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Permanence for Lotka-Volterra systems with multiple delays

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Abstract. In this paper, we consider a class of several species Lotka.Volterra systems with time delays and establish sufficient conditions which ensure the systems to be permanent. We improve and extend the known conditions of the permanence in Lu and Lu [4], Lu et al. [5] and Nakata and Muroya [7]. Moreover, we give the conditions of the permanence for a class of three species Lotka-Volterra predator-prey systems which need no restriction on the size of time delays and improve the result in [2]. Some examples for comparison with the previous results are given to illustrate the main results.

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1 Introduction

Many authors have since studied the dynamical behavior of some ecological models governed by functional (ordinary) differential equations (see, e.g., [1–10] and the references therein). The way of interactions between species is also varied depending on situations. For instance, a species may eat others, may be eaten by others and may compete or cooperate with others. From the biological aspects, under the above circumstances, it is quite important to obtain the condition which ensures coexistence of all species in multispecies communities.

In this paper, we consider the following n-dimensional Lotka-Volterra system with multiple delays:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)\left[r_1 - \displaystyle\sum_{\substack{j=1\\ j\neq 2}}^{n}\sum_{l=0}^{m} a_{1j}^l x_j(t-l\tau) + a_{12}^1 x_2(t-\tau)\right], \\ \dfrac{dx_i(t)}{dt} = x_i(t)\left[r_i - \displaystyle\sum_{\substack{j=1\\ j\neq i+1}}^{n}\sum_{l=0}^{m} a_{ij}^l x_j(t-l\tau) + \sum_{l=0}^{m} a_{i\,i+1}^l x_{i+1}(t-l\tau)\right],\ i=2,\dots,n-1, \\ \dfrac{dx_n(t)}{dt} = x_n(t)\left[r_n - \displaystyle\sum_{j=2}^{n}\sum_{l=0}^{m} a_{nj}^l x_j(t-l\tau) + \sum_{l=0}^{m} a_{n1}^l x_1(t-l\tau)\right],\ t\geq 0, \end{cases} \tag{1.1}$$

with initial conditions

$$x_i(\theta) = \phi_i(\theta) \geq 0,\ \ \theta\in[-m\tau,0],\ \ \phi_i(0)>0,\ 1,2,\dots,n, \tag{1.2}$$

where $\tau \geq 0$, each r_i, a_i and a_{ij} are constants with ϕ is continuous on $[-m\tau, 0]$. It is said that system (1.1) is permanent if there is a compact set K in the interior of $\mathbb{R}^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_i > 0, i = 1, 2, \dots, n\}$ such that all the

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solutions $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of system (1.1) with initial conditions (1.2) ultimately enter K. That is, there exists m and M for any solutions $x(t)$ such that

$$0 < m \leq \liminf_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_i(t) \leq M < +\infty.$$

To examine the population dynamics of the ecological systems composed of such a variety of species, competitive or prey-predator Lotka-Volterra systems have been widely discussed in the literature. For example, Ahmad and Lazer [1] have established the average conditions for persistence on the nonautonomous Lotka-Volterra competitive systems with no delays and Xu and Chen [10] have studied the delayed nonautonomous three species Lotka-Volterra predator-prey systems without dominating instantaneous negative feedback. For two species Lotka-Volterra predator-prey and competitive systems with dominating instantaneous negative feedback, Wang and Ma [9], Lu and Takeuchi [6] obtained that delays are harmless for the permanence. Recently, for the following three species Lotka-Volterra predator-prey system:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_{11}x_1(t) + a_{12}x_2(t-\tau) - a_{13}x_3(t-2\tau)], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_{21}x_1(t) - a_{22}x_2(t) + a_{23}x_2(t-\tau)], \\ \dfrac{dx_3(t)}{dt} = x_3(t)[r_3 + a_{31}x_1(t) - a_{32}x_2(t-\tau) - a_{33}x_3(t-2\tau)], \end{cases} \tag{1.3}$$

where $r_i > 0$, $a_{ij} > 0$ $1 \leq i, j \leq 3$ and $\tau \geq 0$ are constants, Enatsu [2, Corollary 2.1] has obtained the following result depending on the length of the delays:

Theorem 1.1. *System* (1.3) *is permanent if* $a_{13} \geq a_{23}$, $a_{22} > a_{12}$ *and* $r_1 - a_{13}\hat{M}_3 > 0$, $r_2 - a_{21}\hat{M}_1 > 0$, $r_3 - a_{32}\hat{M}_2 > 0$, *where*

$$\hat{M}_1 = -\frac{a_{12}\hat{P}}{r_1} + \Big\{\frac{a_{12}\hat{P}}{r_1} + \frac{1}{a_{11}}\Big(r_1 + \frac{a_{12}\hat{P}}{\hat{x}_1^*}\Big)\Big\}e^{2r_1\tau}, \ \hat{M}_2 = \frac{r_2 + a_{23}\hat{M}_3}{a_{22}}, \ \hat{M}_3 = \frac{r_3 + a_{31}\hat{M}_1}{a_{33}}e^{2(r_3 + a_{31}\hat{M}_1)\tau},$$

$$\hat{m}_1 = \frac{r_1 - a_{13}\hat{M}_3}{a_{11}}e^{2(r_1 - a_{13}\hat{M}_3 - a_{11}\hat{M}_1)\tau}, \ \hat{m}_2 = \frac{r_2 - a_{21}\hat{M}_1}{a_{22}}, \ \hat{m}_3 = \frac{r_3 - a_{32}\hat{M}_2}{a_{33}^2}e^{2(r_3 - a_{32}\hat{M}_2 - a_{33}\hat{M}_3)\tau},$$

and $x = \hat{x}_1^*$ *is a unique positive solution of the following equation:*

$$x(r_1 - a_{11}x) + a_{12}\hat{P} = 0, \ \ \hat{P} = \frac{(r_1 + r_2)^2}{4a_{11}(a_{22} - a_{12})} > 0.$$

On the other hand, some authors have recently argued that cooperation is also an important interaction among species, which is commonly seen in social animals and in human society. In addition to the above statements, in the real system, the feedback of interspecific interactions and intraspecific competitions on the population dynamics are generally delayed.

However, there are few papers concerning multispecies Lotka-Volterra cooperative systems with delays to compare with competitive and prey-predator systems. Lin and Lu [3] consider the following two species Lotka-Volterra cooperative system with delays and obtain sufficient conditions which ensure the system to be permanent:

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)(r_1 - a_1x_1(t) - a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12})), \\ \dfrac{dx_2(t)}{dt} = x_2(t)(r_2 - a_2x_2(t) + a_{21}x_1(t-\tau_{21}) - a_{22}x_2(t-\tau_{22})), \end{cases} \tag{1.4}$$

where $a_i > 0$, $a_{ij} > 0$, $\tau_{ij} \geq 0$ and $r_i > 0$ $(i, j = 1, 2)$ are constants.

Theorem 1.2. *System* (1.4) *is permanent if* $r_i > 0$ *and* $a_1a_2 - a_{12}a_{21} > 0$.

On the other hand, Lu and Lu [4] have investigated the permanence for the two species case of system (1.1), namely,

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)(r_1 - a_{11}^1x_1(t-\tau) - a_{11}^2x_1(t-2\tau) + a_{12}^1x_2(t-\tau)), \\ \dfrac{dx_2(t)}{dt} = x_2(t)(r_2 + a_{21}^0x_1(t) + a_{21}^1x_1(t-\tau) - a_{22}^0x_2(t) - a_{22}^1x_2(t-\tau)), \end{cases} \tag{1.5}$$

where $a_{ij} > 0$, $r_i > 0$ $(i, j = 1, 2)$ and $\tau \geq 0$ are constants. The following result is obtained in [4, Theorem 1.3].

Theorem 1.3. *Let* $a_{21}^0 = 0$. *Then system* (1.5) *is permanent if there exist constants* $C_i > 0$, $D_i \geq 0$ *such that* $\frac{dx_i(t)}{dt} \leq C_ix_i(t) + D_i$ $(i = 1, 2)$.

$$\left\{a_{11}^2(1 - 2r_1\tau) + a_{11}^1(1 - r_1\tau)\right\}a_{22}^0 - a_{12}^1a_{21}^1 > 0. \tag{1.6}$$

From Theorem 1.3, the delays may harm the permanence for Lotka-Volterra cooperative systems. For the similar analysis of delay effect, Lu et al. [5] obtained the following result:

Theorem 1.4. *For system* (1.4)*, let* $\tau_{21} = 0$ *and* $\tau_{12} = \tau_{22} \geq 0$*. Then system* (1.4) *is permanent if* $r_i > 0$ $(i = 1, 2)$*,* $a_1 > a_{21}$ *and* $a_{22} > a_{12}$*.*

Recently, Nakata and Muroya [7] establish new sufficient conditions for system (1.5) to be permanent. Remarkably, their conditions no longer depend on the size of time delays. They obtain the following result (see [8, Corollary 1.2]):

Theorem 1.5. *System* (1.5) *is permanent if*

$$a_{11}^1 > a_{21}^0, \ a_{11}^2 > a_{21}^1, \ a_{22}^0 > a_{12}^1. \tag{1.7}$$

Later, Enatsu [2] extended their result for a n-dimensional system, in which the following result is also obtained.

Theorem 1.6. *System* (1.5) *is permanent if*

$$a_{11}^1 > a_{21}^0, \ a_{11}^2 \geq a_{21}^1, \ a_{22}^0 \geq a_{12}^1. \tag{1.8}$$

In the present paper, by using techniques of Nakata and Muroya [7] and a boundary Lyapunov functional method in [6,9], we give the improved permanence conditions for systems (1.3)-(1.5). The main results are as follows:

Theorem 1.7. *System* (1.5) *is permanent if* $r_1 a_{12}^0 + r_2 a_{11}^1 > 0$*,* $r_1 a_{21}^1 + r_2 a_{11}^2 > 0$*,* $a_{11}^1 a_{22}^0 > a_{21}^0 a_{12}^1$ *and* $a_{11}^2 a_{22}^0 > a_{21}^1 a_{12}^1$*.*

Moreover, we can extend the above techniques to the three species case for system (1.3) as follows.

Theorem 1.8. *System* (1.3) *is permanent if* $a_{13}a_{22} > a_{23}a_{12}$*,* $r_2 a_{11} - r_1 a_{21} > 0$*,* $r_3 a_{22} - r_2 a_{32} > 0$ *and* $r_1 A_{11} + r_2 A_{12} + r_3 A_{13} > 0$ *hold. Here,* $A_{11} = a_{22}a_{33} + a_{23}a_{32}$*,* $A_{12} = a_{12}a_{33} + a_{32}a_{13}$ *and* $A_{13} = a_{12}a_{23} - a_{13}a_{22}$*.*

Remark 1.1. Theorem 1.7 improves Theorems 1.3, 1.5 and 1.6 since our permanence conditions are valid if either of the following three cases hold:

$$\text{i) } a_{11}^2 < a_{23}^1, \ \text{ii) } a_{22}^0 < a_{12}^1, \ \text{iii) } r_2 < 0.$$

In addition, Theorem 1.8 generalizes the results by Enatsu [2, Corollary 2.1] for three species and our conditions also do not need the restriction on the size of time delay τ.

2 Preliminaries and some lemmas

At first, we introduce some basic lemmas. In particular, Lemma 2.2 plays an important role for illustrating the permanence of the cooperative population system.

Lemma 2.1. *Every solution of system* (1.1) *with initial conditions exists in the interval* $[0, +\infty)$ *and remains positive for all* $t \geq 0$*.*

The following result is obtained in [8].

Lemma 2.2. *Consider the following inequality;*

$$\frac{du(t)}{dt} \leq u(t)[a - bu^\alpha(t - \tau)] + D$$

with initial conditions $u(t) = \varphi(t)$ *for* $t \in [-\tau, 0]$ *and* $\varphi(0) > 0$*, where* $a > 0$*,* $b > 0$*,* $\alpha > 0$ *and* $D \geq 0$ *are constants. Then there exists a positive* $M_u < +\infty$ *such that*

$$\limsup_{t \to +\infty} u(t) \leq M_u \equiv -\frac{D}{a} + \left(\frac{D}{a} + u^*\right) \mathrm{e}^{a\tau} > 0,$$

where $u = u^*$ *is a unique positive solution of* $u(a - bu^\alpha) + D = 0$*.*

Lemma 2.3. *For system* (1.5)*, assume that* $r_1 a_{12}^0 + r_2 a_{11}^1 > 0$*,* $r_1 a_{21}^1 + r_2 a_{11}^2 > 0$*,* $a_{11}^1 a_{22}^0 > a_{21}^0 a_{12}^1$ *and* $a_{11}^2 a_{22}^0 > a_{21}^1 a_{12}^1$ *hold. Then there exists a positive constant* α *such that* $\max\{a_{21}^0 / a_{11}^1, a_{21}^1 / a_{11}^2\} < \alpha < a_{22}^0 / a_{12}^1$ *and*

$$\limsup_{t \to +\infty} x_1(t) x_2(t - \tau) \leq N \equiv \frac{(r_1 \alpha)^{\alpha+1}}{(\alpha a_{11}^1 - a_{21}^0)^\alpha a_{22}^0} \mathrm{e}^{(r_1 \alpha + r_2)\tau} < +\infty, \tag{2.1}$$

Proof. Since $a_{11}^1 a_{22}^0 > a_{21}^0 a_{12}^1$ and $a_{11}^2 a_{22}^0 > a_{21}^1 a_{12}^1$, it is clear that there exists a constant α with $\max\{a_{21}^0/a_{11}^1, a_{21}^1/a_{11}^2\} < \alpha < a_{22}^0/a_{12}^1$. Furthermore, from $r_1 a_{12}^0 + r_2 a_{11}^1 > 0$ and $r_1 a_{21}^1 + r_2 a_{11}^2 > 0$, we have $r_1\alpha + r_2 > 0$ and $\alpha a_{11}^1 - a_{21}^0 > 0$.

In order to show (2.1), we first suppose that $\limsup_{t\to+\infty} x_1^\alpha(t)x_2(t-\tau) = +\infty$. Then there exists a subsequence $\{t_k\}_{k=1}^{+\infty}$ such that

$$\lim_{t_k\to+\infty} x_1^\alpha(t_k)x_2(t_k-\tau) = +\infty, \text{ and } \frac{d}{dt}x_1^\alpha(t)x_2(t-\tau)|_{t=t_k} \geq 0, \; k = 0, 1, 2, \ldots.$$

From (1.5), we obtain

$$\begin{aligned}\frac{d}{dt}x_1^\alpha(t)x_2(t-\tau) &= x_1^\alpha(t)x_2(t-\tau)[r_1\alpha + r_2 - (\alpha a_{11}^1 - a_{21}^0)x_1(t-\tau) - (\alpha a_{11}^2 - a_{21}^1)x_1(t-2\tau) \\ &\quad - (a_{22}^0 - \alpha a_{12}^1)x_2(t-\tau) - a_{22}^0 x_2(t-2\tau)] \\ &= x_1^\alpha(t)x_2(t-\tau)[r_1\alpha + r_2 - (\alpha a_{11}^1 - a_{21}^0)x_1(t-\tau) - a_{22}^0 x_1(t-2\tau)]. \end{aligned} \tag{2.2}$$

From (2.2), it follows that

$$(\alpha a_{11}^1 - a_{21}^0)x_1(t_k - tau) - a_{22}^0 x_1(t_k - 2\tau) \leq r_1\alpha + r_2.$$

Thus, we get $x_1(t_k-\tau) \leq \frac{r_1\alpha+r_2}{\alpha a_{11}^1 - a_{21}^0}$ and $x_2(t_k - 2\tau) \leq \frac{r_1\alpha+r_2}{a_{22}^0}$. By integrating both sides of (2.2) from $t_k-\tau$ to t_k, we obtain

$$x_1(t_k)x_2(t_k-\tau) \leq x_1(t_k-\tau)x_2(t_k-2\tau)\mathrm{e}^{(r_1\alpha+r_2)\tau} < +\infty.$$

This leads to a contradiction. Thus, we have $\limsup_{t\to+\infty} x_1(t)x_2(t-\tau) < +\infty$. Similar to the above discussion, we obtain (2.1). The proof is complete. $\square$

Lemma 2.4. *For system* (1.5)*, assume that* $r_1 a_{12}^0 + r_2 a_{11}^1 > 0$, $r_1 a_{21}^1 + r_2 a_{11}^2 > 0$, $a_{11}^1 a_{22}^0 > a_{21}^0 a_{12}^1$ *and* $a_{11}^2 a_{22}^0 > a_{21}^1 a_{12}^1$ *hold. Then it holds that*

$$\begin{aligned}\limsup_{t\to+\infty} x_1(t) &\leq M_1 \equiv \left[-\frac{a_{12}^1 N}{r_1} + \left(-\frac{a_{12}^1 N}{r_1} + x^*\right)\mathrm{e}^{\alpha r_1\tau}\right]^{\frac{1}{\alpha}} < +\infty, \\ \limsup_{t\to+\infty} x_2(t) &\leq M_2 \equiv \frac{r_2 + (a_{21}^0 + a_{21}^1)M_1}{a_{22}^0} < +\infty,\end{aligned}$$

where $x = x^*$ *is the unique positive solution of* $x(r_1 - a_{11}^1 x^{\frac{1}{\alpha}}) + a_{12}^1 N = 0$.

Proof. At first, we show that $x_1(t)$ is bounded above. From Lemma 2.3, for any positive constant $\varepsilon_1 > 0$, there exists a positive constant T_1 such that $x_1(t)x_2(t-\tau) \leq N + \varepsilon_1$ for $t > T_1$. For the functional $V(t) := x_1^\alpha(t)$, we have

$$\begin{aligned}\frac{dV(t)}{dt} &= \alpha x_1^\alpha(t)[r_1 - a_{11}^1 x_1(t-\tau) - a_{11}^2 x_1(t-2\tau) + a_{12}^1 x_2(t-\tau)] \\ &= \alpha x_1^\alpha(t)[r_1 - a_{11}^1 x_1(t-\tau) - a_{11}^2 x_1(t-2\tau)] + \alpha a_{12}^1(N+\varepsilon_1) \\ &= \alpha V(t)[r_1 - a_{11}^1 V^{\frac{1}{\alpha}}(t-\tau)] + \alpha a_{12}^1(N+\varepsilon_1).\end{aligned}$$

Since ε_1 is arbitrarily chosen, by Lemma 2.2, we obtain

$$\limsup_{t\to+\infty} V(t) \leq -\frac{a_{12}^1 N}{r_1} + \left(-\frac{a_{12}^1 N}{r_1} + x^*\right)\mathrm{e}^{\alpha r_1\tau},$$

from which we obtain $\limsup_{t\to+\infty} x_1(t) \leq M_1$.

Next, we show that $x_2(t)$ is ultimately bounded. For any positive constant $\varepsilon_2 > 0$, there exists a positive constant T_2 such that $x_1(t) \leq M_1 + \varepsilon_2$ for $t > T_2$. Therefore, we have $\limsup_{t\to+\infty} x_2(t) \leq M_2$. $\square$

Similar to Lemmas 2.3 and 2.4, we have

Lemma 2.5. *For system* (1.3)*, assume that* $a_{13}a_{22} > a_{23}a_{12}$ *holds. Then there exist positive constants* α_1 *such that* $a_{23}/a_{13} < \alpha_1 < a_{22}/a_{12}$ *and*

$$\limsup_{t\to+\infty} x_1^{\alpha_1}(t)x_2(t-\tau) \leq N_1.$$

Lemma 2.6. *For system* (1.3)*, if* $a_{13}a_{22} > a_{23}a_{12}$ *then it holds that*

$$\limsup_{t\to+\infty} x_1(t) \le \tilde{M}_1 \equiv \left[-\frac{a_{12}N_1}{r_1} + \left(-\frac{a_{12}N_1}{r_1} + x_1^*\right)\mathrm{e}^{\alpha_1 r_1 \tau}\right]^{\frac{1}{\alpha_1}} < +\infty,$$
$$\limsup_{t\to+\infty} x_3(t) \le \tilde{M}_3 \equiv \frac{r_3 + a_{31}\tilde{M}_1}{a_{33}}\mathrm{e}^{2(r_3+a_{31}\tilde{M}_1)\tau} < +\infty,$$
$$\limsup_{t\to+\infty} x_2(t) \le \tilde{M}_2 \equiv \frac{r_2 + a_{23}\tilde{M}_3}{a_{22}} < +\infty,$$

where $x = x_1^*$ *is the unique positive solution of* $x(r_1 - a_{11}x^{\frac{1}{\alpha_1}}) + a_{12}N_1 = 0$.

3 Proof of Theorem 1.7

In this section, we give a proof of Theorem 1.7.

Proof of Theorem 1.7. From Lemma 2.4, there are two positive constants M_1 and M_2 such that for sufficiently large t, any solution of (1.5) satisfies $0 < x_i(t) \le M_i$ $(i = 1, 2)$. We now consider the following functionals:

$$\begin{cases} V_1(t) = x_1(t)\exp\Big\{-a_{11}^1\int_{t-\tau}^t x_1(s)ds - a_{11}^2\int_{t-2\tau}^t x_1(s)ds\Big\} \\ V_2(t) = (x_1(t))^{a_{21}^0+a_{21}^1}(x_2(t))^{a_{11}^1+a_{11}^2}\exp\Big\{(a_{21}^1a_{11}^2 - a_{11}^1a_{21}^0)\int_{t-\tau}^t x_1(s)ds \\ \qquad -a_{11}^2(a_{21}^0+a_{21}^1)\int_{t-2\tau}^t x_1(s)ds + (a_{12}^1(a_{21}^0+a_{21}^1) - a_{22}^1(a_{11}^1+a_{11}^2))\int_{t-\tau}^t x_2(s)ds\Big\}. \end{cases} \tag{3.3}$$

Then we have

$$V_1'(t) \ge V_1(t)(\delta_1 - \Delta_1 x_1(t)),\ V_2'(t) = V_2(t)(\delta_2 - \Delta_2 x_2(t)),$$

where $\delta_1 = r_1$, $\Delta_1 = a_{11}^1 + a_{11}^2$, $\delta_2 = (a_{11}^1 + a_{11}^2)r_2 + (a_{21}^0 + a_{21}^1)r_1$ and $\Delta_2 = (a_{11}^1 + a_{11}^2)(a_{22}^0 + a_{22}^1) - a_{12}^1(a_{21}^0 + a_{21}^1)$. Let us fix $0 < h_i < \frac{\delta_i}{2\Delta_i}$ for $(i = 1, 2)$. If $x_i(t) \le h_i$ holds for some i, then we have

$$\frac{dV_i(t)}{dt} \ge \frac{\delta_i}{2}V_i(t), \tag{3.4}$$

from which we obtain

$$\underline{m}_1 x_1(t) \le V_1(t) \le x_1(t), \tag{3.5}$$
$$\underline{m}_2(x_1(t))^{a_{21}^0+a_{21}^1}(x_2(t))^{a_{11}^1+a_{11}^2} \le V_2(t) \le \overline{m}_2(x_1(t))^{a_{21}^0+a_{21}^1}(x_2(t))^{a_{11}^1+a_{11}^2}, \tag{3.6}$$

where

$$\underline{m}_1 = \exp[-(a_{11}^1 + 2a_{11}^2)M_1\tau],$$
$$\underline{m}_2 = \exp[-(a_{11}^1a_{21}^0 + 2a_{11}^2a_{12}^0 + 2a_{11}^2a_{21}^0)M_1\tau - (a_{11}^1a_{21}^0 + a_{11}^2a_{22}^1)],$$
$$\underline{m}_3 = \exp[a_{21}^1a_{11}^2M_1\tau + a_{12}^1(a_{21}^0 + a_{21}^1)M_2\tau].$$

We now consider the two curves l_1 and l_2 in the region $\{X \in \mathbb{R}_{+0}^2 | X_i \le M_i, i = 1, 2\}$ defined by

$$l_1 : X_1 = \lambda h_1,$$

with $\lambda \le \underline{m}_1$. Moreover, we suppose that the intersection point of curve l_1 with $X_2 = h_2$ is $(\tilde{X}_1, h_2)$. We here choose C such that

$$0 < C < \underline{m}_2(\tilde{X}_1)^{a_{21}^0+a_{21}^1}(h_2)^{a_{11}^1+a_{11}^2}$$

and then define l_2 by

$$l_2 : X_1^{a_{21}^0+a_{21}^1}X_2^{a_{11}^1+a_{11}^2} = C/\overline{m}_2.$$

By using the techniques in [6,9], we first show that if there is a $t_0^* > t_0$ such that $x_1(t_0^*) > h_1$, then the orbits will remain on the right side of curve l_1 for all $t \geq t_0^*$. In fact, if $x(t)$ meets l_1 at t_2 then there exists a $t_1 \in (t_0^*, t_2)$ such that $x_1(t_1) = h_1$ and $x_1(t) < h_1$ for any $t \in (t_1, t_2]$. From (3.4), we have $V_1(t_1) < V_1(t_2)$. On the other hand, it holds that

$$V_1(t_2) < x_1(t_2) = \lambda h_1 \leq \underline{m}_1 h_1 = \underline{m} x_1(t_1) \leq V_1(t_1).$$

This is a contradiction.

Similar to the above discussion, we secondly show that if there is a $t_3 > t_0$ such that $x_1(t_3) > h_1$ then $x(t)$ cannot meet l_2 for all $t > t_3$. In fact, if $x(t)$ meets l_2 at t_5, then there exists a $t_4 \in (t_3, t_5]$ such that $x_2(t_4) = h_2$ and $x_2(t) < h_2$ for $t \in (t_4, t_5]$. By (3.4), we have

$$V_2(t_4) < V_2(t_5) \leq \overline{m}_2 (x_1(t_5))^{a_{21}^0 + a_{21}^1} (x_2(t_5))^{a_{11}^1 + a_{11}^2} = C.$$

However, since $x(t)$ lies on the right side of l_1 and $x_2(t_4) = h_2$, we have

$$V_2(t_4) \geq \underline{m}_2 (x_1(t_4))^{a_{21}^0 + a_{21}^1} (x_2(t_4))^{a_{11}^1 + a_{11}^2} \geq \underline{m}_2 \tilde{X}_1^{a_{21}^0 + a_{21}^1} h_2^{a_{11}^1 + a_{11}^2} > C.$$

This is a contradiction.

Finally, We check that for any solution $x(t)$ and any $t_0 > 0$, there is a $t_6 > t_0$ such that either $x_1(t_6) > h_1$ or $x_2(t_6) > h_2$. Otherwise, $x_1(t) \leq h_1$ and $x_2(t) \leq h_2$ for all $t > t_0$. By integrating (3.4), we have $V_1(t) \to +\infty$ as $t \to +\infty$, this contradicts the boundedness of $x_1(t)$. Therefore, there is a t_6 such that $x_1(t_6) > h_1$. Now we show that there is a $t_7 > t_6$ such that $x_2(t_7) > h_2$. Otherwise, for all $t \geq t_7$, $x_2(t) \leq h_2$, then we have $V_2(t) \to +\infty$ as $t \to +\infty$, this contradicts the boundedness of $x(t)$. Therefore, we have proved that for any $t > t_0$, there $t_7 > t_6 > t_0$ such that $x_1(t_6) > h_1$, $x_2(t_7) > h_2$.

The above steps show that any solution $x(t) = (x_1(t), x_2(t))$ will enter a smaller region $\{X = (X_1, X_2) \in \mathbb{R}_{+0}^2 | h_i \leq X_i \leq M_i, i = 1, 2\}$ and will not leave the larger one. This shows the permanence. $\square$

4 Proof of Theorem 1.8

In this section, we give a proof of Theorem 1.8.

Proof of Theorem 1.8. From Lemma 2.6, there are positive constants $\tilde{M}_i$ $(i = 1, 2, 3)$ such that for sufficiently large t, any solution of (1.5) satisfies $0 < x_i(t) \leq \tilde{M}_i$ $(i = 1, 2, 3)$. We now consider the following functionals:

$$\begin{cases} V_1(t) = x_1(t)^{A_{11}} x_2(t)^{A_{12}} x_3(t)^{A_{13}} \exp\Big[(a_{12}A_{11} - a_{32}A_{13}) \int_{t-\tau}^t x_2(s)ds + (a_{31}A_{13} - a_{21}A_{12}) \int_{t-\tau}^t x_1(s)ds \\ \qquad -(a_{13}A_{11} - a_{33}A_{13}) \int_{t-2\tau}^t x_3(s)ds + a_{23}A_{12} \int_{t-\tau}^t x_3(s)ds\Big] \\ V_2(t) = (x_1(t))^{-a_{21}} (x_2(t))^{a_{11}} \exp\Big[-a_{11}a_{21} \int_{t-\tau}^t x_1(s)ds - a_{21}a_{12} \int_{t-\tau}^t x_2(s)ds \\ \qquad +a_{21}a_{13} \int_{t-2\tau}^t x_3(s)ds + a_{11}a_{23} \int_{t-\tau}^t x_3(s)ds\Big], \\ V_3(t) = (x_2(t))^{-a_{32}} (x_3(t))^{a_{22}} \exp\Big[(a_{32}a_{21} + a_{22}a_{31}) \int_{t-\tau}^t x_1(s)ds \\ \qquad -a_{32}a_{23} \int_{t-\tau}^t x_3(s)ds - a_{32}a_{22} \int_{t-2\tau}^t x_2(s)ds - a_{22}a_{33} \int_{t-2\tau}^t x_3(s)ds\Big], \end{cases} \tag{4.7}$$

Then we have

$$V_1'(t) \geq V_1(t)(\delta_1 - \Delta_1 x_1(t)),$$

where $\delta_1 = r_1 A_{11} + r_2 A_{12} + r_3 A_{13}$ and $\Delta_1 = \det(A)$ with the matrix

$$\begin{pmatrix} -a_{11} & -a_{21} & a_{31} \\ a_{12} & -a_{22} & -a_{32} \\ -a_{13} & a_{32} & -a_{33} \end{pmatrix}$$

and

$$V_i'(t) \geq V_i(t)(\delta_i - \Delta_i x_i(t)), \ i = 2, 3,$$

where $\delta_2 = r_2a_{11} - r_1a_{21}$, $\Delta_2 = a_{11}a_{22} + a_{12}a_{21}$ and $\delta_3 = r_3a_{22} - r_2a_{32}$, $\Delta_3 = a_{22}a_{33} + a_{23}a_{32}$. Let us fix $0 < h_i < \frac{\delta_i}{2\Delta_i}$ for $(i = 1, 2, 3)$. If $x_i(t) \le h_i$ for some i, then we have

$$\frac{dV_i(t)}{dt} \ge \frac{\delta_i}{2} V_i(t), \tag{4.8}$$

from which we obtain

$$\underline{m}_1 x_1(t)^{A_{11}} x_2(t)^{A_{12}} x_3(t)^{A_{13}} \le V_1(t) \le \overline{m} x_1(t)^{A_{11}} x_2(t)^{A_{12}} x_3(t)^{A_{13}}, \tag{4.9}$$

$$\underline{m}_2 (x_1(t))^{-a_{21}} (x_2(t))^{a_{11}} \le V_2(t) \le \overline{m}_2 (x_1(t))^{-a_{21}} (x_2(t))^{a_{11}}, \tag{4.10}$$

$$\underline{m}_3 (x_2(t))^{-a_{32}} (x_3(t))^{a_{22}} \le V_3(t) \le \overline{m}_3 (x_2(t))^{-a_{32}} (x_3(t))^{a_{22}}, \tag{4.11}$$

where

$$\begin{aligned}
\underline{m}_1 &= \exp[(a_{31}A_{13} - a_{21}A_{12})\tilde{M}_1\tau - 2a_{13}A_{11}\tilde{M}_3\tau], \\
\overline{m}_1 &= \exp[(a_{23}A_{12} - 2a_{33}A_{13})\tilde{M}_3\tau + (a_{12}A_{11} - a_{32}A_{13})\tilde{M}_2\tau], \\
\underline{m}_2 &= \exp[-a_{11}a_{21}\tilde{M}_1\tau - a_{21}a_{12}\tilde{M}_2\tau], \\
\overline{m}_2 &= \exp[(2a_{21}a_{13} + a_{11}a_{23})\tilde{M}_3\tau], \\
\underline{m}_3 &= \exp[-a_{32}a_{22}\tilde{M}_2\tau - (2a_{22}a_{33} + a_{32}a_{33})\tilde{M}_3\tau], \\
\overline{m}_3 &= \exp[(a_{32}a_{21} + a_{22}a_{31})\tilde{M}_1\tau]
\end{aligned}$$

We now consider the three surfaces L_1, L_2 and L_3 in the region $\{X \in \mathbb{R}^3_{+0} | X_i \le \tilde{M}_i, i = 1, 2, 3\}$ defined by

$$\begin{aligned}
L_2 &: X_1^{-a_{21}} X_2^{a_{11}} = (\lambda_2 h_2)^{a_{11}} \tilde{M}_1^{-a_{21}}, \\
L_3 &: X_2^{-a_{32}} X_3^{a_{22}} = (\lambda_3 h_3)^{a_{22}} \tilde{M}_2^{-a_{32}},
\end{aligned}$$

with $\lambda_i \le \underline{m}_i / \overline{m}_i$ $(i = 2, 3)$. Moreover, we suppose that the intersection of L_2 and L_3 with $X_1 = h_1$ is $X_2 = \tilde{X}_2$. We here choose C' such that

$$0 < C' < \underline{m}_1 h_1^{A_{11}} \tilde{X}_2^{A_{12}} \tilde{M}_3^{A_{13}}$$

and then define L_1 by

$$L_1 : X_1^{A_{11}} X_2^{A_{12}} X_3^{A_{13}} = C'/\overline{m}_1.$$

We first show that if there is a $t_0^* > t_0$ such that $x(t) = (x_1(t), x_2(t), x_3(t))$ lies on the right side of L_2, then the orbits will remain on the right side of curve L_2 for all $t \ge t_0^*$. In fact, if $x(t)$ meets L_2 at t_2 then there exists a $t_1 \in (t_0^*, t_2)$ such that $x_2(t_1) = h_2$ and $x_2(t) < h_2$ for $t \in (t_1, t_2]$. From (4.8), we have $V_2(t_1) < V_2(t_2)$. On the other hand, it holds that

$$\begin{aligned}
V_2(t_2) &\le \underline{m}_2 (x_1(t_2))^{-a_{21}} (x_2(t_2))^{a_{11}} = \underline{m}_2 (\tilde{M}_1)^{-a_{21}} (\lambda_2 h_2)^{a_{11}} \\
&\le \underline{m}_2 (\tilde{M}_1)^{-a_{21}} h_2^{a_{11}} = \overline{m}_2 (x_1(t_1))^{-a_{21}} (x_2(t_1))^{a_{11}} \le V_2(t_1),
\end{aligned}$$

provided $\lambda \le \underline{m}_2 / \overline{m}_2$.

Similar to the above discussion, we secondly show that if there is a $t_3 > t_0$ such that $x_3(t_3) > h_3$ and $x(t)$ lies on the right side of L_3 for all $t \ge t_3$, then $x(t)$ cannot meet L_3 for all $t \ge t_3$.

In this step, we show that if there is a $t_4 > t_0$ such that $x_2(t_4) > h_2$, $x_3(t_4) > h_3$ and $x(t)$ lies on the right side of curves L_2 and L_3, then $x(t)$ cannot meet L_1 for all $t > t_4$. In fact, if $x(t)$ meets L_1 at t_6, then there exists a $t_5 \in (t_4, t_6]$ such that $x_1(t_5) = h_1$ for $t \in (t_5, t_6]$, $x_1(t) < h_1$. By (4.9), we have

$$V_1(t_5) < V_1(t_6) \le \overline{m}_1 x_1(t_6)^{A_{11}} x_2(t_6)^{A_{12}} x_3(t_6)^{A_{13}} = C'.$$

However, since $x(t)$ lies on the right side of L_2, L_3 and $x_1(t_5) = h_1$, we have

$$\begin{aligned}
V_1(t_5) &\ge \underline{m}_1 x_1(t_5)^{A_{11}} x_2(t_5)^{A_{12}} x_3(t_5)^{A_{13}} \\
&> \underline{m}_1 h_1^{A_{11}} \tilde{X}_2^{A_{12}} \tilde{M}_3^{A_{13}} = C'.
\end{aligned}$$

This is a contradiction.

Finally, we check that for any solution $x(t)$ and any $t_0 > 0$, there is a $t_6 > t_0$ such that either $x_1(t_6) > h_1$ or $x_2(t_6) > h_2$ or $x_3(t_6) > h_3$. Otherwise, $x_i(t) \le h_i$ holds for all $t > t_0$, $i = 1, 2, 3$. By integrating (4.8), we have $V_i(t) \to +\infty$ as $t \to +\infty$ and this yields

$$V_1(t)(V_3(t))^{a_{13}} \le \overline{m}_1 (\overline{m}_3)^{a_{13}} x_1(t)^{A_{11}} x_2(t)^{a_{12}a_{33}} x_3(t)^{a_{12}a_{23}} \longrightarrow +\infty$$

as $t \to +\infty$. This contradicts the boundedness of $x_i(t)$. Therefore, there is a t_6 such that $x_1(t_6) > h_1$. Now we show that there is a $t_7 > t_6$ such that $x_2(t_7) > h_2$ or $x_3(t_7) > h_3$. Otherwise, for all $t \geq t_7$, $x_2(t) \leq h_2$, $x_3(t_7) \leq h_3$ and we have $V_2(t) \to +\infty$ as $t \to +\infty$. This yields from (4.10) that $V_2(t) \leq x_1(t)^{-a_{21}} \overline{m}_2 \tilde{M}_2^{a_{11}}$ holds for sufficiently large t. Hence $x_1(t) \to 0$ as $t \to +\infty$. However, we know that there is a sequence $t_n \to \infty$ as $n \to +\infty$ such that $x_1(t_n) > h_1$. Therefore, we have proved that for any $t > t_0$, there are $t_7 > t_6 > t_0$ such that $x_1(t_6) > h_1$, $x_2(t_7) > h_2$.

Finally, we show that there is a $t_8 > t_7$ such that $x_3(t_8) > h_3$. Otherwise, for all $t \geq t_8$, $x_3(t) \leq h_3$, we have $V_3(t) \to +\infty$. This also yields from (4.10) that $V_3(t) \leq x_2(t)^{-a_{32}} \overline{m}_3 \tilde{M}_3^{a_{23}}$ holds for sufficiently large t. Hence $x_2(t) \to 0$ as $t \to +\infty$. However, we know that there is a sequence $t_n \to +\infty$ as $n \to +\infty$ such that $x_2(t_n) > h_2$. Hence, for any $t > t_0$, there $t_7 > t_6 > t_0$ such that $x_1(t_6) > h_1$, $x_2(t_7) > h_2$.

The above steps show that any solution $x(t) = (x_1(t), x_2(t), x_3(t))$ will enter a smaller region $\{X = (X_1, X_2, X_3) \in \mathbb{R}^3_{+0} | h_i \leq X_i \leq M_i, i = 1, 2, 3\}$ and will not leave the larger one A. This shows the permanence. $\square$

5 Concluding remarks and some examples

In most natural systems, biological models (such as population dynamics, epidemiology, neural networks) involving time delays have been studied by a lot of authors, since time delays can express the maturation period of a species, the incubation time of a disease, the maturation of blood cells, etc.

In the present paper, we consider the permanence property for a class of Lotka-Volterra cooperative systems with multiple delays and obtain sufficient conditions without any restrictions on the size of time delays which improve the results of [2, 4, 5, 7]. We illustrate our criteria by the following examples:

Example 5.1.

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[1 - ax_1(t-2\tau) - \mathrm{e}x_1(t-\tau) + 2\mathrm{e}^{-\frac{11}{4}} x_2(t-\tau)], \\ \dfrac{dx_2(t)}{dt} = x_2(t)\left[r + bx_1(t) + \dfrac{2}{3}\mathrm{e}^{\frac{1}{4}} x_1(t-\tau) - \dfrac{1}{2}x_2(t) - \dfrac{1}{6}\mathrm{e}^2 x_2(t-\tau)\right]. \end{cases} \tag{5.1}$$

If $a = 1$, $b = 0$, $r = 1$, in [4], system (5.1) is permanent if τ satisfies $0 < \tau < \tau^* := \frac{1+\mathrm{e}-\frac{8}{3}\mathrm{e}^{--\frac{5}{2}}}{2+\mathrm{e}} \approx 0.74$ and in [2, 7], the permanence result is improved for any $\tau > 0$ and $a > b > 0$. In this case, our criterion in Theorem A also holds true. But if either of the following parameter set:

(5-I) $a = 1$, $b = \frac{1}{2}$, $-\frac{2}{3}\mathrm{e}^{-\frac{3}{4}} < r < 0$,

(5-II) $r > 0$ and $a = 1 < b < \frac{1}{4}\mathrm{e}^{\frac{11}{4}}$,

then conditions of Theorems 1.2, 1.4 and 1.5 fail, but Theorem A guarantees that system (5.1) remains permanent.

Example 5.2.

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[1 - \alpha_0 x_1(t) - 2x_1(t-\tau_{11}) + x_2(t-\tau_{12})], \\ \dfrac{dx_2(t)}{dt} = x_2(t)\left[1 - \dfrac{1}{4}x_2(t) + \dfrac{1}{2}x_1(t) - 2x_2(t-\tau_{22})\right]. \end{cases} \tag{5.2}$$

If $\tau_{12} = \tau_{22} = \tau$ and α_0, from [5, Theorem 1.3], system (5.2) is permanent, but if $\alpha_0 \in (1/4, 1/2)$, the conditions in [5, Theorem 1.3] fails. On the other hand, we see that the condition of Theorem B holds, thus system (5.2) is permanent for $\alpha_0 \in (1/4, 1/2)$. satisfies $0 < \tau < \tau^* := \frac{1+\mathrm{e}-\frac{8}{3}\mathrm{e}^{-\frac{5}{2}}}{2+\mathrm{e}} \approx 0.74$ and in [2, 4],

Example 5.3.

$$\begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)\left[\dfrac{11}{4} - 3x_1(t) + x_2(t-\tau) - \dfrac{3}{4}x_3(t-2\tau)\right], \\ \dfrac{dx_2(t)}{dt} = x_2(t)\left[\dfrac{5}{2} - x_1(t-\tau) - 2x_2(t) + \dfrac{1}{2}x_3(t-\tau)\right], \\ \dfrac{dx_3(t)}{dt} = x_3(t)[3 + x_1(t-\tau) - x_2(t-\tau) - 3x_3(t-2\tau)]. \end{cases} \tag{5.3}$$

From Enatsu [2], we see that if $\tau \in (0, 0.0759]$, system (5.3) is permanent, but the conditions in the present paper need no restriction on the size τ. It is clear that the condition of Theorem C is satisfied. Hence, our criterion guarantees that system (5.3) is permanent for any sizes of time delays.

Recently, Nakata [8] considered the general case of system (1.5) and gave some sufficient conditions for the permanence, in fact, our technique can also work in those cases. Some ideas in this paper are also applicable to the nonautonomous Lotka-Volterra cooperative systems with delays and a class of the n-species Lotka-Volterra cooperative population systems with delays. These become our future topics.

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