject Classification. Primary: 34K20, 34K25; Secondary: 92D30.

1 Introduction

Multi-group epidemic models have been studied in the literature of mathematical epidemiology to describe transmission dynamics of various infectious diseases such as measles, mumps, gonorrhea, West-Nile virus and HIV/AIDS. A heterogeneous host population can be divided into several homogeneous groups according tomodes of transmission, contact patterns, or geographic distributions, so that within-group and inter-group interactions could be modeled separately.

There are several group models, see, e.g., patch models [1,25] and as transport-related models [14,15,18] and references therein. In 2006, Guo et al. [8] have first succeeded to establish the complete global dynamics for a multi-group SIR model, by making use of the theory of non-negative matrices, Lyapunov functions and a subtle grouping technique in estimating the derivatives of Lyapunov functions guided by graph theory.

However, some researchers on multi-group SIR epidemic models, commonly follow this research approach to analyze the global stability of various multi-group SIR epidemic models. On the other hand, recently, Nakata et al. [19] and Enatsu et al. [6] proposed a simple idea to extend Lyapunov functional techniques in McCluskey [16] for SIR epidemic models to SIRS epidemic models, and Muroya et al. [17] succeeded to prove the global stability for a class of multi-group SIR epidemic models without use of the grouping technique by graph theory in Guo et al. [8].

Motivated by these facts, we are interested in the global stability of the following multi-group SIRS epidemic model which has cross patch infection between different groups:

$$\begin{cases} \dfrac{dS_k}{dt} = b_k - \mu_{k1} S_k - S_k \left(\displaystyle\sum_{j=1}^{n} \beta_{kj} I_j \right) + \delta R_k, \\ \dfrac{dI_k}{dt} = S_k \left(\displaystyle\sum_{j=1}^{n} \beta_{kj} I_j \right) - (\mu_{k2} + \gamma_k) I_k, \\ \dfrac{dR_k}{dt} = \gamma_k I_k - (\mu_{k3} + \delta_k) R_k, \quad k = 1, 2, \cdots, n. \end{cases} \tag{1.1}$$

*Corresponding author.

$S_k(t)$, $I_k(t)$ and $R_k(t)$, $k = 1, 2, \ldots, n$ denote the numbers of susceptible, infected and recovered individuals in city k at time t, respectively. b_k, $k = 1, 2, \ldots, n$ is the recruitment rate of the population, μ_{ki}, $i = 1, 2, 3$ is the natural death rates of susceptible, infected and recovered individuals in city k, $k = 1, 2, \ldots, n$, and γ_k denotes the natural recovery rate of the infected individuals in city k, $k = 1, 2, \ldots, n$, respectively. Functions describing the dynamics within city k of each population of individuals, might involve all populations of individuals that are present in each city. We suppose that there are no between-city interactions and two cities are connected by the direct transport such as airplanes. Therefore, for the model (1.1), the only input is the recruitment. Moreover, not only for infective individuals I_k in city k, disease is transmitted to the susceptible individuals S_k by the incidence rate $\beta_{kk} S_k I_k$ with a transmission rate β_{kk}, but also we consider cross patch infection between different groups such that for each I_j, $j \neq k$ who travel from other city j into city k, disease is transmitted by the incidence rate $\beta_{kj} S_k I_j$ with a transmission rate β_{kj}.

The initial conditions of system (1.1) take the form

$$S_k(0) = \phi_1^k > 0,\ I_k(0) = \phi_2^k > 0,\ R_k(0) = \phi_3^k > 0,\ k = 1, 2, \ldots, n \tag{1.2}$$

By the biological meanings, we may assume that

$$\mu_{k1} \leq \min(\mu_{k2}, \mu_{k3}), \qquad k = 1, 2, \ldots, n. \tag{1.3}$$

Moreover, for simplicity in this paper, we assume that

$$\text{the } n \times n \text{ matrix } \mathbf{B} = (\beta_{kj})_{n\times n} \text{ is irreducible,} \tag{1.4}$$

that is, an infected individual in the first group can cause infection to a susceptible individual in the second group through an infection path. Put

$$\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)), \tag{1.5}$$

where the positive n-column vector $\mathbf{S}^0 = (S_1^0, S_2^0, \ldots, S_n^0)^T = (b_1/\mu_{11}, b_2/\mu_{21}, \ldots, b_n/\mu_{n1})^T$ and $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$ denotes a spectral radius of the matrix $\tilde{\mathbf{M}}(\mathbf{S}^0)$ defined by

$$\tilde{\mathbf{M}}(\mathbf{S}^0) = \left(\frac{\beta_{kj} S_k^0}{\mu_{k2} + \gamma_k}\right)_{n\times n}. \tag{1.6}$$

Observe that if $\delta_k = 0$, $k = 1, 2, \ldots, n$, then the variables R_k, $k = 1, 2, \ldots, n$ do not appear in (1.1) and hence, in this case, we may consider only the reduced system for S_k and I_k, $k = 1, 2, \ldots, n$ as follows.

$$\begin{cases} \dfrac{dS_k}{dt} = b_k - \mu_{k1} S_k - S_k\left(\displaystyle\sum_{j=1}^n \beta_{kj} I_j\right), \\ \dfrac{dI_k}{dt} = S_k\left(\displaystyle\sum_{j=1}^n \beta_{kj} I_j\right) - (\mu_{k2} + \gamma_k) I_k. \end{cases} \tag{1.7}$$

For system (1.7), the result of Guo *et al.* [8] is as follows.

Theorem A *For* (1.7)*, assume that* $\mu_{k1} \leq \mu_{k2}$, $k = 1, 2, \ldots, n$ *and* (1.4) *holds. Then, for* $\hat{R}_0 \leq 1$*, the disease-free equilibrium* $\hat{\mathbf{E}}^0 = (\hat{S}_1^0, 0, \hat{S}_2^0, 0, \ldots, \hat{S}_n^0, 0)$ *of system (1.7) is globally asymptotically stable in* $\hat{\Gamma}$*, and for* $\hat{R}_0 > 1$*, there exists an endemic equilibrium* $\hat{\mathbf{E}}^* = (\hat{S}_1^*, \hat{I}^*, \hat{S}_2^*, \hat{I}_2^*, \ldots, \hat{S}_n^*, \hat{I}_n^*)$ *of system (1.1) (see [5]) which is globally asymptotically stable in* $\hat{\mathbf{\Gamma}}^0$*, where* $\hat{\mathbf{\Gamma}}^0$ *is the interior of the feasible region* $\hat{\mathbf{\Gamma}}$ *defined by*

$$\hat{\mathbf{\Gamma}} = \left\{(S_1, I_1, S_2, I_2, \ldots, S_n, I_n) \in \mathbf{R}_{+0}^{2n} \middle|\ S_k \leq \frac{b_k}{\mu_{k1}}, S_k + I_k \leq \frac{b_k}{\mu_{k1}},\ k = 1, 2, \ldots, n\right\},$$

and $\mathbf{R}_{+0}^m = \{(x_1, \ldots, x_m) : x_k \geq 0,\ k = 1, 2, \ldots, m\}$. The main theorem in this paper is as follows.

Theorem 1.1. *For system (1.1), assume that* (1.3) *and* (1.4) *hold. Then, for* $\tilde{R}_0 \leq 1$*, the disease-free equilibrium* $\mathbf{E}^0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_n^0, 0, 0)$ *is globally asymptotically stable in* $\mathbf{\Gamma}$*, and for* $\tilde{R}_0 > 1$*, system (1.1) is uniformly persistent in* $\mathbf{\Gamma}^0$ *and there exists at least one endemic equilibrium* $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_n^*)$ *in* $\mathbf{\Gamma}^0$*, where* $\mathbf{\Gamma}^0$ *is the interior of the feasible region* $\mathbf{\Gamma}$ *defined by*

$$\mathbf{\Gamma} = \left\{(S_1, I_1, R_1, S_2, I_2, R_2, \ldots, S_n, I_n, R_n) \in \mathbf{R}_+^{3n} \middle|\ S_k \leq S_k^0,\ S_k + I_k + R_k \leq \frac{b_k}{\mu_{k1}},\ k = 1, 2, \ldots, n\right\}. \tag{1.8}$$

Moreover, for $\tilde{R}_0 > 1$*, if*

$$\mu_{k1} S_k^* - \delta_k R_k^* \geq 0, \quad \text{for any } k = 1, 2, \ldots, n, \tag{1.9}$$

then $\mathbf{E}^*$ *is globally asymptotically stable in* $\mathbf{\Gamma}^0$*.*

The organization of this paper is as follows. In order to prove Theorem 1.1, we consider the reduced system of (1.1). In Section 2, we offer eventual boundedness of the solutions. In Section 3, following the proof techniques in Guo *et al.* [8], we similarly prove the global asymptotic stability of the disease-free equilibrium for $\tilde{R}_0 \leq 1$ and uniform persistence and the existence of the endemic equilibrium $\mathbf{E}^*$ of system (1.1) for $\tilde{R}_0 > 1$ (see Proposition 3.1 and Corollary 3.1). In Section 4, for $\tilde{R}_0 > 1$, using Lyapunov function techniques, under the condition (1.9), we prove the global asymptotic stability of the endemic equilibrium of (1.1).

2 Positiveness and eventual boundedness

In this section, we consider positiveness and eventual boundedness of solutions of (1.1). Let $\mathbf{S}^0 = (S_1^0, S_2^0, \ldots, S_n^0)^T = (b_1/\mu_{11}, b_2/\mu_{21}, \ldots, b_n/\mu_{n1})^T$ be a positive n-column vector and $N_k = S_k + I_k + R_k$ be the total population in city k, $k = 1, 2, \ldots, n$. Then, we have the following lemma:

Lemma 2.1. *It holds that*

$$S_k(t) > 0,\ I_k(t) > 0,\ R_k(t) > 0,\ \textit{for any } k = 1, 2, \ldots, n \textit{ and } t \geq 0, \tag{2.1}$$

and under the condition (1.3),

$$\limsup_{t \to +\infty} N_k(t) \leq S_k^0,\ k = 1, 2, \ldots, n \tag{2.2}$$

holds.

Proof. First, by (1.2), we have $S_k(0) > 0$, $I_k(0) > 0$ and $R_k(0) > 0$ for any $k = 1, 2, \ldots, n$. Suppose that there exist a positive t_1 and a positive integer k_1 such that $S_{k_1}(t_1) = 0$ and $S_k(t) > 0,\ I_k(t) > 0,\ R_k(t) > 0$ for any $k = 1, 2, \ldots, n$ and $0 \leq t < t_1$. However, by (1.1), we have $\frac{d}{dt} S_{k_1}(t_1) \geq b_{k_1} > 0$, which is a contradiction to the fact that $S_{k_1}(t) > 0 = S_{k_1}(t_1)$ for any $0 \leq t < t_1$. Similarly, suppose that there exists a positive t_2 such that there exists a positive integer k_2 such that $I_{k_2}(t_2) = 0$ and $S_k(t) > 0,\ I_k(t) > 0,\ R_k(t) > 0$ for any $k = 1, 2, \ldots, n$ and $0 \leq t < t_2$. But by (1.1), we have that $\frac{d}{dt} I_{k_2}(t_2) \geq 0$, which is a contradiction to the fact that $S_{k_2}(t) > 0 = S_{k_2}(t_2)$ for any $0 \leq t < t_2$. Similarly, we can obtain the similar result for $R_k(t)$. Thus, we obtain (2.1).

By (1.1) and (1.3), we have

$$\frac{d}{dt} N_k(t) = \frac{d}{dt}\{S_k(t) + I_k(t) + R_k(t)\} = b_k - \mu_{k1} S_k(t) - \mu_{k2} I_k(t) - \mu_{k3} R_k(t)\} \leq b_k - \mu_{k1} N_k(t), \quad k = 1, 2, \ldots, n,$$

from which we obtain (2.2). □

3 Global stability of the disease-free equilibrium $\mathbf{E}^0$ for $\tilde{R}_0 \leq 1$

We can obtain the following proposition, whose proof is similar to that of Guo *et al.* [8, Proposition 3.1] (see the proof of Proposition 3.1 in Appendix).

Proposition 3.1. (1) *If $\tilde{R}_0 \leq 1$, then the disease-free equilibrium $\mathbf{E}^0 = (S_1^0, 0, 0, S_2^0, 0, 0, \ldots, S_n^0, 0, 0)$ of system (1.1) is the unique equilibrium of (1.1) and it is globally asymptotically stable in $\mathbf{\Gamma}$.*
(2) *If $\tilde{R}_0 > 1$, then $\mathbf{E}^0$ is unstable and system (1.1) is uniformly persistent in $\mathbf{\Gamma}^0$.*

Uniform persistence of (1.1) together with uniform boundedness of solutions in $\mathbf{\Gamma}^0$ (follows from the positive invariance of the bounded region $\mathbf{\Gamma}$) implies the existence of an endemic equilibrium of (1.1) in $\mathbf{\Gamma}^0$ (see [20, Theorem D.3] or Bhatia *et al.* [2, Theorem 2.8.6]).

Corollary 3.1. *If $\tilde{R}_0 > 1$, then (1.1) has at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, , S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_3^*)$ such that*

$$(\tilde{\mathbf{F}}(\mathbf{S}^*) - \tilde{\mathbf{V}})\mathbf{I}^* = \mathbf{0}, \tag{3.1}$$

where

$$\tilde{\mathbf{F}}(\mathbf{S}) = (\beta_{kj} S_k)_{n \times n} \textit{ and } \tilde{\mathbf{V}} = \mathrm{diag}(\mu_{12} + \gamma_1, \mu_{22} + \gamma_2, \ldots, \mu_{n2} + \gamma_n), \tag{3.2}$$

and

$$\mathbf{S} = (S_1, S_2, \ldots, S_n)^T,\ \mathbf{S}^* = (S_1^*, S_2^*, \ldots, S_n^*)^T,\ \mathbf{I}^* = (I_1^*, I_2^*, \ldots, I_n^*)^T. \tag{3.3}$$

Now, we consider a relation between the reproduction number R_0 and $\tilde{R}_0$ in (3.4) (see also [23]). Put

$$R_0 = \rho(\mathbf{M}(\mathbf{S}^0)), \tag{3.4}$$

where the positive n-column vector $\mathbf{S} = (S_1, S_2, \ldots, S_n)^T$ and $\rho(\mathbf{M}(\mathbf{S}^0))$ denotes the spectral radius of the matrix $\mathbf{M}(\mathbf{S}^0)$ defined by

$$\mathbf{M}(\mathbf{S}) = \left(\frac{\beta_{kj} S_k}{\mu_{j2} + \gamma_j} \right)_{n \times n}, \tag{3.5}$$

Then, we have the following lemma (see the proof of Lemma 3.1 in Appendix):

Lemma 3.1.

$$\begin{cases} R_0 < 1, & \textit{if and only if}, \quad \tilde{R}_0 < 1, \\ R_0 = 1, & \textit{if and only if}, \quad \tilde{R}_0 = 1, \\ R_0 > 1, & \textit{if and only if}, \quad \tilde{R}_0 > 1. \end{cases} \tag{3.6}$$

Therefore, for convenience, we may use $\tilde{R}_0$ defined by (1.5) as a threshold parameter (see Guo *et al.* [8]) in place of the basic reproduction number R_0 defined by (3.4).

4 Global stability of the endemic equilibrium $\mathbf{E}^*$ for $\tilde{R}_0 > 1$

In this section, we assume $\tilde{R}_0 > 1$, and we prove that an endemic equilibrium of (1.1) is globally asymptotically stable in $\mathbf{\Gamma}^0$. By Corrollary 3.1, there exists an endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*, \ldots, S_n^*, I_n^*, R_n^*) \in \mathbf{\Gamma}^0$ such that

$$\begin{cases} b_k = \mu_{k1} S_k^* + \displaystyle\sum_{j=1}^n \beta_{kj} S_k^* I_j^* - \delta_k R_k^*, \\ (\mu_{k2} + \gamma_k) I_k^* = \displaystyle\sum_{j=1}^n \beta_{kj} S_k^* I_j^*, \\ \gamma_k I_k^* - (\mu_{k3} + \delta_k) R_k^* = 0, \quad k = 1, 2, \ldots, n. \end{cases} \tag{4.1}$$

We rewrite (1.1) as

$$\begin{cases} \dfrac{dS_k}{dt} = b_k - \mu_{k1} S_k - \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j + \delta_k R_k, \\ \dfrac{dI_k}{dt} = \displaystyle\sum_{j=1}^n \beta_{kj} S_k I_j - (\mu_{k2} + \gamma_k) I_k, \\ \dfrac{dR_k}{dt} = \gamma_k I_k - (\mu_{k3} + \delta_k) R_k, \quad k = 1, 2 \ldots, n. \end{cases} \tag{4.2}$$

Now, for some positive constants $v_1, v_2, \cdots, v_n$, let us consider

$$U_1 = \sum_{k=1}^n v_k \left\{ S_k^* g\left(\frac{S_k}{S_k^*} \right) + I_k^* g\left(\frac{I_k}{I_k^*} \right) \right\}, \ g(x) = x - 1 - \ln x \geq g(1) = 0, \quad \text{for any } x > 0. \tag{4.3}$$

Differentiating U_1, we have

$$\frac{dU_1(t)}{dt} = \sum_{k=1}^n v_k \left\{ \left(1 - \frac{S_k^*}{S_k} \right) \frac{dS_k}{dt} + \left(1 - \frac{I_k^*}{I_k} \right) \frac{dI_k}{dt} \right\}.$$

Put

$$x_k = \frac{S_k}{S_k^*}, \ y_k = \frac{I_k}{I_k^*}, \ z_k = \frac{R_k}{R_k^*}, \ k = 1, 2, \ldots, n. \tag{4.4}$$

Lemma 4.1. *Assume the conditions* (1.4) *and* $\tilde{R}_0 > 1$. *Then,*

$$\begin{aligned} \frac{dU_1(t)}{dt} = &\sum_{k=1}^n v_k \left\{ -\mu_{k1} S_k^* \left(1 - \frac{1}{x_k} \right) (x_k - 1) + \delta_k R_k^* \left(1 - \frac{1}{x_k} \right) (z_k - 1) \right\} \\ &- \sum_{k=1}^n v_k \sum_{j=1}^n \beta_{kj} S_k^* I_j^* \left\{ g\left(\frac{1}{x_k} \right) + g\left(\frac{x_k y_j}{y_k} \right) \right\} + \sum_{k=1}^n \left\{ \sum_{j=1}^n v_j \beta_{jk} S_j^* - v_k (\mu_{k2} + \gamma_k) \right\} I_k^* g(y_k). \end{aligned} \tag{4.5}$$

Proof. By (4.1) and (4.2), we have

$$\begin{aligned}\frac{dS_k}{dt} &= b_k - \mu_{k1} S_k - \sum_{j=1}^{n} \beta_{kj} S_k I_j + \delta R_k \\ &= -\mu_k (S_k - S_k^*) - \sum_{j=1}^{n} \beta_{kj} (S_k I_j - S_k^* I_j^*) + \delta_k (R_k - R_k^*) \\ &= -\mu_k S_k^* (x_k - 1) - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* (x_k y_j - 1) + \delta_k R_k^* (z_k - 1),\end{aligned}$$

and

$$\frac{dI_k}{dt} = \sum_{j=1}^{n} \beta_{kj} S_k I_j - (\mu_{k2} + \gamma_k) I_k = \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* x_k y_j - (\mu_{k2} + \gamma_k) I_k^* y_k = \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* (x_k y_j - y_k).$$

Then,

$$\begin{aligned}\frac{dU_1(t)}{dt} &= \sum_{k=1}^{n} v_k \left[\left(1 - \frac{1}{x_k}\right) \left\{ -\mu_{k1} S_k^* (x_k - 1) - \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* (x_k y_j - 1) + \delta_k R_k^* (z_k - 1) \right\} \right. \\ &\quad \left. + \left(1 - \frac{1}{y_k}\right) \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* (x_k y_j - y_k) \right] \\ &= \sum_{k=1}^{n} v_k \left[-\mu_{k1} S_k^* \left(1 - \frac{1}{x_k}\right)(x_k - 1) + \delta_k R_k^* \left(1 - \frac{1}{x_k}\right)(z_k - 1) \right] \\ &\quad + \sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \left\{ \left(1 - \frac{1}{x_k}\right)(1 - x_k y_j) + \left(1 - \frac{1}{y_k}\right)(x_k y_j - y_k) \right\} \right]. \end{aligned} \tag{4.6}$$

Next, for the last equation of (4.6), we have

$$\begin{aligned}\left(1 - \frac{1}{x_k}\right)(1 - x_k y_j) + \left(1 - \frac{1}{y_k}\right)(x_k y_j - y_k) &= \left(1 - \frac{1}{x_k} - x_k y_j + y_j\right) + \left(x_k y_j - \frac{x_k y_j}{y_k} - y_k + 1\right) \\ &= 2 - \frac{1}{x_k} + y_j - \frac{x_k y_j}{y_k} - y_k \\ &= -g\left(\frac{1}{x_k}\right) - g\left(\frac{x_k y_j}{y_k}\right) + g(y_j) - g(y_k).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{k=1}^{n} v_k \left[\sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \left\{ \left(1 - \frac{1}{x_k}\right)(1 - x_k y_j) + \left(1 - \frac{1}{y_k}\right)(x_k y_j - y_k) \right\} \right] \\ &= -\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* I_j^* \left\{ g\left(\frac{1}{x_k}\right) + g\left(\frac{x_k y_j}{y_k}\right) \right\} \right] + \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \{ g(y_j) - g(y_k) \},\end{aligned} \tag{4.7}$$

and by (4.1), we have

$$\begin{aligned}\sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* \{ g(y_j) - g(y_k) \} &= \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* g(y_j) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* g(y_k) \\ &= \sum_{j=1}^{n} v_j \sum_{k=1}^{n} \beta_{jk} S_j^* I_k^* g(y_k) - \sum_{k=1}^{n} v_k (\mu_{k2} + \gamma_k) I_k^* g(y_k) \\ &= \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j \beta_{jk} S_j^* - v_k (\mu_{k2} + \gamma_k) \right\} I_k^* g(y_k).\end{aligned} \tag{4.8}$$

Hence, from (4.6)-(4.8), one can obtain this lemma. □

Moreover, let us consider the following condition:

$$\sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j \beta_{jk} S_j^* - v_k (\mu_{k2} + \gamma_k) \right\} I_k^* g(y_k) = 0. \tag{4.9}$$

Then, we have the following lemma:

Lemma 4.2. *The following system*

$$\sum_{j=1}^{n} v_j \beta_{jk} S_j^* = v_k(\mu_{k2} + \gamma_k), \quad k = 1, 2, \ldots, n, \tag{4.10}$$

has a positive solution $(v_1, v_2, \ldots, v_n)$ *defined by*

$$(v_1, v_2, \cdots, v_n) = (C_{11}, C_{22}, \ldots, C_{nn}), \tag{4.11}$$

where

$$\tilde{\beta}_{kj} = \beta_{kj} S_k^* I_j^*, \quad 1 \leq k, j \leq n, \tag{4.12}$$

and

$$\tilde{\mathbf{B}} = \begin{bmatrix} \sum_{j\neq 1} \tilde{\beta}_{1j} & -\tilde{\beta}_{21} & \cdots & -\tilde{\beta}_{n1} \\ -\tilde{\beta}_{12} & \sum_{j\neq 2} \tilde{\beta}_{2j} & \cdots & -\tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\beta}_{1n} & -\tilde{\beta}_{2n} & \cdots & \sum_{j\neq n} \tilde{\beta}_{nj} \end{bmatrix}, \tag{4.13}$$

and C_{kk} *denotes the cofactor of the* k*-th diagonal entry of* $\tilde{\mathbf{B}}$, $1 \leq k \leq n$.

Proof. Consider a basis for the solution space of the linear system

$$\tilde{\mathbf{B}}\mathbf{v} = 0, \tag{4.14}$$

which can be written as (4.11) (see for example, Berman and Plemmons [3]). By the irreducibility of $\mathbf{B}$, we know that $(\tilde{\beta}_{kj})_{n\times n}$ is irreducible and $v_k = C_{kk} > 0,\ k = 1, 2, \ldots, n$. Then, by (4.14), we have that

$$\begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{21} & \cdots & \tilde{\beta}_{n1} \\ \tilde{\beta}_{12} & \tilde{\beta}_{22} & \cdots & \tilde{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\beta}_{1n} & \tilde{\beta}_{2n} & \cdots & \tilde{\beta}_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \left(\sum_{j=1}^{n} \tilde{\beta}_{1j}\right) v_1 \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{2j}\right) v_2 \\ \vdots \\ \left(\sum_{j=1}^{n} \tilde{\beta}_{nj}\right) v_n \end{bmatrix},$$

from which we have

$$\sum_{j=1}^{n} v_j \tilde{\beta}_{jk} = v_k \sum_{j=1}^{n} \tilde{\beta}_{kj}, \quad k = 1, 2, \ldots, n,$$

which is equivalent to

$$\sum_{j=1}^{n} v_j \beta_{jk} S_j^* I_k^* = v_k \sum_{j=1}^{n} \beta_{kj} S_k^* I_j^* = v_k(\mu_{k2} + \gamma_k) I_k^*, \quad k = 1, 2, \ldots, n.$$

Hence, by $I_k^* > 0$, we obtain that (4.10) has a positive solution $(v_1, v_2, \ldots, v_n)$ defined by (4.11). $\square$

We offer a key inequality in [19] to obtain our main result in this section.

Lemma 4.3.

$$\left(1 - \frac{1}{x_k}\right)(z_k - 1) + (z_k - 1)(1 - x_k) = \left(1 - \frac{1}{x_k}\right)(1 - x_k)(z_k - 1) = -\frac{(x_k - 1)^2}{x_k}(z_k - 1). \tag{4.15}$$

Now, for some positive constants $\tilde{c}_k,\ k = 1, 2, \ldots, n$, consider

$$\tilde{N}_k = S_k + I_k + \tilde{c}_k R_k, \quad \tilde{N}_k^* = S_k^* + I_k^* + \tilde{c}_k R_k^*, \text{ and } \tilde{n}_k = \frac{\tilde{N}_k}{\tilde{N}_k^*},\ k = 1, 2, \ldots, n.$$

Then,

$$\begin{cases} N_k - N_k^* = (\tilde{N}_k - \tilde{N}_k^*) - (\tilde{c}_k - 1)(R_k - R_k^*), \\ I_k - I_k^* = (N_k - N_k^*) - (S_k - S_k^*) - (R_k - R_k^*) = (\tilde{N}_k - \tilde{N}_k^*) - (S_k - S_k^*) - \tilde{c}_k(R_k - R_k^*). \end{cases} \tag{4.16}$$

Since by (1.1),

$$\begin{aligned} \frac{dR_k}{dt} &= \gamma_k I_k - (\mu_{k3} + \delta_k)R_k \\ &= \gamma_k(I_k - I_k^*) - (\mu_{k3} + \delta_k)(R_k - R_k^*), \\ &= \gamma_k\{(\tilde{N}_k - \tilde{N}_k^*) - (S_k - S_k^*) - \tilde{c}_k(R_k - R_k^*)\} - (\mu_{k3} + \delta_k)(R_k - R_k^*), \\ &= \gamma_k(\tilde{N}_k - \tilde{N}_k^*) - \gamma_k(S_k - S_k^*) - (\gamma_k\tilde{c}_k + \mu_{k3} + \delta_k)(R_k - R_k^*), \end{aligned}$$

and hence,

$$\begin{aligned} \frac{d}{dt}\left(\frac{(R_k - R_k^*)^2}{2}\right) &= (R_k - R_k^*)\frac{dR_k}{dt} \\ &= (R_k - R_k^*)\{\gamma_k(\tilde{N}_k - \tilde{N}_k^*) - \gamma_k(S_k - S_k^*) - (\gamma_k\tilde{c}_k + \mu_{k3} + \delta_k)(R_k - R_k^*)\} \\ &= \gamma_k R_k^*\tilde{N}_k^*(z_k - 1)(\tilde{n}_k - 1) - \gamma_k R_k^* S_k^*(z_k - 1)(x_k - 1) - (\gamma_k\tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2(z_k - 1)^2. \end{aligned} \tag{4.17}$$

Moreover, it holds that

$$\begin{aligned} \frac{d\tilde{N}_k}{dt} &= b - \mu_{k1}S_k - \{(\mu_{k2} + \gamma_k) - \tilde{c}_k\gamma_k\}I_k - \{\tilde{c}_k(\mu_{k3} + \delta_k) - \delta_k\}R_k \\ &= -\mu_{k1}(S_k - S_k^*) - \{(\mu_{k2} + \gamma_k) - \tilde{c}_k\gamma_k\}(I_k - I_k^*) - \{\tilde{c}_k(\mu_{k3} + \delta_k) - \delta_k\}(R_k - R_k^*), \\ &= -\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - \{(\mu_{k2} + \gamma_k - \mu_{k1}) - \mu_{k1} - \tilde{c}_k\gamma_k\}(I_k - I_k^*) - \{\tilde{c}_k(\mu_{k3} + \delta_k) - (\mu_{k1} + \delta_k)\}(R_k - R_k^*). \end{aligned}$$

Therefore, if we choose $\tilde{c}_k$ and $\tilde{\varepsilon}_k$ as

$$\tilde{c}_k = \frac{\mu_{k2} - \mu_{k1}}{\gamma_k} + 1 \geq 1,\ \tilde{\varepsilon}_k = \tilde{c}_k(\mu_{k3} + \delta_k) - (\mu_{k1} + \delta_k) \geq \mu_{k3} - \mu_{k1} \geq 0,\ k = 1, 2, \ldots, n, \tag{4.18}$$

then

$$\frac{d\tilde{N}_k}{dt} = -\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - \tilde{\varepsilon}_k(R_k - R_k^*), \tag{4.19}$$

and hence,

$$\begin{aligned} \frac{d}{dt}\left(\frac{(\tilde{N}_k - \tilde{N}_k^*)^2}{2}\right) &= (\tilde{N}_k - \tilde{N}_k^*)\frac{d\tilde{N}_k}{dt} \\ &= (\tilde{N}_k - \tilde{N}_k^*)\{-\mu_{k1}(\tilde{N}_k - \tilde{N}_k^*) - \tilde{\varepsilon}_k(R_k - R_k^*)\} \\ &= -\mu_{k1}(N_k^*)^2(\tilde{n}_k - 1)^2 - \tilde{\varepsilon}_k\tilde{N}_k R_k^*(\tilde{n}_k - 1)(z_k - 1). \end{aligned} \tag{4.20}$$

As a result, one can easily obtain the following lemma:

Lemma 4.4. *Under the condition* (1.3)*,*

$$\frac{d}{dt}\left(\frac{(R_k - R_k^*)^2}{2}\right) = -\gamma_k R_k^* S_k^*(z_k - 1)(x_k - 1) - (\gamma_k\tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2(z_k - 1)^2 + \gamma_k R_k^*\tilde{N}_k^*(z_k - 1)(\tilde{n}_k - 1). \tag{4.21}$$

Moreover, if we choose $\tilde{c}_k$ and $\tilde{\varepsilon}_k$ as (4.18)*, then*

$$\frac{d}{dt}\left(\frac{(\tilde{N}_k - \tilde{N}_k^*)^2}{2}\right) = -\mu_{k1}(N_k^*)^2(\tilde{n}_k - 1)^2 - \tilde{\varepsilon}_k\tilde{N}_k R_k^*(\tilde{n}_k - 1)(z_k - 1). \tag{4.22}$$

In paticular, if $\mu_{k1} = \mu_{k2} = \mu_{k3} = \mu$ for $k = 1, 2, \ldots, n$, then $\tilde{c}_k = 1$, $\tilde{\varepsilon}_k = 0$ and $\tilde{N}_k = N_k$ for $k = 1, 2, \ldots, n$. Hence, (4.22) holds. For $N_k^* = S_k^* + I_k^* + R_k^*$ and W_k defined by

$$W_k = \frac{(R_k - R_k^*)^2}{2} + \frac{1}{\mu}\left\{\frac{\gamma_k}{2(\mu + \gamma_k + \delta_k)}\right\}^2\frac{(N_k - N_k^*)^2}{2}, \tag{4.23}$$

it holds that

$$\frac{dW_k(t)}{dt} = -(\mu + \gamma_k + \delta_k)\left\{R_k^*(z_k - 1) - \frac{\gamma_k N_k^*}{2(\mu + \gamma_k + \delta_k)}(n_k - 1)\right\}^2 + \gamma_k R_k^* S_k^*(z_k - 1)(1 - x_k) \tag{4.24}$$

Otherwise, $\tilde{\varepsilon} > 0$ and for W_k defined by

$$W_k = \frac{(R_k - R_k^*)^2}{2} + \frac{\gamma_k}{\tilde{\varepsilon}} \frac{(\tilde{N}_k - \tilde{k}^*)^2}{2}, \tag{4.25}$$

it holds that

$$\frac{dW_k(t)}{dt} = -\gamma_k R_k^* S_k^* (z_k - 1)(x_k - 1) - (\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2 (z_k - 1)^2 - \frac{\gamma_k \mu_{k1}}{\tilde{\varepsilon}} (N_k^*)^2 (\tilde{n}_k - 1)^2. \tag{4.26}$$

Next, consider

$$U = U_1 + U_2, \text{ and } U_2 = \sum_{k=1}^{n} v_k \frac{\delta_k}{\gamma_k S_k^*} W_k. \tag{4.27}$$

Then, by Lemmas 4.1-4.4, one can easily obtain the following lemma:

Lemma 4.5. *Assume the conditions* (1.4) *and* $\tilde{R}_0 > 1$. *Then,*

$$\begin{aligned} \frac{dU(t)}{dt} =& -\sum_{k=1}^{n} v_k \{\mu_{k1} S_k^* + \delta R_k^* (z_t - 1)\} \left(1 - \frac{1}{x_k}\right)(x_k - 1) - \sum_{k=1}^{n} v_k \sum_{j=1}^{n} \left[\beta_{kj} S_k^* I_j^* \left\{ g\left(\frac{1}{x_k}\right) + g\left(\frac{x_k y_j}{y_k}\right)\right\}\right] \\ &+ \sum_{k=1}^{n} \left\{ \sum_{j=1}^{n} v_j \beta_{jk} S_j^* - v_k (\mu_{k2} + \gamma_k) \right\} I_k^* g(y_k) - \sum_{k=1}^{n} v_k \frac{\delta_k}{\gamma_k S_k^*} W_{k0}, \end{aligned} \tag{4.28}$$

where

$$W_{k0} = \begin{cases} (\mu + \gamma_k + \delta_k) \left\{ R_k^* (z_k - 1) - \dfrac{\gamma_k N_k^*}{2(\mu + \gamma_k + \delta_k)} (n_k - 1) \right\}^2, & \text{if } \mu_{k1} = \mu_{k2} = \mu_{k3} = \mu, \\ -(\gamma_k \tilde{c}_k + \mu_{k3} + \delta_k)(R_k^*)^2 (z_k - 1)^2 - \dfrac{\gamma_k \mu_{k1}}{\tilde{\varepsilon}} (N_k^*)^2 (\tilde{n}_k - 1)^2, & \text{otherwise.} \end{cases} \tag{4.29}$$

Proof of Theorem 1.1. If $\tilde{R}_0 \le 1$, then by Proposition 3.1, we can obtain the first part $\tilde{R}_0 \le 1$ of Theorem 1.1. We now consider the case $\tilde{R}_0 > 1$. Then, by Proposition 3.1, system (1.1) is uniformly persistent in $\mathbf{\Gamma}^0$, and by Corollary 3.1, there exists at least one endemic equilibrium $\mathbf{E}^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$.

By Lemma 4.2, we have that there exists a positive n column vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ such that (4.10) holds. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be chosen as in (4.10) and suppose that (1.9) hold. Then, for (4.28), it holds (4.9) and

$$\mu_{k1} S_k^* + \delta R_k^* (z_t - 1) \ge \mu_{k1} S_k^* - \delta R_k^* \ge 0,$$

and hence, we obtain $\frac{dU(t)}{dt} \le 0$. Moreover, $\frac{dU(t)}{dt} = 0$ if and only if

$$x_k = 1, \ y_k = 1, \text{ and } z_k = 1 \text{ for any } t > 0, \ j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, n. \tag{4.30}$$

Therefore, the only compact invariant subset where $\frac{dU(t)}{dt} = 0$ is the singleton $\{\mathbf{E}^*\}$. By Proposition 3.1 and a similar argument as in Section 3, $\mathbf{E}^*$ is globally asymptotically stable in $\mathbf{\Gamma}^0$, if $\tilde{R}_0 > 1$. Hence, the proof is complete. □

Acknowledgments

The authors wish to express their gratitude to an anonymous referee and also the editor for their helpful comments which improve the quality of the paper in the present style. The first author 's research was supported by Scientific Research (c), No. 24540219 of Japan Society for the Promotion of Science. The second author 's research was supported by JSPS Fellows, No. 237213 of Japan Society for the Promotion of Science. The third author 's research was supported by JSPS Fellows, No. 222176 of Japan Society for the Promotion of Science.

References

[1] J. Arino, Disease in metapopulations in: Z. Ma, Y. Zhou and J. Wu (eds), Modeling and Dynamics of Infectious Diseases, Higher Education Press, Beijing (2009) 65-123.

[2] N. P. Bhatia and G. P. Szegö, Dynamical Systems: Stability Theory and Applications, *Lecture Notes in Mathematics* **35**, Springer, Berlin, 1967.

[3] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.

[4] H. Chen and J. Sun, Global stability of delay multigroup epidemic models with group mixing nonlinear incidence rates, *Appl. Math. Comput.* **218** (2011) 4391-4400.

[5] O. Diekmann and J. A. P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases, Wiley, Chichester, UK, 2000.

[6] Y. Enatsu, Y. Nakata and Y. Muroya, Lyapunov functional techniques for the global stability analysis of a delayed SIRS epidemic model, *Nonlinear Analysis RWA* **13** (2012) 2120-2133.

[7] H. I. Freedman, M. X. Tang and S. G. Ruan, Uniform persistence and flows near a closed positively invariant set, *J. Dynam. Diff. Equat.* **6** (1994) 583-600.

[8] H. Guo, M. Y. Li and Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, *Canadian Appl. Math. Quart.* **14** (2006) 259-284.

[9] H. Guo, M. Y. Li and Z. Shuai, A graph-theoretic approach to the method of global Lyapunov functions, *Proc. Amer. Math. Soc.* **136** (2008) 2793-2802.

[10] J. P. LaSalle, The stability of dynamical systems, *Regional Conference Series in Applied Mathematics*, SIAM, Philadelphia, 1976.

[11] M. Y. Li, J. R. Graef, L. Wang and J. Karsai, Global dynamics of a SEIR model with varying total population size, *Math. Biosci.* **160** (1999) 191-213.

[12] M. Y. Li and Z. Shuai, Global-stability problem for coupled systems of differential equations on networks, *J. Diff. Equat.* **284** (2010) 1-20.

[13] M. Y. Li, Z. Shuai and C. Wang, Global stability of multi-group epidemic models with distributed delays, *J. Math. Anal. Appl.* **361** (2010) 38-47.

[14] J. Liu and Y. Zhou, Global stability of an SIRS epidemic model with transport-related infection, *Chaos Solitons and Fractals* **40** (2009) 145-158.

[15] X. Liu and Y. Takeuchi, Spread of disease with transport-related infection, *J. Theor. Biol.* **242** (2006) 517-528

[16] C. C. McCluskey, Complete global stability for an SIR epidemic model with delay-Distributed or discrete, *Nonlinear Analysis RWA* **11** (2010) 55-59.

[17] Y. Muroya, Y. Enatsu and T. Kuniya, Global stability for a class of multi-group SIR epidemic models with patches through migration and cross patch infection, *Acta Mathematica Scientia* **33** (2013) 341-361.

[18] Y. Nakata, On the global stability of a delayed epidemic model with transport-related infection, *Nonlinear Analysis RWA* **12** (2011) 3028-3034.

[19] Y. Nakata, Y. Enatsu and Y. Muroya, On the global stability of an SIRS epidemic model with distributed delays, *Disc. Cont. Dyn. Sys. Supplement* (2011) 1119-1128.

[20] H. L. Smith and P. Waltman, The Theory of the Chemostat: Dynamics of Microbial Competition, Cambridge University Press, Cambridge, 1995.

[21] H. Shu, D. Fan and J. Wei, Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission, *Nonlinear Analysis RWA* **13** (2012) 1581-1592.

[22] R. Sun, Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence, *Comput. Math. Appl.* **60** (2010) 2286-2291.

[23] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, *Math. Biosci.* **180** (2002) 29-48.

[24] R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, 1962.

[25] W. Wang and X. Zhao, An epidemic model in a patchy environment, *Math. Biosci.* **190** (2004) 97-112.

[26] Z. Yuan and L. Wang, Global stability epidemiological models with group mixing and nonlinear incidence rates, *Nonlinear Analysis RWA.* **11** (2010) 995-1004.

[27] Z. Yuan and X. Zou, Global threshold property in an epidemic models for disease with latency spreading in a heterogeneous host population, *Nonlinear Analysis RWA.* **11** (2010) 3479-3490.

Appendix

Proof of Proposition 3.1. Let $\mathbf{S} = (S_1, S_2, \cdots, S_n)^T$ and $\mathbf{S}^0 = (S_1^0, S_2^0, \cdots, S_n^0)^T$, and put

$$\tilde{\mathbf{M}}(\mathbf{S}) = \left(\frac{\beta_{kj} S_k}{\mu_k + \gamma_k}\right)_{n\times n}. \tag{A.1}$$

Since in $\mathbf{\Gamma}$, it holds that $0 \le S_k \le S_k^0$ for $1 \le k \le n$ and $\emptyset \le \tilde{\mathbf{M}}(\mathbf{S}) \le \tilde{\mathbf{M}}(\mathbf{S}^0)$. Since $\mathbf{B}$ is irreducible, we know $\tilde{\mathbf{M}}(\mathbf{S})$ and $\tilde{\mathbf{M}}(\mathbf{S}^0)$ are irreducible. Therefore, $\rho(\tilde{\mathbf{M}}(\mathbf{S})) < \rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$, provided $\mathbf{S} \ne \mathbf{S}^0$ (see, for example, [24, Lemma 2.3]).

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) \le 1$, then for $\mathbf{S} \ne \mathbf{S}^0$, by the above, $\rho(\tilde{\mathbf{M}}(\mathbf{S})) < 1$, and

$$\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} = \mathbf{I}$$

has only the trivial solution $\mathbf{I} = \mathbf{0}$. Thus, $\mathbf{E}^0$ is then only equilibrium of system (1.1) in $\mathbf{\Gamma}$ if $\tilde{R}_0 \le 1$.

Let $(\omega_1, \omega_2, \ldots, \omega_n)$ be a left eigenvector of $\tilde{\mathbf{M}}(\mathbf{S}^0)$ corresponding to $\rho(\tilde{\mathbf{M}}(\mathbf{S}^0))$, i.e.,

$$(\omega_1, \omega_2, \ldots, \omega_n)\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) = (\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0).$$

Since $\tilde{\mathbf{M}}(\mathbf{S}^0)$ is irreducible, we know $\omega_k > 0$ for $k = 1, 2, \ldots, n$. Set

$$L = (\omega_1, \omega_2, \ldots, \omega_n)\begin{bmatrix} \mu_1+\gamma & 0 & \cdots & 0 \\ 0 & \mu_2+\gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n+\gamma_n \end{bmatrix}^{-1}\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix}.$$

Differentiations gives

$$\begin{aligned} L' &= (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} - \mathbf{I}] \le (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S}^0)\mathbf{I} - \mathbf{I}] \\ &= \{\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1\}(\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I} \le 0, \quad \text{if } \tilde{R}_0 \le 1. \end{aligned}$$

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) < 1$, then $L' = 0 \iff \mathbf{I} = \mathbf{0}$. If $\tilde{R}_0 = 1$, then $L' = 0$ implies

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} = (\omega_1, \omega_2, \ldots, \omega_n)\mathbf{I}. \tag{A.2}$$

If $\mathbf{S} \ne \mathbf{S}^0$, then

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}) < (\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0) = (\omega_1, \omega_2, \ldots, \omega_n).$$

Thus, (A.2) has only the trivial solution $\mathbf{I} = \mathbf{0}$. Therefore, $L' = 0 \iff I = \mathbf{0}$ or $\mathbf{S} = \mathbf{S}^0$ provided $\tilde{R}_0 \le 1$. It can be verified that the only compact invariant subset of the set , where $L' = 0$ is the singleton $\{\mathbf{E}^0\}$. By LaSalle's Invariance Principle (see [10]), $\mathbf{E}^0$ is globally asymptotically stable in $\mathbf{\Gamma}$ if $\tilde{R}_0 \le 1$.

If $\tilde{R}_0 = \rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) > 1$ and $\mathbf{I} \ne \mathbf{0}$, we know that

$$(\omega_1, \omega_2, \ldots, \omega_n)\tilde{\mathbf{M}}(\mathbf{S}^0) - (\omega_1, \omega_2, \ldots, \omega_n) = \{\rho(\tilde{\mathbf{M}}(\mathbf{S}^0)) - 1\}(\omega_1, \omega_2, \ldots, \omega_n) > 0.$$

and thus $L' = (\omega_1, \omega_2, \ldots, \omega_n)[\tilde{\mathbf{M}}(\mathbf{S})\mathbf{I} - \mathbf{I}] > 0$ in a neighborhood of $\mathbf{E}^0$ in $\mathbf{\Gamma}^0$, by continuity. This implies $\mathbf{E}^0$ is unstable.

Using a uniform persistence result from Freedman *et al.* [7] and a similar argument as in the proof of Li *et al.* [11, Proposition 3.3], we can show that, when $\tilde{R}_0 > 1$, the instability of $\mathbf{E}^0$ implies the uniform persistence of (1.1). This completes the proof of Proposition 3.1. □

Proof of Lemma 3.1. Since

$$S_k^*\left(\sum_{j=1}^n \beta_{kj} I_j^*\right) - (\mu_k + \gamma_k)I_k^* = 0, \quad k = 1, 2, \ldots, n, \tag{A.3}$$

we have

$$\begin{cases} \left(\dfrac{\beta_{kj} S_k^*}{\mu_j + \gamma_j}\right)_{n\times n}((\mu_1+\gamma_1)I_1^*, (\mu_2+\gamma_2)I_2^*, \ldots, (\mu_n+\gamma_n)I_n^*)^T = ((\mu_1+\gamma_1)I_1^*, (\mu_2+\gamma_2)I_2^*, \ldots, (\mu_n+\gamma_n)I_n^*)^T, \\ \left(\dfrac{\beta_{kj} S_k^*}{\mu_k + \gamma_k}\right)_{n\times n}(I_1^*, I_2^*, \ldots, I_n^*)^T = (I_1^*, I_2^*, \ldots, I_n^*)^T, \end{cases} \tag{A.4}$$

from which we obtain

$$\rho\left(\left(\frac{\beta_{kj} S_k^*}{\mu_j + \gamma_j}\right)_{n\times n}\right) = \rho\left(\left(\frac{\beta_{kj} S_k^*}{\mu_k + \gamma_k}\right)_{n\times n}\right) = 1, \tag{A.5}$$

that is,

$$\rho(\mathbf{M}(\mathbf{S}^*)) = \rho(\tilde{\mathbf{M}}(\mathbf{S}^*)) = 1, \tag{A.6}$$

where for $\mathbf{S} = (S_1, S_2, \ldots, S_n)$ and $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{V}}$ defined by (3.2), we set

$$\mathbf{M}(\mathbf{S}) = \tilde{\mathbf{F}}(\mathbf{S})\tilde{\mathbf{V}}^{-1} = \left(\frac{\beta_{kj} S_k}{\mu_j + \gamma_j}\right)_{n\times n} \text{ and } \tilde{\mathbf{M}}(\mathbf{S}) = \tilde{\mathbf{V}}^{-1}\tilde{\mathbf{F}}(\mathbf{S}) = \left(\frac{\beta_{kj} S_k}{\mu_k + \gamma_k}\right)_{n\times n}. \tag{A.7}$$

Hence by (1.5), (3.4), Lemma 2.1 and the above discussions in the first part of proof of Proposition 3.1 on irreducible non-negative matrices theory (see for example, Varga [24, Chapter 2]), we can easily obtain (3.6). □