AAECC manuscript No.
(will be inserted by the editor)

Effective Algorithm for Computing Noetherian Operators of Zero-Dimensional Ideals

Katsusuke Nabeshima · **Shinichi Tajima**

Received: date / Accepted: date

Abstract We consider Noetherian operators in the context of symbolic computation. Upon utilizing the theory of holonomic $\mathcal{D}$-modules, we present a new method for computing Noetherian operators associated to a zero-dimensional ideal. An effective algorithm that consists mainly of linear algebra techniques is proposed for computing them. Moreover, as applications, new computation methods of polynomial ideals are discussed by utilizing the Noetherian operators.

Keywords Noetherian operators · holonomic $\mathcal{D}$-module · primary ideals · zero-dimensional ideals · partial differential operators

Mathematics Subject Classification (2020) 33F10 · 68W30 · 16S32

1 Introduction

We introduce an effective algorithm for computing Noetherian operators of zero-dimensional primary ideals and present new computation methods of polynomial ideals as the applications.

Describing ideals in polynomial rings by using systems of differential operators is one of the major approaches to study them. In [30], F. S. Macaulay brought the notion of an inverse system, a system of differential conditions

This work has been partly supported by JSPS Grant-in-Aid for Science Research (C) (18K03320 and 18K03214).

Katsusuke Nabeshima
Department of Applied Mathematics, Tokyo University of Science,
1-3, Kagurazaka, Tokyo, Japan
E-mail: nabeshima@rs.tus.ac.jp

Shinichi Tajima
Graduate School of Science and Technology, Niigata University,
8050, Ikarashi 2-no-cho, Nishi-ku, Niigata, Japan
E-mail: tajima@emeritus.niigata-u.ac.jp

that describes an ideal. In 1937, W. Gröbner mentioned the importance of the Macaulay's inverse system in the study of linear differential equations with constant coefficients, and in 1938, he introduced differential operators to characterize ideals that are primary to a rational maximal ideal [22,23]. After that the important results and the terminology came from L. Ehrenpreis and V. P. Palamodov in [15,16,45], that is the characterization of primary ideals by the differential operators. The differential operators allow one to characterize the primary ideal by differential conditions on the associated characteristic variety. The differential operators are called *Noetherian operators*.

Subsequent algebraic and computational approaches to characterize primary ideals with the use of differential operators were given in [3,6,12,42]. Studies of Noetherian operators by local cohomology and $\mathcal{D}$-modules have been given in [49–52].

Recently, Y. Cid-Ruiz, J. Chen et al. have studied the Noetherian operators in the context of symbolic computation, and have given an algorithm and the Macaulay2 implementation in [8–11]. They use the Hilbert scheme and Macaulay dual space for studying and computing the Noetherian operators.

In this paper, we consider Noetherian operators of zero-dimensional ideals. In the case of a zero-dimensional primary ideal, the set of Noetherian operators becomes a finite dimensional vector space over a field. Therefore, in order to compute Noetherian operators, it is possible to utilize the same framework of [34] and its variants as FGLM algorithm [17], Mourrain's notion in [36,37, "connected to 1"], Buchberger-Möller algorithm [1] [34, Algorithm 1], both the soft Möller and the hard Möller algorithms [34, Algorithm 2] and algorithms for computing algebraic local cohomology classes [38,39,53]. We adapt the approach given in [38,39,53] and design an algorithm for computing a basis of the vector space of Noetherian operators. Accordingly, the resulting algorithm of the present paper is constructed by mainly linear algebra techniques, and much more effective than the algorithms presented in [8,11]. Moreover, the resulting algorithm is free from computing a primary decomposition of a zero-dimensional ideal I and its primary component Q. The resulting algorithm is able to compute the Noetherian operators of Q from the generators of I and the associated prime of Q.

In the latter half of this paper, we discuss representations of polynomial ideals and we make use of Noetherian operators to compute polynomial ideals e.g. sum of ideals, intersection of ideals and ideal quotients. More specifically, we consider Noetherian representation.

In integers, prime factorization of any number means to represent that number as a product of prime numbers, namely, any number can be represented by "prime numbers" and "exponents", for instance, $31752 = 2^3 \cdot 3^4 \cdot 7^2$.

What corresponds to the prime factorization in polynomial ideals?

A primary decomposition of any ideal means to represent that ideal as an intersection of primary ideals, namely, any polynomial ideal can be represented by primary ideals, for instance,

$$\langle (x^2+y^2)^2+3x^2y-y^3, x^2+y^2-1\rangle = \langle x^2, y-1\rangle \cap \langle 4y^2+4y+1, 4x^2-4y-5\rangle,$$

that corresponds to $31752 = 8 \cdot 81 \cdot 49$ in integers. An answer of the above question is a multiplicity variety, that is a collection of pairs consisting of algebraic varieties and Noetherian operators, introduced by L. Ehrenpreis [16]. In this paper, we introduce the concept of Noetherian representation, as a counterpart in algebraic setting of multiplicity variety, that is a collection of pairs consisting of the associated prime and Noetherian operators of primary components of a zero-dimensional ideal. See Table 1 that shows a correspondence between integers and polynomial ideals. We discuss the Noetherian representation and present new computational methods of polynomial ideals as applications of the Noetherian representations.

Table 1 Correspondence representations between integers and polynomial ideals

Integer	Polynomial ideal
31752	$I = \langle (x^2+y^2)^2 + 3x^2y - y^3, x^2+y^2-1 \rangle \subset \mathbb{Q}[x,y]$
$31752 = 8 \cdot 81 \cdot 49$	primary decomposition $I = \langle x^2, y-1 \rangle \cap \langle 4y^2+4y+1, 4x^2-4y-5 \rangle$ where $\sqrt{\langle x^2, y-1 \rangle} = \langle x, y-1 \rangle$, $\sqrt{\langle 4y^2+4y+1, 4x^2-4y-5 \rangle} = \langle 4x^2-3, 2y+1 \rangle$
prime factorization $31752 = 2^3 \cdot 3^4 \cdot 7^2$	"Noetherian representation" $\{\langle x, y-1 \rangle, \{1, \partial_x\}), (\langle 4x^2-3, 2y+1 \rangle, \{1, \partial_x + 2x\partial_y\})\}$

This paper is organized as follows. Section 2 briefly reviews Noetherian operators, and gives notations and definitions that will be used in this paper. Section 3 gives basic properties of Noetherian operators. Section 4 describes the new algorithm for computing Noetherian operators of zero-dimensional ideals, and Noetherian representation of ideals. As the new algorithm has been implemented in the computer algebra system Risa/Asir, Section 5 gives results of benchmark tests. Section 6 presents an algorithm for computing generators of an ideal from a Noetherian representation. Section 7 illustrates applications of Noetherian operators and Noetherian representations.

2 Preliminaries

Here first we briefly recall basic notations and definitions that are used in this paper. We refer the reader to [3,23,31–35,47]. Secondly, we review a definition of Noetherian operators.

2.1 Notations

Throughout this paper, we use the notation x as the abbreviation of n variables $x_1, \dots, x_n$, K as a subfield of the field $\mathbb{C}$ of complex numbers and $\mathbb{Q}$

as the field of rational numbers. The set of natural numbers $\mathbb{N}$ includes zero. For $f_1, \ldots, f_t \in K[x]$, let $\langle f_1, \ldots, f_t \rangle$ denote the ideal in $K[x]$ generated by $f_1, \ldots, f_t$, and $\sqrt{\langle f_1, \ldots, f_t \rangle}$ denote the radical of the ideal $\langle f_1, \ldots, f_t \rangle$.

An ideal $I \subset K[x]$ is *prime* whenever $f, g \in K[x]$ and $fg \in I$, then either $f \in I$ or $g \in I$. An ideal $I \subset K[x]$ is *primary* if $fg \in I$ implies either $f \in I$ or some power $g^m \in I$ (for some $m > 0$). If $I \subset K[x]$ is primary, $\sqrt{I}$ is prime.

Definition 1 If an ideal $I \subset K[x]$ is primary and $\sqrt{I} = \mathfrak{p}$, then we say that I is $\mathfrak{p}$-primary.

Definition 2 A *primary decomposition* of an ideal $I \subset K[x]$ is an expression of I as an intersection of primary ideals: $I = \bigcap_{i=1}^{r} Q_i$ where $Q_1, \ldots, Q_r$ are primary. If the decomposition satisfies

(i) the prime ideals $\sqrt{Q_1}, \ldots, \sqrt{Q_r}$ are pairwise distinct, and

(ii) for all $j = 1, \ldots, r$, we have $Q_j \not\supset \bigcap_{i \neq j}^{r} Q_i$,

then $\bigcap_{i=1}^{r} Q_i$ is said to be *minimal.* Each Q_i is called a primary component of I.

Currently, several algorithms and implementations for computing primary decomposition of ideals are known [2,4,5,7,13,14,19,25,26,35,41,48].

It is reported by the authors of [4,26,48] that their algorithms of computing a prime decomposition of the radical $\sqrt{I}$ are much faster than those of computing primary decomposition of a polynomial ideal I in $K[x]$. Recently in [4], T. Aoyama and M. Noro introduce a quite effective algorithm for computing the prime decomposition of $\sqrt{I}$. We apply the algorithm of [4] for computing Noetherian operators in Section 4.

Let $\mathcal{D} = K[x][\partial]$ denote the ring of partial differential operators with coefficients in $K[x]$, where $\partial = \{\partial_1, \partial_2, \ldots, \partial_n\}$, $\partial_i = \frac{\partial}{\partial x_i}$ with relations $x_i \bullet x_j := x_j x_i, \partial_i \bullet \partial_j := \partial_j \partial_i$, $\partial_j \bullet x_i := x_i \partial_j$ $(i \neq j)$, $x_i \bullet \partial_j = x_i \partial_j$ and $\partial_i \bullet x_i = x_i \partial_i + 1$ $(1 \leq i, j \leq n)$, i.e.

$$\mathcal{D} = \left\{ \sum_{\beta \in \mathbb{N}^n} c_\beta \partial^\beta \,\middle|\, c_\beta \in K[x] \right\}$$

where $\partial^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n}$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. Throughout the paper we assume that a linear partial differential operator is always represented in the canonical form: each power product of a partial differential operator is written as $x^\alpha \partial^\beta$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \beta \in \mathbb{N}^n$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. For instance,

$$\begin{aligned}
(\partial_1 + x_1\partial_2)(x_1^2 x_2) &= x_1^3 + 2x_1x_2 \text{ in } K[x_1, x_2], \\
(\partial_1 + x_1\partial_2) \bullet (x_1^2 x_2) &= x_1^3 x_2 \partial_2 + x_1^2 x_2 \partial_1 + x_1^3 + 2x_1x_2 \text{ in } K[x_1, x_2][\partial_1, \partial_2].
\end{aligned}$$

Let us fix a term order $\succ$ on $\mathbb{N}^n$. For a given partial differential operator of the form

$$\psi = c_\alpha \partial^\alpha + \sum_{\alpha \succ \beta} c_\beta \partial^\beta \qquad (c_\alpha, c_\beta \in K[x]),$$

we call ∂^α the *head term*, c_α the *head coefficient*, α the *head exponent*, ∂^β the lower terms and β the lower exponents. We denote the head term by $\mathrm{ht}(\psi)$, the head coefficient by $\mathrm{hc}(\psi)$ and the head exponent by $\mathrm{hex}(\psi)$. Furthermore, we denote the set of terms of ψ as $\mathrm{Term}(\psi) = \{\partial^\lambda \mid \psi = \sum_{\lambda \in \mathbb{N}^n} c_\lambda \partial^\lambda,\ c_\lambda \neq 0\}$, the set of lower terms of ψ as $\mathrm{LL}(\psi) = \left\{\partial^\lambda \in \mathrm{Term}(\psi) \,\middle|\, \partial^\lambda \neq \mathrm{ht}(\psi)\right\}$ and the set of exponents of $\mathrm{Term}(\psi)$ as $\mathrm{Expo}(\mathrm{Term}(\psi)) = \{\lambda \in \mathbb{N}^n \,|\partial^\lambda \in \mathrm{Term}(\psi)\}$. For a finite subset $\Psi \subset \mathcal{D}$, $\mathrm{ht}(\Psi) = \{\mathrm{ht}(\psi) \mid \psi \in \Psi\}$, $\mathrm{LL}(\Psi) = \bigcup_{\psi \in \Psi} \mathrm{LL}(\psi)$.

For instance, let $\psi = x_1^3x_2^2\partial_1^3\partial_2^2\partial_3 + x_3^2\partial_1^2\partial_3 + x_1x_3\partial_2\partial_3 + x_1^2x_2x_3$ be a partial differential operator in $\mathbb{Q}[x_1, x_2, x_3][\partial_1, \partial_2, \partial_3]$ and $\succ$ the graded lexicographic term order with $(1, 0, 0) \succ (0, 1, 0) \succ (0, 0, 1)$ where $(1, 0, 0), (0, 1, 0)$, $(0, 0, 1)$ corresponds to $\partial_1, \partial_2, \partial_3$. Then,

$\mathrm{ht}(\psi) = \partial_1^3\partial_2^2\partial_3$,
$\mathrm{hc}(\psi) = x_1^3x_2^2$,
$\mathrm{hex}(\psi) = (3, 2, 1)$,
$\mathrm{LL}(\psi) = \{\partial_1^2\partial_3, \partial_2\partial_3, 1\}$,
$\mathrm{Term}(\psi) = \{\partial_1^3\partial_2^2\partial_3, \partial_1^2\partial_3, \partial_2\partial_3, 1\}$
$\mathrm{Expo}(\mathrm{Term}(\psi)) = \{(3, 2, 1), (2, 0, 1), (0, 1, 1), (0, 0, 0)\}$,
$\mathrm{Expo}(\mathrm{LL}(\psi)) = \{(2, 0, 1), (0, 1, 1), (0, 0, 0)\}$.

For each $1 \leq i \leq n$, we write the standard unit vector as

$$\boldsymbol{e}_i = (0, \ldots, 0, \overset{i\text{th}}{1}, 0, \ldots, 0).$$

2.2 Noetherian Operators

The main purpose of this paper is to construct an effective algorithm for computing Noetherian operators. Here we give a definition of Noetherian operators.

In 1960, L. Ehrenpreis announced his fundamental principle, which states that the solutions of a system of linear partial differential equations with constant coefficients can be represented in terms of certain integrals [15]. See also [16,45]. At the core of the fundamental principle, one has the following theorem.

Theorem 1 (Ehrenpreis-Palamodov[15, 16, 45]) *Let Q be a $\mathfrak{p}$-primary ideal in $K[x]$ and $Q \neq \langle 1 \rangle$. There exist partial differential operators $\psi_1, \psi_2, \ldots,$ ψ_ℓ in $\mathcal{D}$ with the following property. A polynomial $g \in K[x]$ lies in the ideal Q if and only if $\psi_1(g), \psi_2(g), \ldots, \psi_\ell(g) \in \mathfrak{p}$.*

Remark that the theorem above was discussed for $K[x]$-modules in [15,16, 45]. We rewrite the theorem, given by [15,16,45], for ideals in $K[x]$ because we focus our attention to the case of ideals in polynomial rings.

Definition 3 The partial differential operators $\psi_1, \psi_2, \dots, \psi_\ell$ that satisfy Theorem 1 are called *Noetherian operators* for the primary ideal Q.

Theorem 1 says that the pairs consisting of the partial differential operators $\psi_1, \psi_2, \dots, \psi_\ell$ and the prime ideal $\mathfrak{p}$ characterize the structure of the primary ideal Q. Moreover, the partial differential operators can be seen as a multiplicity of the primary ideal Q at the prime ideal $\mathfrak{p}$. (See Definition 5.) Thus, Noetherian operators are important ingredients for analyzing the primary ideal.

Recently, in the papers [8–11], Noetherian operators have been studied in the context of symbolic computation, and algorithms for computing Noetherian operators are given. They use the Hilbert scheme and Macaulay dual space for studying them.

In this paper, we study Noetherian operators of zero-dimensional primary ideals in a different fashion and give a new algorithm for computing Noetherian operators of zero-dimensional primary ideals.

3 Noetherian Operators for Zero-dimensional Ideals

Here we discuss properties of Noetherian operators that are utilized to construct the new algorithm.

Note that, any primary ideal Q in this paper is assumed to be proper, i.e. $Q \neq K[x]$.

3.1 Noetherian Operators and Local Cohomology

Here we present the relations between Noetherian operators and local cohomology classes. See [43, 44, 49–52] for details.

Let I be a zero-dimensional ideal in $K[x]$ and let $H^n_{[Z]}(K[x])$ denote algebraic local cohomology group, with support on $Z = \mathbb{V}(I) = \{a \in \mathbb{C}^n \mid g(a) = 0, \forall g \in I\}$, defined by

$$H^n_{[Z]}(K[x]) = \lim_{k\to\infty} \mathrm{Ext}^n_{K[x]}(K[x]/(\sqrt{I})^k, K[x]).$$

Let $I = \bigcap_{i=1}^r Q_i$ be a minimal primary decomposition where $Q_1, \dots, Q_r$ are primary. According to the primary decomposition, $Z = \mathbb{V}(I)$ can be written as an union of irreducible affine varieties $Z = \bigcup_{i=1}^r Z_i$ where $Z_i = \mathbb{V}(\sqrt{Q_i})$. Then, $H^n_{[Z]}(K[x])$ is decomposed to a direct sum

$$H^n_{[Z]}(K[x]) = H^n_{[Z_1]}(K[x]) \oplus \cdots \oplus H^n_{[Z_r]}(K[x]).$$

Let

$$\begin{aligned} H_{Q_i} &= \{\eta \in H^n_{[Z_i]}(K[x]) \mid q\eta = 0, \forall q \in Q_i\},\\ H_{\sqrt{Q_i}} &= \{\eta \in H^n_{[Z_i]}(K[x]) \mid q\eta = 0, \forall q \in \sqrt{Q_i}\},\\ Ann_{K[x]}(H_{Q_i}) &= \{g \in K[x] \mid g\eta = 0, \forall \eta \in H_{Q_i}\},\\ Ann_{K[x]}(H_{\sqrt{Q_i}}) &= \{g \in K[x] \mid g\eta = 0, \forall \eta \in H_{\sqrt{Q_i}}\}. \end{aligned}$$

According to the Grothendieck duality, we regard the K-vector space H_{Q_i} and $H_{\sqrt{Q_i}}$ as the dual spaces of $N_{Q_i} = K[x]/Q_i$ and $N_{\sqrt{Q_i}} = K[x]/\sqrt{Q_i}$ respectively. Then,

$$H_{Q_i} \simeq \mathcal{H}om_{K[x]}\left(N_{Q_i}, H^n_{[Z_i]}(K[x])\right),\ H_{\sqrt{Q_i}} \simeq \mathcal{H}om_{K[x]}\left(N_{\sqrt{Q_i}}, H^n_{[Z_i]}(K[x])\right)$$

$$Ann_{K[x]}(H_{Q_i}) = Q_i,\ Ann_{K[x]}(H_{\sqrt{Q_i}}) = \sqrt{Q_i}$$

holds.

Set $M_{Q_i} = \mathcal{D}/\mathcal{D}Q_i$ and $M_{\sqrt{Q_i}} = \mathcal{D}/\mathcal{D}\sqrt{Q_i}$. Then, since $K[x] \subset \mathcal{D}$, we also have

$$H_{Q_i} \simeq \mathcal{H}om_{\mathcal{D}}\left(M_{Q_i}, H^n_{[Z_i]}(K[x])\right),\quad H_{\sqrt{Q_i}} \simeq \mathcal{H}om_{\mathcal{D}}\left(M_{\sqrt{Q_i}}, H^n_{[Z_i]}(K[x])\right).$$

That is, H_{Q_i} and $H_{\sqrt{Q_i}}$ can be interpreted as the local cohomology solution spaces of the holonomic $\mathcal{D}$-modules M_{Q_i} and $M_{\sqrt{Q_i}}$ respectively.

$$\begin{tikzcd}
M_{Q_i} \arrow[rr, dashed, "\rho\colon \mathcal{D}\text{-map}"] \arrow[dr, swap, "\mathcal{H}om_{\mathcal{D}}(M_{Q_i}, H^n_{[Z_i]}(K[x])) \ni \varphi \circ \rho"] & & M_{\sqrt{Q_i}} \arrow[dl, "\varphi \in \mathcal{H}om_{\mathcal{D}}(M_{\sqrt{Q_i}}, H^n_{[Z_i]}(K[x]))"] \\
& H^n_{[Z_i]}(K[x]) &
\end{tikzcd}$$

Now, we are able to consider Noetherian operators as follows.

Definition 4 ([51]) The set of $\mathcal{D}$-linear homomorphisms $\mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}})$ between the two left $\mathcal{D}$-modules are called the *Noetherian space* of Q_i.

The Noetherian space has a structure of a right $K[x]/\sqrt{Q_i}$-module. Recall that there is a natural map

$$\mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}}) \times \mathcal{H}om_{\mathcal{D}}(M_{\sqrt{Q_i}}, H^n_{[Z_i]}(K[x])) \to \mathcal{H}om_{\mathcal{D}}(M_{Q_i}, H^n_{[Z_i]}(K[x])).$$

Therefore, the Noetherian space describes the relation between H_{Q_i} and $H_{\sqrt{Q_i}}$.

Definition 5 The ratio $\ell_i = \dim_K(K[x]/Q_i)/\dim_K(K[x]/\sqrt{Q_i})$ of dimensions of vector spaces is called the *multiplicity* of the ideal Q_i.

Since the Noetherian space is an ℓ_i-dimensional vector space over the field $K[x]/\sqrt{Q_i}$, we have the following lemma.

Lemma 1 ([50,51]) *There exist $\rho_1, \rho_2, \ldots, \rho_{\ell_i} \in \mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}})$ such that any $\delta \in \mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}})$ can be written in a unique way as*

$$\delta = c_1\rho_1 + c_2\rho_2 + \cdots + c_{\ell_i}\rho_{\ell_i}$$

where $c_1, c_2, \ldots, c_{\ell_i} \in K[x]/\sqrt{Q_i}$.

Since $1 \in M_{Q_i}$ is an equivalent class of differential operators, a representative ϱ_j of the image $\rho_j(1)$ is a differential operator, $j = 1, 2, \ldots, \ell_i$. We call the set $\{\varrho_1, \ldots, \varrho_{\ell_i}\}$ the *Noetherian operator basis*. Let ψ_j denote the formal adjoint of ϱ_j. The Grothendieck duality yields the following.

Theorem 2 ([52]) *The differential operators $\psi_1, \psi_2, \ldots, \psi_{\ell_i}$ satisfy*

$$Q_i = \{h \in K[x] | \psi_1(h), \psi_2(h), \ldots, \psi_{\ell_i}(h) \in \sqrt{Q_i}\}.$$

The theorem says that the differential operators above are *Noetherian operators* introduced in Definition 3.

Let us go back to the zero-dimensional ideal I. Since

$$\mathcal{H}om_{\mathcal{D}}(M_I, H^n_{[Z_i]}(K[x])) = \mathcal{H}om_{\mathcal{D}}(M_{Q_i}, H^n_{[Z_i]}(K[x])),$$

we obtain the following theorem.

Theorem 3 *Let $M_I = \mathcal{D}/\mathcal{D}I$. Then,*

$$\mathcal{H}om_{\mathcal{D}}(M_I, M_{\sqrt{Q_i}}) \simeq \mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}}).$$

This theorem says that the primary ideal Q_i can be defined by I and the prime ideal $\sqrt{Q_i}$, i.e. we do not need a basis of the primary ideal Q_i for $\mathcal{H}om_{\mathcal{D}}(M_{Q_i}, M_{\sqrt{Q_i}})$. This consequence is utilized in Theorem 5 and powerfully works when we compute Noetherian operators of Q_i.

3.2 Properties of Noetherian Operators

Here we introduce important properties of Noetherian operators of primary ideals. Note that we adopt the classical definition (i.e. Defition 3) for Noetherian operators. The following theorem is from L. Hörmander [24].

Theorem 4 (L. Hörmander, Theorem 7.7.6 and pp. 235 of [24])
Let Q be a primary ideal and $\sqrt{Q} = \mathfrak{p}$ and s a natural number that satisfies $\mathfrak{p}^s \subset Q$. Let $\mathcal{N}_s(Q)$ be the set of all partial differential operators $\psi = \sum_{|\beta|<s} c_\beta \partial^\beta$ $(c_\beta \in K[x])$, such that $\psi(h) \in \mathfrak{p}$ for all $h \in Q$ where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ and $|\beta| = \beta_1 + \cdots + \beta_n$.

Then,

(i) $g \in K[x]$, $\psi(g) \in \mathfrak{p}$ *for all* $\psi \in \mathcal{N}_s(Q) \iff g \in Q$.

(ii) One can choose $\psi_1, \psi_2, \ldots, \psi_\ell \in \mathcal{N}_s(Q)$ *such that*

$$g \in K[x], \psi_j(g) \in \mathfrak{p} \text{ for } j = 1, 2, \ldots, \ell \iff g \in Q.$$

(iii) For each $j \in \{1, \ldots, n\}$, *the commutator* $[\psi, x_j] = \psi \bullet x_j - x_j \bullet \psi \in \mathcal{N}_s(Q)$.

Note that for $\psi = \sum_{\beta\in\mathbb{N}^n} c_\beta \partial^\beta$ then,

$$[\psi, x_j] = \sum_{(\beta_1,\ldots,\beta_j,\ldots,\beta_n)\in\mathbb{N}^n,\ \beta_j\geq 0} \beta_j c_\beta \partial_1^{\beta_1}\cdots\partial_j^{\beta_j-1}\cdots\partial_n^{\beta_n},$$

and it is of lower order than ψ. We call an element of $\mathcal{N}_s(Q)$ *Noetherian* w.r.t. Q.

We turn to a zero-dimensional ideal I. As we described in Section 3.1, we showed that a primary component Q_i of I can be characterized by I and $\sqrt{Q_i}$, in Theorem 3. By applying Theorem 3 to Theorem 4, we can extend Theorem 4.

Lemma 2 *Let I be a zero-dimensional ideal generated by $f_1,\ldots,f_t$ in $K[x]$ and Q a primary component of a minimal primary decomposition of I with $\sqrt{Q}=\mathfrak{p}$. Let $\mathcal{N}_s(I)$ be the set of all partial differential operators*

$$\psi = \sum_{\beta\in\mathbb{N}^n,|\beta|<s} c_\beta\partial^\beta\ (c_\beta\in K[x]),\ \textit{such that}\ \psi(f)\in\mathfrak{p}\ \textit{for all}\ f\in I$$

where s is from Theorem 4. Then, the following are equivalent:

(i) $\psi\in\mathcal{N}_s(I)$.
(ii) $[\psi,x_i]\in\mathcal{N}_{s-1}(I)$ *for* $i=1,2,\ldots,n$ *and* $\psi(f_j)\in\mathfrak{p}$ *for* $j=1,2,\ldots,t$.

Proof ((*i*) $\Longrightarrow$ (*ii*)) Since $f_j\in Q$ for each $j=1,2,\ldots,t$, thus we have $\psi(f_j)\in\mathfrak{p}$. By applying Theorem 3 to Theorem 4, we can change $\mathcal{N}_s(Q)$ as the set $\mathcal{N}_s(I)$. Moreover, $[\psi,x_i]$ is of lower order than ψ, hence $[\psi,x_i]\in\mathcal{N}_{s-1}(I)$.
((*ii*) $\Longrightarrow$ (*i*)) If $\alpha_1\geq 1$, then

$$\begin{aligned}\psi(x^\alpha f_j) &= \psi(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j)\\ &= (\psi\bullet x_1 - x_1\bullet\psi + x_1\bullet\psi)(x_1^{\alpha_1-1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j)\\ &= [\psi,x_1](x^{\alpha_1-1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j) + x_1\psi(x^{\alpha_1-1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j)\end{aligned}$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n$ and $j=1,2,\ldots,t$. Thus, we have the following

$$\begin{aligned}\psi(x^\alpha f_j) &= \psi(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j)\\ &= \sum_{\alpha_1\geq 1,\beta_1+\gamma_1=\alpha_1-1} x_1^{\beta_1}[\psi,x_1](x_1^{\gamma_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_j)\\ &\quad + \sum_{\alpha_2\geq 1,\beta_2+\gamma_2=\alpha_2-1} x_1^{\alpha_1}x_2^{\beta_2}[\psi,x_2](x_2^{\gamma_2}x_3^{\alpha_3}\cdots x_n^{\alpha_n}f_j)\\ &\quad +\cdots\\ &\quad + \sum_{\alpha_n\geq 1,\beta_n+\gamma_n=\alpha_n-1} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\beta_n}[\psi,x_n](x_n^{\gamma_n}f_j)\\ &\quad + x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}\psi(f_j)\end{aligned}$$

where $a_1, a_2, \dots, a_n, \beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{N}$.

Since $[\psi, x_i] \in \mathcal{N}_{s-1}(I)$ and $\psi(f_j) \in \mathfrak{p}$, we have $\psi(x^\alpha f_j) \in \mathfrak{p}$ and $\psi \in \mathcal{N}_s(I)$.

For all $f \in I$, f can be written as $f = \sum_{i=1}^{t} a_i f_i$ where $a_1, \dots, a_t \in K[x]$. As $a_1, \dots, a_t$ are K-linear combinations of finite terms in $K[x]$, therefore, by the discussion above, $\psi(f) \in \mathfrak{p}$. This means that $\psi \in \mathcal{N}_s(I)$. $\square$

Theorem 3 and Lemma 2 lead us to the following important theorem.

Theorem 5 *Using the same notations as in Lemma 2, let* NT_Q *be the set of all partial differential operators* $\psi = \sum_{\beta \in \mathbb{N}^n, |\beta| < s} c_\beta \partial^\beta$ $(c_\beta \in K[x])$,

such that $[\psi, x_i] \in \mathcal{N}_{s-1}(I)$ *for* $i = 1, 2, \dots, n$ *and* $\psi(f_j) \in \mathfrak{p}$ *for* $j = 1, 2, \dots, t$.

Then,

(i) $g \in K[x]$, $\psi(g) \in \mathfrak{p}$ *for all* $\psi \in \mathrm{NT}_Q \iff g \in Q$.
(ii) One can choose $\psi_1, \psi_2, \dots, \psi_\ell \in \mathrm{NT}_Q$ *such that*

$$g \in K[x], \psi_k(g) \in \mathfrak{p} \text{ for } k = 1, 2, \dots, \ell \iff g \in Q.$$

Proof By applying Theorem 3 and Lemma 2 to Theorem 4, we can change $\mathcal{N}_s(Q)$ of Theorem 4 as the set NT_Q. $\square$

We do not need generators of the primary ideal Q to characterize a set NT_Q of Noetherian operators of Q. We only need the generators of I and $\mathfrak{p}$ for NT_Q of Q.

We emphasize that Theorem 5 is the main tool in the remaining part of this paper.

Proposition 1 *Let* Q *be a zero-dimensional primary ideal in* $K[x]$ *and* $\sqrt{Q} = \mathfrak{p}$. *Then, the set* NT_Q, *that is from Theorem 5, is a finite dimensional vector space over the field* $K[x]/\mathfrak{p}$.

Proof Let $c\partial^\tau$ be a partial differential operators in $\mathcal{D}$ with $c \in \mathfrak{p} \subset K[x]$ where $\tau \in \mathbb{N}^n$. Then, for all $f \in K[x]$, $(c\partial^\tau)(f) \equiv 0 \mod \mathfrak{p}$, namely, $c\partial^\tau$ does not characterize any properties of the primary ideal Q. Thus, we do not consider the such unneeded partial differential operators. (Remark that $c\partial^\tau$ is the formal adjoint of a representative of the zero-mapping in $\mathcal{H}om_{\mathcal{D}}(M_Q, M_{\sqrt{Q}})$.)

For all $\psi, \varphi \in \mathrm{NT}_Q$, $f \in Q$, and $c \in K[x]/\mathfrak{p}$, $c(\psi+\varphi)(f) = c\psi(f) + c\varphi(f) \in \mathfrak{p}$. Hence $c(\psi + \varphi) \in \mathrm{NT}_Q$, thus NT_Q is a vector space.

Since I is zero-dimensional, there exists $u_i \in Q$ that is a univariate polynomial in the variable x_i for $i = 1, 2, \dots, n$. Then, $\partial_i^{k_i}(u_i) \in K[x]/\mathfrak{p}$ and thus $\partial_i^{k_i}(u_i) \notin \mathfrak{p}$. Hence, $\partial_i^{k_i} \notin \mathrm{NT}_Q$. By (ii) of Lemma 2 (or Corollary 1 in Section 4.1), an arbitrary exponent $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathrm{Expo}(\mathrm{Term}(\mathrm{NT}_Q))$ must have the property $\alpha_i \le k_i - 1$ for each $1 \le i \le n$. This leads us the fact that the number of terms in $\mathrm{NT}' = \{\psi \in \mathrm{NT}_Q \,|\psi = \sum_\alpha c_\alpha \partial^\alpha, c_\alpha \notin \mathfrak{p}\}$ is at most $\prod_{i=1}^{n}(k_i - 1) < \infty$ and the number of combinations of terms in $\mathrm{Term}(\mathrm{NT}')$ is also finite. Therefore, the number of linearly independent elements in NT' is finite, namely, NT_Q is a finite dimensional vector space over the field $K[x]/\mathfrak{p}$. $\square$

In what follows, the notation NT_Q, that is introduced in Theorem 5, is utilized as the set of Noetherian operators of Q.

The first aim of this paper is constructing an algorithm for computing the basis of the vector space NT_Q.

Definition 6 Let $\succ$ be a term order in $\mathbb{N}^n$, Q a zero-dimensional primary ideal in $K[x]$ and $\sqrt{Q} = \mathfrak{p}$. Let NB_Q be a basis of the vector space NT_Q over the field $K[x]/\mathfrak{p}$ such that

for all $\psi \in \mathrm{NB}_Q$, $\mathrm{hc}(\psi) = 1$, $\mathrm{ht}(\psi) \notin \mathrm{ht}(\mathrm{NB}_Q \setminus \{\psi\})$ and $\mathrm{ht}(\psi) \notin \mathrm{LL}(\mathrm{NB}_Q)$.

Then, the basis is called a *reduced basis* of NT_Q w.r.t. $\succ$.

4 New Algorithm for Computing Noetherian Operators

Here, we present a new algorithm for computing Noetherian operators of zero-dimensional primary ideals.

This section consists of three parts. In Section 4.1 and Section 4.2, we discuss strategies for selecting head exponents and lower exponents of Noetherian operators. In Section 4.3, we give the new algorithm and examples for Noetherian operators.

4.1 Head Exponents of Noetherian Operators

Here we present strategies for selecting possible candidates of head exponents of NT_Q w.r.t. a term order $\succ$.

Definition 7 Let $T \subset \mathbb{N}^n$. Then, we define the neighbors of T as $\mathrm{Neighbor}(T)$, i.e.

$$\mathrm{Neighbor}(T) = \{\tau + \boldsymbol{e}_i \mid \tau \in T,\ i = 1, \ldots, n\}.$$

The following corollary is a direct consequence of Lemma 2 (ii).

Corollary 1 *Let $Q \subset K[x]$ be a zero-dimensional primary ideal and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$. Let*

$$\Lambda_Q = \left\{\lambda \in \mathbb{N}^n \,\middle|\, \partial^\lambda \in \mathrm{ht}(\mathrm{NT}_Q)\right\} \text{ and } \Lambda_Q^{(\lambda)} = \{\lambda' \in \Lambda_Q \mid \lambda \succ \lambda'\}.$$

If $\lambda \in \Lambda_Q$, then for each $1 \leq i \leq n$, $\lambda - \boldsymbol{e}_i$ is in $\Lambda_Q^{(\lambda)}$, provided $\lambda_i \geq 1$.

If $\lambda \in \Lambda_Q$, then, by Corollary 1, there is a possibility that an element of $\mathrm{Neighbor}(\{\lambda\})$ belongs to Λ_Q.

Lemma 3 *Using the same notations as in Corollary 1, let* $\partial^\alpha + \sum_{\alpha \succ \beta} c_\beta \partial^\beta \notin \mathrm{NT}_Q$ *where* $c_\beta \in K[x]$. *Then, for all* $\lambda \in \{\alpha + \gamma \mid \gamma \in \mathbb{N}^n\}$, $\lambda \notin \Lambda_Q$.

Proof Set $A^{(0)} = \{\alpha\}$ and $A^{(k+1)} = \mathrm{Neighbor}(A^{(k)})$ for $k \in \mathbb{N}$. We prove $A^{(k)} \cap \Lambda_Q = \emptyset$ with induction on k. If $k = 0$, clearly $A^{(0)} \cap \Lambda_Q = \emptyset$. Assume that, for k, $A^{(k)} \cap \Lambda_Q = \emptyset$. Then, for all $\lambda \in A^{(k+1)}$, there exists $i \in \{1, \ldots, n\}$ such that $\lambda - \boldsymbol{e}_i \in A^{(k)}$. As $A^{(k)} \cap \Lambda_Q = \emptyset$, hence, by Corollary 1, $\lambda \notin \Lambda_Q$, namely, $A^{(k+1)} \cap \Lambda_Q = \emptyset$. Therefore, for all $k \in \mathbb{N}$, $A^{(k)} \cap \Lambda_Q = \emptyset$.

Since $\bigcup_{k=0}^{\infty} A^{(k)} = \{\alpha + \gamma \mid \gamma \in \mathbb{N}^n\}$, the lemma holds. □

Let $\lambda \notin \Lambda_Q$, then, a set of candidates of head exponents can be reduced by λ. This fact makes up the following algorithm.

Sub-algorithm (Headcandidate)

Specification: Headcandidate($\tau, \Lambda^{(\tau)}, \mathrm{FL}$)
Making new candidates for head exponents.
Input: $\tau \in \Lambda \subset \mathbb{N}^n$, $\Lambda^{(\tau)} = \{\lambda' \in \Lambda \mid \tau \succ \lambda'\}$, $\mathrm{FL} = \{\alpha \in \mathbb{N}^n \mid \alpha \notin \Lambda^{(\tau)}\}$.
Output: CT: set of new candidates for head exponents.
BEGIN
$\mathrm{CT} \leftarrow \emptyset$; $B \leftarrow \mathrm{Neighbor}(\{\tau\})$; $B \leftarrow B \backslash (B \cap \{\alpha + \gamma \mid \alpha \in \mathrm{FL}, \gamma \in \mathbb{N}^n\})$;
  while $B \neq \emptyset$ **do**
    select $\tau' = (\tau'_1, \tau'_2, \ldots, \tau'_n)$ from B; $B \leftarrow B \backslash \{\tau'\}$;
    for each i **from** 1 **to** n **do** Flag$\leftarrow 1$;
      if $\tau'_i \neq 0$ **then**
        if $\tau' - \boldsymbol{e}_i \notin \Lambda^{(\tau)}$ **then**
         Flag$\leftarrow 0$; **break**;
        end-if
      end-if
    end-for
    if Flag$= 1$ **then**
      $\mathrm{CT} \leftarrow \mathrm{CT} \cup \{\tau'\}$;
    end-if
  end-while
return CT;
END

Remark 1 The main algorithm "**Noether**" that will be introduced in Section 4.3 decides head exponents of NT_Q, from bottom to up w.r.t. a term order $\succ$. Hence, the sets $\Lambda^{(\tau)}$ and FL are already obtained when the algorithm above makes the set of the candidates.

4.2 Lower Exponents of Noetherian Operators

Here we make a set of possible candidates of lower exponents in NT_Q.

Lemma 2 (ii) yields the following corollary.

Corollary 2 *Using the same notations as in Corollary 1, let Γ_Q denote the set of exponents of lower terms in* NT_Q *and $\Gamma_Q^{(\lambda)}$ denote a subset of Γ_Q : $\Gamma_Q^{(\lambda)} = \{\lambda' \in \Gamma_Q \mid \lambda \succ \lambda'\}$. If $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_n) \in \Gamma_Q$, then*

(C) *for each $i = 1, 2, \ldots, n$, $\lambda - \boldsymbol{e}_i$ is in $\Gamma_Q^{(\lambda)} \cup \Lambda_Q^{(\lambda)}$, provided $\lambda_i \geq 1$.*

Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_\ell\}$ be the reduced basis of the vector space of Noetherian operators for the zero-dimensional primary ideal Q w.r.t. a term order $\succ$ and

$$\mathrm{ht}(\psi_\ell) \succ \cdots \succ \mathrm{ht}(\psi_{i+1}) \succ \mathrm{ht}(\psi_i) \succ \cdots \succ \mathrm{ht}(\psi_2) \succ \mathrm{ht}(\psi_1).$$

Set $\Psi^{(\psi_{i+1})} = \{\psi_i, \ldots, \psi_1\}$,

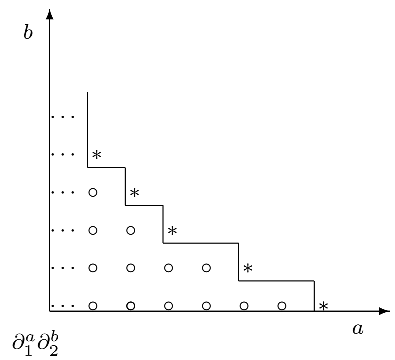
$$\begin{aligned}
B_h &= \mathrm{Neighbor}(\mathrm{Expo}(\mathrm{ht}(\Psi^{(\psi_{i+1})}))) \setminus \mathrm{Expo}(\mathrm{ht}(\Psi^{(\psi_{i+1})})),\\
B_l &= \mathrm{Neighbor}(\mathrm{Expo}(\mathrm{LL}(\Psi^{(\psi_{i+1})}))) \setminus \mathrm{Expo}(\mathrm{ht}(\Psi^{(\psi_{i+1})})),\\
\mathrm{FL} &= \{\lambda \in B_h \mid \mathrm{hexp}(\mathrm{ht}(\psi_{i+1})) \succ \lambda, \gamma \in B_h \setminus \{\lambda\}, \gamma \nmid \lambda\},\\
\mathrm{EL} &= \{\beta \in B_l \mid \text{ if } \beta_i \geq 1, \beta - \boldsymbol{e}_j \in \mathrm{Expo}(\mathrm{Term}(\Psi^{(\psi_{i+1})}))\}.
\end{aligned}$$

Since an element of $\mathrm{Expo}(\mathrm{LL}(\Psi^{(\psi_i)})) \cup \mathrm{FL} \cup \mathrm{EL}$ satisfies the condition **(C)** of Corollary 2, the lower exponents of ψ_{i+1} are in

$$\mathrm{Expo}(\mathrm{LL}(\Psi^{(\psi_i)})) \cup \mathrm{FL} \cup \mathrm{EL}\,.$$

Note that, an element of the set FL is given by the sub-algorithm **Head-candidate**, namely, the element becomes a candidate of a head exponent but the element can not become a head term of Ψ w.r.t. $\succ$. In **Fig.** 1, an element of $\mathrm{ht}(\Psi^{(\psi_i)})$ are displayed as "$\circ$" and an exponent of $\mathrm{FL} \subset \mathbb{N}^2$ are displayed as "$*$". In general, an element of FL is said to be a corner.

Fig. 1 example of exponents in $\mathrm{ht}(\Psi^{(\psi_i)})$ and FL



In fact, since an element of FL is from a set CT of candidates of head exponents, thus, we make only the set EL for deciding lower exponents.

Remark 2 There is a possibility that an element in $B_l \setminus \mathrm{EL}$ becomes a lower exponent of partial differential operators, because the element may be smaller than the head exponents. Thus, we need to keep the set $B_l \setminus \mathrm{EL}$.

Remark 3 We do not need to select head exponents that already obtained as candidates of lower terms because we are considering a reduced basis of the vector space. Thus, elements of $\mathrm{Expo}(\mathrm{ht}(\Psi^{(\psi_{i+1})}))$ are deleted from the set $\mathrm{Neighbor}(\mathrm{Expo}(\mathrm{LL}(\Psi^{(\psi_{i+1})})))$.

4.3 Algorithm for Computing Noetherian Operators

Here, we present an algorithm for computing Noetherian operators. Broadly speaking, although there are some differences in the details, the plot of the algorithm is very similar to FGLM algorithm [17,34,35].

We utilize Theorem 5 (Remark 4), Corollary 1 and Corollary 2 to compute the Noetherian operators. The main algorithm **Noether** consists of mainly three blocks, computing candidates for head exponents, computing candidates for lower exponents and solving a system of linear equations. For each block, the algorithm makes use of several sets as intermediate data. As this is a dynamic algorithm, each intermediate data is often renewed in the algorithm. We fix the meaning of the sets as follows.

$\mathrm{CT} := \{\lambda \in \mathbb{N}^n \,|\lambda$ is a candidate for head exponents of a basis$\}$,
$\mathrm{CL} := \{\tau \in \mathbb{N}^n \,|\tau$ is a candidate for lower exponents for some $\lambda \in \mathrm{CT}\}$,
$\mathrm{FL} := \{\lambda \in \mathbb{N}^n \,|\lambda$ is a failed candidate for head exponents$\}$,
EL: described in Section 4.2.

In the following algorithm, sets UU, E are used for algorithmic consistency, to decide lower exponents. The sub-algorithm "**DetermineP**" that is utilized in Algorithm 1 "**Noether**", determines indeterminates c_τs that are coefficients of the partial differential operator ψ.

Remark 4 Let $I = \langle f_1, \ldots, f_t\rangle \subset K[x]$ and Q a primary component of I where $\sqrt{Q} = \mathfrak{p}$. If a partial differential operator ψ is in the reduced basis NB_Q of NT_Q w.r.t. a term order $\succ$, then ψ satisfies the following condition $(N^{(*)})$

$(N^{(*)})$ "$\psi(f_i) \in \mathfrak{p}$ and $[\psi, x_j] \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_Q)$"

where $1 \leq i \leq t$, $1 \leq j \leq n$.

Remark 5 It is obvious that $\partial^{(0,0,\ldots,0)} = 1 \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_Q)$, and hence, $\boldsymbol{e}_1$, $\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ become candidates of head exponents of the Noetherian operators.

Algorithm 1 (Noether)

Input: $I = \langle f_1, f_2, \ldots, f_t\rangle \subset K[x]$: zero-dimensional ideal,
$\succ$: term order on $\mathbb{N}^n$.
Output: $\{(\mathfrak{p}_i, \mathrm{NB}_{Q_i}) \mid \mathfrak{p}_i = \sqrt{Q_i}, \mathrm{NB}_{Q_i}$ is a reduced basis of the Noetherian operators of $Q_i, 1 \leq i \leq r\}$ where $\bigcap_{i=1}^r Q_i$ is a minimal primary decomposition of I and Q_is are primary ideals.

BEGIN
$\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\} \leftarrow$ Compute the prime decomposition of $\sqrt{I}$;
  /* i.e. $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$ where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are prime ideals */
 for each i **from** 1 **to** r **do**
  $\mathrm{NB}_{Q_i} \leftarrow \{1\}$; $\mathrm{CT} \leftarrow \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$; $\mathrm{CL} \leftarrow \emptyset$; $\mathrm{FL} \leftarrow \emptyset$; $\mathrm{UU} \leftarrow \emptyset$;
   while $\mathrm{CT} \neq \emptyset$ **do**
    $\lambda \leftarrow$ Take the smallest element in CT w.r.t. $\succ$; $\mathrm{CT} \leftarrow \mathrm{CT} \setminus \{\lambda\}$;
    $\mathrm{E} \leftarrow \{\gamma \in \mathrm{UU} \,|\, \lambda \succ \gamma\}$;
    $\mathrm{UU} \leftarrow \mathrm{UU} \setminus \mathrm{E}$;
    $\mathrm{EL} \leftarrow \{\gamma \in \mathrm{E} |$ if $\gamma_i \geq 1, \gamma - \boldsymbol{e}_i \in \mathrm{Expo}(\mathrm{Term}(\mathrm{NB}_i)), 1 \leq i \leq n\}$;
    $\mathrm{CL} \leftarrow \mathrm{CL} \cup \mathrm{EL}$;
    $\psi \leftarrow \partial^\lambda + \sum_{\tau \in \mathrm{CL}} c_\tau \partial^\tau$;   /* ($c_\tau$s are indeterminates) */
    $\psi' \leftarrow$ **DetermineP**$(\{f_1, \ldots, f_t\}, \psi, \mathfrak{p}_i, \mathrm{NB}_{Q_i}, \{c_\tau | \tau \in \mathrm{CL}\})$;
      /*by checking the condition $(N^{(*)})$, determine c^τ*/
      /* if $\psi' = 0$, then ψ is not a Noetherian operator */
    if $\psi' \neq 0$ **then**
     $\mathrm{NB}_{Q_i} \leftarrow \mathrm{NB}_{Q_i} \cup \{\psi'\}$;
     $\mathrm{CT} \leftarrow$ **Headcandidate(**$\lambda, \mathrm{Expo}(\mathrm{ht}(\mathrm{NB}_i)), \mathrm{FL}) \cup \mathrm{CT}$;
     $\mathrm{UU} \leftarrow \mathrm{Neighbor}(\mathrm{Expo}(\mathrm{LL}(\psi'))) \cup \mathrm{UU}$;
    else
     $\mathrm{FL} \leftarrow \mathrm{FL} \cup \{\lambda\}$; $\mathrm{CL} \leftarrow \mathrm{CL} \cup \{\lambda\}$;
    end-if
   end-while
 end-for
return $\{(\mathfrak{p}_1, \mathrm{NB}_{Q_1}), \ldots, (\mathfrak{p}_r, \mathrm{NB}_{Q_r})\}$;
END

Sub-algorithm (DetermineP)

Specification: DetermineP$(\{f_1, f_2, \ldots, f_t\}, \psi, \mathfrak{p}, \mathrm{NB}_Q, \{c_\tau | \tau \in \mathrm{CL}\})$
Determining c_τs that are coefficients of the partial differential operator ψ.

Input: $\{f_1, f_2, \ldots, f_t\}, \psi, \mathfrak{p}, \mathrm{NB}_Q, \{c_\tau | \tau \in \mathrm{CL}\}$: described in Algorithm 1.
Output: P': if $\psi' = 0$, then ψ is not a Noetherian operator for Q_i, otherwise ψ' is a Noetherian operator for Q_i where $\mathrm{ht}(\psi') = \mathrm{ht}(\psi)$.

BEGIN
$L \leftarrow \emptyset$; $C \leftarrow \emptyset$; $\{q_1, q_2, \ldots, q_s\} \leftarrow \mathrm{NB}_Q$; /* $|\,\mathrm{NB}_Q\,| = s$ */
 for each i **from** 1 **to** s **do**
  $g \leftarrow$ Compute the normal form of $\psi(f_i)$ w.r.t. $\mathfrak{p}$;
   if $g \neq 0$ **then**
    $L \leftarrow L \cup \{g = 0\}$;
   end-if
 end-for
 for each j **from** 1 **to** n **do**
  $b_j \leftarrow [\psi, x_j]$;
  $C \leftarrow C \cup \{b_j\}$;

end-for
$\boldsymbol{v}=(\partial^{\alpha 1}\ \partial^{\alpha 2}\cdots\ \partial^{\alpha \ell})\leftarrow$ Make a vector from $\mathrm{Term}(C)=\{\partial^{\alpha 1},\cdots,\partial^{\alpha \ell}\}$;
$M\leftarrow$ Get the $\ell\times(s+n)$ matrix that satisfies $(q_1\cdots q_s\ b_1\ \cdots b_n)=\boldsymbol{v}M$;
$\left(\begin{array}{c|c} E_s & \cdots \\ \hline 0 & A \end{array}\right)\leftarrow$ Reduce M by elementary operations of matrix over $K[x]/\mathfrak{p}$;
$L\leftarrow L\cup\{a'=0\mid a'$ is a component of the matrix $A\}$;
if the system of linear equations L has no solution **then** $(\triangle)$
return 0;
else
$\psi'\leftarrow$ Get the solution of L and substitute the solution into c_τs of ψ;
return ψ';
end-if
END

Remark 6 As we mentioned in Section 2.1, algorithms of [4,26,48] for computing the prime decomposition of $\sqrt{I}$, is much faster than algorithms for computing a primary decomposition of I. One can utilize the effective algorithm introduced in [4].

Remark 7 In the sub-algorithm, E_s represents the identity matrix of size s. Then, A is the zero matrix if and only if $b_1,b_2,\ldots,b_n\in\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_Q)$. This fact comes from linear algebra. Hence, the sub-algorithm, actually, check the condition $(N^{(*)})$ (see *Remark* 4). Notice that the sub-algorithm consists of linear algebra computations except for computing a normal form of $\psi(f_i)$ w.r.t $\mathfrak{p}$ where $1\leq i\leq s$. This is a big advantage of Algorithm 1. We give the results of benchmark tests in Section 5.

Theorem 6 *Algorithm 1 terminates and outputs correctly.*

Proof As the input I is zero-dimensional, thus each primary component of the minimal primary decomposition $I=\bigcap_{i=1}^r Q_i$ is zero-dimensional, too. By utilizing an algorithm [4] for computing the prime decomposition of $\sqrt{I}$, all prime components $\sqrt{Q_1}=\mathfrak{p}_1,\ldots,\sqrt{Q_r}=\mathfrak{p}_r$ are obtained by finite steps. For each $\mathfrak{p}_i$, we compute a basis of the vector space NT_{Q_i}.

Algorithm 1 is designed for computing a reduced basis of the finite dimensional vector space NT_{Q_i} from bottom to up w.r.t. $\succ$. It is obviously that the reduced basis is unique if we fix $\succ$.

The smallest element 1 must be contained in NB_{Q_i}, and other partial differential operators of NB_{Q_i} are decided from bottom to up w.r.t. $\succ$ by utilizing the condition $(N^{(*)})$. As we mentioned in Theorem 5, NB_{Q_i} is decided by the condition "$[\psi,x_i]\in\mathrm{Span}_{K[x]/\mathfrak{p}_i}(\mathrm{NB}_{Q_i})$ and $\psi(f_j)\in\mathfrak{p}_i$". Furthermore, if we select ∂^λ as a head term, then ∂^λ never appear subsequent CT. Thus, NB_{Q_i} does not contain elements ψ,φ such that $\mathrm{ht}(\psi)=\mathrm{ht}(\varphi)$, and $\mathrm{ht}(\mathrm{NB}_{Q_i})\cap\mathrm{LL}(\mathrm{NB}_{Q_i})=\emptyset$ in the computation. Hence, the system L, at $(\triangle)$, does not have multiple solutions in sub-algorithm **DetermineP**. This means that L has the unique solution or no solution. Hence, the sub-algorithm **DetermineP** works correctly.

Since NT_{Q_i} is finite dimensional, therefore Algorithm 1 terminates and outputs correctly. $\square$

We illustrate the algorithm **Noether** with the following example.

Example 1 Let us consider a zero-dimensional ideal $I = \langle f, g\rangle$ in $\mathbb{Q}[x, y]$ where $f = (x^2+y^2)^2+3x^2y-y^3, g = x^2+y^2-1$. Let $\succ$ be the graded lexicographic term order with $(1, 0) \succ (0, 1)$ on $\mathbb{N}^2$ where $(1, 0), (0, 1)$ correspond to $\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}$.

By utilizing the algorithm introduced in [4], we obtain the prime decomposition of $\sqrt{I}$ as follows

$$\sqrt{I} = \langle x, y-1\rangle \cap \langle 4x^2-3, 2y+1\rangle$$

where $\langle x, y-1\rangle$ and $\langle 4x^2-3, 2y+1\rangle$ are prime ideals in $\mathbb{Q}[x, y]$. Let $\mathfrak{p}_1 = \langle x, y-1\rangle$ and $\mathfrak{p}_2 = \langle 4x^2-3, 2y+1\rangle$. As I is zero-dimensional, I can be written as

$$I = Q_1 \cap Q_2$$

where Q_1 is $\mathfrak{p}_1$-primary and Q_2 is $\mathfrak{p}_2$-primary.

Let us compute bases of Noetherian operators for Q_1 and Q_2.

(1) First, we compute the reduced basis NB_{Q_1} of the vector space NT_{Q_1} w.r.t. $\succ$. Remark that we do not have the generators of the primary ideal Q_1. Set

$$\mathrm{NB}_{Q_1} = \{1\},\ \mathrm{CT} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\},\ \mathrm{FL} = \emptyset,\ \mathrm{CL} = \emptyset.$$

(1-i) Take the smallest exponent $\boldsymbol{e}_2 = (0, 1)$ in CT and update $\mathrm{CT} = \{(0, 1)\}$. Set $\psi = \partial_x^0\partial_y^1 = \partial_y$ and check the condition $(N^{(*)})$, then

$$\psi(f) = 4x^2y + 3x^2 + 4y^3 - 3y^2 \notin \mathfrak{p}_1,\ \psi(g) = 2y \notin \mathfrak{p}_1,$$
$$[\psi, x] = 0 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}),\quad [\psi, y] = 1 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}).$$

Hence, ψ does not satisfy the condition $(N^{(*)})$. Update $\mathrm{FL} = \{(0, 1)\}$ and $\mathrm{CL} = \{(0, 1)\}$.

(1-ii) Take the smallest exponent $\lambda = (1, 0)$ in CT and update $\mathrm{CT} = \emptyset$. Set $\psi = \partial^\lambda + \sum_{\tau\in\mathrm{CL}} c_\tau\partial^\tau = \partial_x + c_{(0,1)}\partial_y$ and check the condition $(N^{(*)})$ where $c_{(0,1)}$ is an indeterminate. Then,

$$\psi(f) \equiv c_{(0,1)} \mod \mathfrak{p}_1,\ \psi(g) \equiv 2c_{(0,1)} \mod \mathfrak{p}_1,$$
$$[\psi, x] = 1 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}),\quad [\psi, y] = c_{(0,1)} \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}).$$

Hence, when $c_{(0,1)} = 0$, ψ satisfies the condition $(N^{(*)})$. Therefore, ∂_x is a Noetherian operator of Q_1. Renew

$$\mathrm{CT} := \mathbf{Headcandidate}((1, 0), \mathrm{Expo}(\mathrm{ht}(\mathrm{NB}_{Q_1})), \mathrm{FL}) \cup \mathrm{CT} = \{(2, 0)\},$$

and update $\mathrm{NB}_{Q_1} = \{1, \partial_x\}$.

(1-iii) Take the smallest exponent $\lambda = (2, 0)$ in CT and update $\mathrm{CT} = \emptyset$. Set $\psi = \partial^\lambda + \sum_{\tau\in\mathrm{CL}} c_\tau\partial^\tau = \partial_x^2 + c_{(1,0)}\partial_y$ and check the condition $(N^{(*)})$. Then,

$$\psi(f) \equiv c_{(0,1)} + 10 \mod \mathfrak{p}_1,\ \psi(g) \equiv 2c_{(0,1)} + 2 \mod \mathfrak{p}_1,$$
$$[\psi, x] = 2\partial_x \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}),\quad [\psi, y] = c_{(0,1)} \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_1}).$$

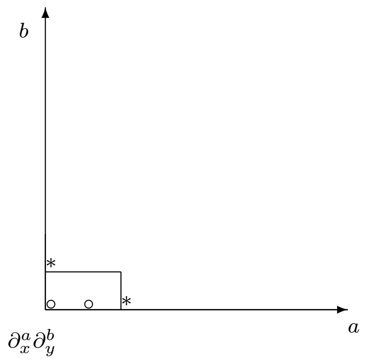
Hence, we need to solve the system of linear equations $c_{(0,1)} + 10 = 0$, $2c_{(0,1)} + 2 = 0$ over $\mathbb{Q}[x,y]/\mathfrak{p}_1$. Clearly, the system does not have any solution. Hence, ψ does not satisfy the condition $(N^{(*)})$. Update $\mathrm{FL} = \{(2,0),(0,1)\}$

Since, now, CT is empty, we stop the computation for Q_1. The reduced basis of the Noetherian operators of Q_1 is

$$\mathrm{NB}_{Q_1} = \{1, \partial_x\}.$$

In **Fig.** 2, an element of $\mathrm{ht}(\mathrm{NB}_{Q_1})$ is displayed as $\circ$ and an element of FL is displayed as $*$.

Fig. 2 Exponents of $\mathrm{ht}(\mathrm{NB}_{Q_1})$ and FL



(2) Second, we compute a reduced basis NB_{Q_2} of the vector space NT_{Q_2}. Set

$$\mathrm{NB}_{Q_2} = \{1\};\ \mathrm{CT} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\};\ \mathrm{FL} = \emptyset;\ \mathrm{CL} = \emptyset;$$

(2-i) Take the smallest exponent $\boldsymbol{e}_2 = (0,1)$ in CT and update $\mathrm{CT} = \{(1,0)\}$. Set $\psi = \partial_x^0\partial_y^1 = \partial_y$ and check the condition $(N^{(*)})$, then

$$\psi(f) = 4x^2y + 3x^2 + 4y^3 - 3y^2 \notin \mathfrak{p}_2,\ \psi(g) = 2y \notin \mathfrak{p}_2,$$
$$[\psi, x] = 0 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_2}),\quad [\psi, y] = 1 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_1}(\mathrm{NB}_{Q_2}).$$

Hence, ψ does not satisfy the condition $(N^{(*)})$. Update $\mathrm{FL} = \{(0,1)\}$ and $\mathrm{CL} = \{(0,1)\}$.

(2-ii) Take the smallest exponent $\lambda = (1,0)$ in CT and update $\mathrm{CT} = \emptyset$. Set $\psi = \partial^\lambda + \sum_{\tau \in \mathrm{CL}} c_\tau \partial^\tau = \partial_x + c_{(0,1)}\partial_y$ and check the condition $(N^{(*)})$ where $c_{(0,1)}$ is an indeterminate. Then,

$$\psi(f) \equiv x - \tfrac{1}{2}c_{(0,1)} \mod \mathfrak{p}_2,\ \psi(g) \equiv 4x - 2c_{(0,1)} \mod \mathfrak{p}_2,$$
$$[\psi, x] = 1 \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_2}(\mathrm{NB}_{Q_2}),\quad [\psi, y] = c_{(0,1)} \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_2}(\mathrm{NB}_{Q_2}).$$

Hence, we need to solve the system of linear equations $x - \frac{1}{2}c_{(0,1)} = 0,\ 4x - 2c_{(0,1)} = 0$ over $\mathbb{Q}[x,y]/\mathfrak{p}_2$. The solution is $c_{(0,1)} = 2x$. Therefore, $\partial_x + 2x\partial_y$ is a Noetherian operator of Q_2. Renew

$$\mathrm{CT} := \textbf{Headcandidate}((1,0), \mathrm{Expo}(\mathrm{ht}(\mathrm{NB}_{Q_2})), \mathrm{FL}) \cup \mathrm{CT} = \{(2,0)\},$$

and update $\mathrm{NB}_{Q_2} = \{1, \partial_x + 2x\partial_y\}$. Moreover, as $\mathrm{LL}(\partial_x + 2x\partial_y) = \{\partial_y\} = \{\partial^{(0,1)}\} \neq \emptyset$, we have to make new candidates of lower terms. Since $\mathrm{Neighbor}(\{(0,1)\}) = \{(1,1),(0,2)\}$ and $(1-1,0),(1,1-1),(0,2-1) \in \mathrm{Expo}(\mathrm{Term}(\mathrm{NB}_{Q_2}))$, we obtain

$$\mathrm{EL} = \{(1,1),(0,2)\},\ \mathrm{CL} := \mathrm{EL} \cup \mathrm{CL} = \{(1,1),(0,2),(1,0)\}.$$

(2-iii) Take the smallest exponent $\lambda = (2,0)$ in CT and update $\mathrm{CT} = \emptyset$. Set $\psi = \partial^\lambda + \sum_{\tau \in \mathrm{CL}} c_\tau \partial^\tau = \partial_x^2 + c_{(1,1)}\partial_x\partial_y + c_{(0,2)}\partial_y^2 + c_{(0,1)}\partial_y$ where $c_{(1,1)}, c_{(0,2)}, c_{(0,1)}$ are indeterminates. We check whether $\psi(f), \psi(g) \in \mathfrak{p}_2$ or not. As

$\psi(f) \equiv 2c_{(0,2)} - c_{(0,1)} + 2 \mod \mathfrak{p}_2$,
$\psi(g) \equiv 16x^2c_{(0,2)} + 4x^2c_{(0,1)} + 48x^2 + 8xc_{(1,1)} + 24c_{(0,2)} - 5c_{(0,1)} - 8 \mod \mathfrak{p}_2$,

we obtain linear equations $\{2c_{(0,2)} - c_{(0,1)} + 2 = 0,\ 16x^2c_{(0,2)} + 4x^2c_{(0,1)} + 48x^2 + 8xc_{(1,1)} + 24c_{(0,2)} - 5c_{(0,1)} - 8 = 0\}$ over $\mathbb{Q}[x,y]/\mathfrak{p}_2$.
Next, we check whether $[\psi, x], [\psi, y] \in \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}_2}(\mathrm{NB}_{Q_2})$ or not. We have

$$[\psi, x] = 2\partial_x + c_{(1,1)}\partial_y, \quad [\psi, y] = c_{(1,1)}\partial_x + 2\partial_y + c_{(0,1)},$$

and

$$(1, \partial_x + 2x\partial_y, [\psi,x], [\psi,y]) = (1\ \partial_x\ \partial_y)\begin{pmatrix} 1 & 0 & 0 & c_{(0,1)} \\ 0 & 1 & 2 & c_{(1,1)} \\ 0 & 2x & c_{(1,1)} & 2 \end{pmatrix}.$$

By utilizing Gaussian elimination method over $\mathbb{Q}[x,y]/\mathfrak{p}_2$, the matrix above can be transformed as

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & c_{(0,1)} \\ 0 & 1 & 2 & c_{(1,1)} \\ \hline 0 & 0 & c_{(1,1)} - 4x & 2 - 2xc_{(1,1)} \end{array}\right).$$

If ψ is a Noetherian operator, then the 4th row vector must be zero. Thus, we have to solve the following system of linear equations over $\mathbb{Q}[x,y]/\mathfrak{p}_2$

$$\begin{cases} 2c_{(0,2)} - c_{(0,1)} + 2 = 0, \\ 16x^2c_{(0,2)} + 4x^2c_{(0,1)} + 48x^2 + 8xc_{(1,1)} + 24c_{(0,2)} - 5c_{(0,1)} - 8 = 0, \\ c_{(1,1)} - 4x = 0, \\ 2 - 2xc_{(1,1)} = 0. \end{cases}$$

One can solve this system of linear equations by utilizing Gaussian elimination method over $\mathbb{Q}[x,y]/\mathfrak{p}_2$ or by computing a Gröbner basis. Here, let us compute the reduced Gröbner basis of

$\langle\{2c_{(0,2)}-c_{(0,1)}+2, 16x^2c_{(0,2)}+4x^2c_{(0,1)}+48x^2+8xc_{(1,1)}+24c_{(0,2)}-5c_{(0,1)}-8, c_{(1,1)}-4x, 2-2xc_{(1,1)}\}\cup\{4x^2-3, 2y+1\}\rangle\subset\mathbb{Q}[c_{(1,1)}, c_{(0,2)}, c_{(0,1)}, x, y]$

w.r.t. a block term order with $\{c_{(1,1)}, c_{(0,2)}, c_{(0,1)}\}\gg\{x, y\}$. Then, the Gröbner basis is $\{1\}$, therefore the system has no solution. Update FL $= \{(2,0),(0,1)\}$.

Since, now CT is empty, we stop the computation. The reduced basis of the Noetherian operators for Q_2 is

$$\mathrm{NB}_{Q_2}=\{1, \partial_x+2x\partial_y\}.$$

Therefore, we get the following set of pairs for I

$$\{(\mathfrak{p}_1,\{1,\partial_x\}),(\mathfrak{p}_2,\{1,\partial_x+2x\partial_y\})\}.$$

The result $\{(\mathfrak{p}_1,\{1,\partial_x\}),(\mathfrak{p}_2,\{1,\partial_x+2x\partial_y\})\}$ can be regard as a primary decomposition of I because the pairs $(\mathfrak{p}_1,\{1,\partial_x\})$ and $(\mathfrak{p}_2,\{1,\partial_x+2x\partial_y\})$ are essentially same as the primary ideals Q_1 and Q_2, respectively. Hence, we can regard I in the same light as $\{(\mathfrak{p}_1,\{1,\partial_x\}),\ (\mathfrak{p}_2,\ \{1,\partial_x+2x\partial_y\})\}$.

Definition 8 (Noetherian representation) Let I be a zero-dimensional ideal in $K[x]$, $I=\bigcap_{i=1}^r Q_i$ a minimal primary decomposition of I where for each $i\in\{1,2,\dots,r\}$, Q_i is $\mathfrak{p}_i$-primary. Moreover, let NB_{Q_i} be a basis of the vector space NT_{Q_i} in $\mathcal{D}$. Then, a finite set $\{(\mathfrak{p}_1,\mathrm{NB}_{Q_1}),\dots,(\mathfrak{p}_r,\mathrm{NB}_{Q_r})\}$ of pairs is called a *Noetherian representation* of I and written by Noether(I).

Remark 8 A Noetherian representation $\{(\mathfrak{p}_1,\mathrm{NB}_{Q_1}),\dots,(\mathfrak{p}_r,\mathrm{NB}_{Q_r})\}$ of I is essentially the same as a *multiplicity variety* introduced by L. Ehrenpreis that is the collection $\{(\mathbb{V}(\mathfrak{p}_1),\mathrm{NB}_{Q_1}),\dots,(\mathbb{V}(\mathfrak{p}_r),\mathrm{NB}_{Q_r})\}$ where for each $1\le i\le r$, $\mathbb{V}(\mathfrak{p}_i)=\{a\in K^n \mid g(a)=0,\forall g\in\mathfrak{p}_i\}$. Since we make use it for several symbolic computations, we adopt the name *Noetherian representation* for polynomial ideals.

Note that the concept of Macaulay basis which plays a key role in the study of zero-dimensional varieties, introduced in [3], is similar to that of the Noetherian representation.

Algorithm 1 that outputs a Noetherian representation of a zero-dimensional ideal has been implemented in a computer algebra system Risa/Asir [40] by the first author[1].

Example 2 Let us consider zero-dimensional ideal

$I=\langle 16x^3y^6+4x^2y^6+16x^3y^3+4y^6+4x^2y^3+4x^3+4y^3+x^2+1, 48x^8+24x^7+3x^6+24x^5+2x^2y^3+6x^4+4x^2\rangle$

in $\mathbb{Q}[x,y]$. We use the total degree lexicographic term order with $\partial_x\succ\partial_y$. Then, the Risa/Asir implementation outputs the following Noetherian representation of I:

[1] Currently, the Risa/Asir implementation is in the following URL
`https://www.rs.tus.ac.jp/~nabeshima/softwares.html`

$\{(\mathfrak{p}_1, \{\partial_x^4 + (12x^2y - 432xy - 36y)\partial_x^2\partial_y + (96xy - 1728y)\partial_x\partial_y + (62787/4x^2y^2 + 2589xy^2 + 1299/4y^2)\partial_y^2 + (-51267/2x^2y - 4218xy - 1251/2y)\partial_y, \partial_x^3 + (6x^2y - 216xy - 18y)\partial_x\partial_y + (24xy - 432y)\partial_y, \partial_x^2 + (2x^2y - 72xy - 6y)\partial_y, \partial_x, 1\}),$
$(\mathfrak{p}_2, \{\partial_x\partial_y, \partial_x, \partial_y, 1\})\}$

where $\mathfrak{p}_1 = \langle 2y^3+1, 4x^3+x^2+1\rangle$, $\mathfrak{p}_2 = \langle 2y^3+1, x\rangle$ are prime and $\sqrt{I} = \mathfrak{p}_1 \cap \mathfrak{p}_2$.

There are several results of zero-dimensional polynomial ideals defined by a dual basis in [3,31,32,34,35], in the context of symbolic computation.

The method for computing a Macaulay basis of a zero-dimensional ideal introduced in [3] requires "computing the roots of the zero-dimensional ideal (i.e. an algebraic extension field of $\mathbb{Q}$ is required to solve them)" and "making a change of coordinates so that the root becomes the origin". However, the new algorithm does not need the roots (i.e. an algebraic extensions field of $\mathbb{Q}$) and any change of coordinates, although it needs a prime decomposition instead of them. Moreover, the new algorithm works over field $\mathbb{Q}$ without any change of coordinates, and outputs prime ideals and Noetherian operators in $\mathbb{Q}[x]$ and in $\mathbb{Q}[x][\partial]$, respectively. Hence, one can directly utilize the outputs for computing a sum of ideals, an intersection of ideals and an ideal quotient, that will be discussed in Section 7. These are advantages of the new algorithm.

Another advantage comes from the size of the output. The sum of all bit lengths of the output of the new algorithm is often smaller than a Gröbner basis of an input ideal. For instance, let us consider the following set G

$$G = \{4x^4 + 4x^2y + 4x^2 + y^2 + 2y + 1, -2x^2y^{10} - y^{11} - y^{10} + 4x^2y^7 + 8x^2y^6 + 2y^8 + 20x^2y^5 + 6y^7 - 2x^2y^4 + 14y^6 - 8x^2y^3 + 9y^5 - 28x^2y^2 - 5y^4 - 40x^2y - 18y^3 - 50x^2 - 34y^2 - 45y - 25, y^{15} - 3y^{12} - 6y^{11} - 15y^{10} + 3y^9 + 12y^8 + 42y^7 + 59y^6 + 69y^5 - 27y^4 - 68y^3 - 135y^2 - 150y - 125\}.$$

The set G is the reduced Gröbner basis of the ideal generated by itself w.r.t. the graded lexicographic term order with $x \succ y$ in $\mathbb{Q}[x, y]$, and $\langle G\rangle$ is a primary ideal. Then, the new algorithm outputs the following pair

$$\{\langle y^5 - y^2 - 2y - 5, 2x^2 + y + 1\rangle, \{1, \partial_x, \partial_y, -4x\partial_y^2 + \partial_x\partial_y, (8y + 8)\partial_y^2 + \partial_x^2\}\}.$$

The size of the bit lengths is obviously smaller than that of G. Notice that the number of the Noetherian operators is 5, i.e. the dimension of the vector space NT is 5. This can be interpreted that the multiplicity of a point of $\mathbb{V}(G)$ is 5.

Let us compare a rational univariate representation (RUR) to the Noetherian representations for the zero-dimensional ideal. It is known that the RUR is a wonderful method to represent and analyze a zero-dimensional ideal [46].

The computer algebra system Maple 2020 has the command

`RationalUnivariateRepresentation`

that outputs a RUR of a zero-dimensional ideal.

Maple 2020 outputs the following RUR of $\langle G\rangle$.

```
> RationalUnivariateRepresentation(G,u)
33554432*u^50 + 419430400*u^48 + 2516582400*u^46 + 9667870720*u^
44 + 26738688000*u^42+ 56727961600*u^40 + 96054804480*u^38 + 133
398528000*u^36 + 155061452800*u^34 + 153317212160*u^32 + 1307033
60000*u^30 + 97187430400*u^28 + 63643279360*u^26 + 36982272000*u
^24 + 19172147200*u^22 + 8897588224*u^20 + 3702860800*u^18 + 138
1606400*u^16 + 461024000*u^14 + 136936000*u^12 + 35920000*u^10 +
 8220000*u^8 + 1600000*u^6 + 262500*u^4 + 31250*u^2 + 3125 = 0,

{x = (1600*u^10 + 3200*u^8 + 2400*u^6 + 880*u^4 + 100*u^2)/(1600
*u^9 + 3200*u^7 + 2400*u^5 + 880*u^3 + 100*u), y = (240*u^5 - 80
*u^3 + 400*u)/(1600*u^9 + 3200*u^7 + 2400*u^5 + 880*u^3 + 100*u)}
```

The irreducible factorization of the first univariate polynomial above is factorized as

$$(32u^{10} + 80u^8 + 80u^6 + 44u^4 + 10u^2 + 5)^5.$$

Hence, this means that the multiplicity of the point is 5. However, the output of `RationalUnivariateRepresentation(G,u)` does not give the information, directly. Furthermore, the size of the bit lengths is bigger than the Noetherian representation.

5 Benchmark Tests

Here, we give results of benchmark tests. All results in this paper have been computed on a PC with [OS: Ubuntu Linux, CPU: Intel(R) Core(TM) i5-8265U CPU @ 1.60 GHz, RAM: 16 GB]. The time is given in second. In Table 2, “>10m” means it takes more than 10 minutes.

In [9], the computer algebra system Macaulay2 [20] package NoetherianOperators, which implements the algorithms for computing Noetherian operators introduced in [11,8] (as well as the algorithms for Macaulay dual spaces [27–29]), is published. Table 2 shows comparisons of the Risa/Asir implementation of Algorithm 1 with the Macaulay2 implementation (the command “noetherianOperators” with Strategy =>"PunctualHilbert") in computation time (CPU time). Note that since the Macaulay2 implementation allows only a primary ideal as the input, we use, as input data, the following bases of zero-dimensional primary ideals in $\mathbb{Q}[x, y, z]$ (or $\mathbb{Q}[x, y]$) for the comparisons. We use the total degree lexicographic term order with $\partial_x \succ \partial_y \succ \partial_z$ (or $\partial_x \succ \partial_y$).

```
F1: {(2097152*y+2097152)*x^30+(-15728640*y-15728640)*x^27+(53084
    160*y+53084160)*x^24+(-106168320*y-106168320)*x^21+(13934592
    0*y+139345920)*x^18+(-125411328*y-125411328)*x^15+(78382080*
    y+78382080)*x^12+(-33592320*y-33592320)*x^9+(9447840*y+94478
    40)*x^6+(-1574640*y-1574640)*x^3+32*y^5+80*y^4+80*y^3+40*y^2
    +118108*y+118099,8*y^3+16*y^2+10*y+2}.
F2: {(32768*y+16384)*x^21+(-172032*y-81920)*x^18+(387072*y+17510
```

```
    4)*x^15+(-483840*y-207360)*x^12+(362880*y+146880)*x^9+(-1632
    96*y-62208)*x^6+(40824*y+14580)*x^3+16*y^4+32*y^3+24*y^2-436
    6*y-1457,4096*x^18-18432*x^15+34560*x^12-34560*x^9+19440*x^6
    -5832*x^3+16*y^4+32*y^3+24*y^2+8*y+730)}.
F3: {(y^12-8*y^9+24*y^6-32*y^3+16)*x^12+(-9*y^12+72*y^9-216*y^6+
    288*y^3-144)*x^8+(27*y^12-216*y^9+648*y^6-864*y^3+432)*x^4-2
    7*y^12+216*y^9-648*y^6+864*y^3-432,x^20-15*x^16+90*x^12-270*
    x^8+405*x^4-243,z^18-36*z^15+540*z^12-4320*z^9+19440*z^6-466
    56*z^3+46656,y^18-12*y^15+60*y^12-160*y^9+240*y^6-192*y^3+64
    }.
F4: {x^21-35*x^18+525*x^15-4375*x^12+21875*x^9-65625*x^6+109375*
    x^3-78125,y^10-10*y^8+40*y^6-80*y^4+80*y^2-32,z^20-30*z^18+4
    05*z^16-3240*z^14+17010*z^12-61236*z^10+153090*z^8-262440*z^
    6+295245*z^4-196830*z^2+59049}.
F5: {y^9-6*y^6+12*y^3-8,z^18-36*z^15+540*z^12-4320*z^9+19440*z^6
    -46656*z^3+46656,x^24-18*x^20+135*x^16-540*x^12+1215*x^8-145
    8*x^4+(y^6-4*y^3+4)*x+729}.
F6: {dF/dx,(dF/dy)^2,F^2}
    where F=x^4+y^9+x*y^7+x^2*y^5+x^2*y^6+x^2*y^7.
F7: {(dF/dx)^2,(dF/dy)^2,F^2}
    where F=x^3+y^17+x*y^12+x*y^13+x*y^14+x*y^15.
F8: {(dF/dx)^2,(dF/dy)^2,F^2}
    where F=x^3+y^16+x*y^11+x*y^12+x*y^13+x*y^14.
```

Table 2 Comparisons of Noetherian operators

	Macaulay2	Risa/Asir (Algorithm 1)
F1	1.9101	0.0312
F2	18.429	0.1563
F3	71.774	3.641
F4	680.00	33.22
F5	>10m	5.094
F6	>10m	1.512
F7	>10m	5.172
F8	>10m	5.063

As is evident from Table 2, the Risa/Asir implementation of Algorithm 1 is better in comparison with Macaulay2 implementation. As a reason for that, the main part of Algorithm 1, that is the sub-algorithm "**DetermineP**", consists of linear algebra techniques. That's why Algorithm 1 is quite effective. This is the big advantage of Algorithm 1.

6 Algorithm for Computing Generators

Here we discuss how to compute generators of a primary ideal from the Noetherian representation.

Definition 9 Let $\mathfrak{p} = \langle p_1, \dots, p_\ell \rangle$ be a prime ideal in $K[x]$ and $\mathfrak{p} \neq \langle 1 \rangle$, we let $(K[x]/\mathfrak{p})[p_1, \dots, p_\ell]$ denote the subset of $K[x]$ consisting of all polynomial expression in $p_1, \dots, p_\ell$ with coefficients in $K/\mathfrak{p}$.

Let $\{p_1, \dots, p_\ell\} \subset K[x]$ be the reduced Gröbner basis of a prime ideal $\mathfrak{p}$ w.r.t. a term order $\succ_x$ and $Q \subset K[x]$ a primary ideal with $\sqrt{Q} = \mathfrak{p}$ and $\mathfrak{p} \neq \langle 1 \rangle$. Set

$$p_1 - z_1, p_2 - z_2, \dots, p_\ell - z_\ell$$

where $z_1, \dots, z_n$ are new variables. Then, we can regard $(K[x]/\mathfrak{p})[p_1, \dots, p_\ell]$ as $(K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$.

We apply the tag-variable technique ([35, Section 26.5]) for computing generators of a primary ideal.

Lemma 4 *Let $\succ_{x\cup z}$ be a block term order in $K[x_1, \dots, x_n, z_1, \dots, z_\ell]$ with $\succ_x$ and $x \gg \{z_1, \dots, z_\ell\}$ and G a Gröbner basis of the ideal $\langle p_1 - z_1, p_2 - z_2, \dots, p_\ell - z_\ell \rangle$ w.r.t. $\succ_{x\cup z}$. For $f \in Q$, let $g = \overline{f}^G$ be the remainder of g on division by G. Then,*

(i) $g(z_1, \dots, z_\ell) \in (K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$, *and*
(ii) $f = g(p_1, \dots, p_\ell)$ *is an expression of f as a polynomial in $p_1, \dots, p_\ell$.*

Proof Since $\{p_1, \dots, p_\ell\}$ is the reduced Gröbner basis of an ideal generated by itself w.r.t. $\succ_x$ in $K[x]$, thus the reduced Gröbner basis $G = \{g_1, \dots, g_m\}$, w.r.t. $\succ_{x\cup z}$, contains $p_1 - z_1, p_2 - z_2, \dots, p_\ell - z_\ell$. When we divide $f \in Q$ by the Gröbner basis G, we get an expression of the form

$$f = A_1 g_1 + A_2 g_2 + \cdots A_m g_m + g$$

with $A_1, \dots, A_m, g \in K[x_1, \dots, x_n, z_1, \dots, z_\ell]$. In this case, any coefficient of g in $K[x]$ are not divided by $p_1, \dots, p_\ell$ and any term of g is smaller than $\mathrm{ht}(p_1), \dots, \mathrm{ht}(p_\ell)$. Hence, $g \in (K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$.

From *(i)*, we have $g(z_1, \dots, z_\ell) \in (K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$. For each $1 \leq i \leq \ell$, let substitute p_i for z_i in the above formula for f. This substitution will not affect f since it involves only $x_1, \dots, x_n$, but it sends every polynomial in $\langle g_1, \dots, g_m \rangle$ to zero. Hence, it follows that $f = g(p_1, \dots, p_\ell)$. □

Lemma 5 *Let $I = \langle f_1, \dots, f_t \rangle$ be a zero-dimensional ideal, Q a primary component of I and $\sqrt{Q} = \mathfrak{p} = \langle p_1, \dots, p_\ell \rangle$. Let $\succ_{x\cup z}$ be a block term order with $\succ_x$ and $x \gg \{z_1, \dots, z_\ell\}$, and G a Gröbner basis of the ideal $\langle p_1 - z_1, p_2 - z_2, \dots, p_\ell - z_\ell \rangle$ w.r.t. $\succ_{x\cup z}$. Set $F' = \left\{ \overline{f_1}^G, \overline{f_2}^G, \dots, \overline{f_t}^G \right\}$ in $(K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$. Then, $\mathbb{V}_{K[x]/\mathfrak{p}}(F') \cap X = \{O\}$ where X is a neighborhood of the origin O of $(K[x]/\mathfrak{p})^\ell$, namely, a prime decomposition of $\sqrt{\langle F' \rangle}$ forms*

$$\sqrt{\langle F' \rangle} = \langle z_1, \dots, z_\ell \rangle \cap P'_1 \cap \cdots \cap P'_{\ell'}$$

in $(K[x]/\mathfrak{p})[z_1, \dots, z_\ell]$ where Q'_i is a prime ideal for each $1 \leq i \leq \ell'$.

Proof In order to substitute p_i for z_i in the generators of the prime ideal $\mathfrak{p} = \langle p_1, \ldots, p_\ell \rangle$ in $K[x]$, $\mathfrak{p}$ forms $\mathfrak{p}' = \langle z_1, \ldots, z_\ell \rangle$ in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$. Thus, $\mathbb{V}_{K[x]/\mathfrak{p}}(\mathfrak{p}') = \{O\}$ in $(K[x]/\mathfrak{p})^\ell$.

We show that $O \notin \mathbb{V}_{K[x]/\mathfrak{p}}(\langle F' \rangle : \mathfrak{p}')$ where $\langle F' \rangle : \mathfrak{p}'$ is an ideal quotient. Assume that $\mathbb{V}_{K[x]/\mathfrak{p}}(\langle F' \rangle : \mathfrak{p}')$ contain O in $(K[x]/\mathfrak{p})^\ell$. Then, $\langle z_1, \ldots, z_\ell \rangle \supset \langle F' \rangle : \mathfrak{p}'$. As we just change the variables $z_1, \ldots, z_\ell$ into $p_1, \ldots, p_\ell$, it is clear that $\mathfrak{p} \supset I : \mathfrak{p}$ in $K[x]$. Since I is zero-dimensional, thus this is contradiction. Therefore, $O \notin \mathbb{V}_{K[x]/\mathfrak{p}}(\langle F' \rangle : \mathfrak{p}_i')$, namely, $\mathbb{V}_{K[x]/\mathfrak{p}}(F') \cap X = \{O\}$. □

Now, we are ready to introduce an algorithm for computing the set of generators of Q from the Noetherian representation. We use the sub-algorithm **Headcandidate**, again, that computes candidates of head exponents (neighbors).

Remark that since $z_1 = p_1, z_2 = p_2, \ldots, z_\ell = p_\ell$, we utilize only the symbols $p_1, p_2, \ldots, p_\ell$ (i.e. $p^\lambda = p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_\ell^{\lambda_\ell}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$) in the following algorithm.

Algorithm 2 (inv_Noether)

Input: $\{(\mathfrak{p}, \mathrm{NB})\}$: Noetherian representation of a primary ideal Q where $\{p_1, p_2, \ldots, p_\ell\} \subset K[x]$ is a reduced Gröbner basis of the prime ideal $\mathfrak{p}$ and $\mathrm{NB} = \{\psi_1, \psi_2, \ldots, \psi_r\} \subset \mathcal{D}$. $\succ$: term order on $\mathbb{N}^\ell$.
Output: B: a basis of a primary ideal Q in $K[x]$.

BEGIN
$B \leftarrow \emptyset$; $\mathrm{CT} \leftarrow \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\ell\}$; $S \leftarrow \emptyset$; $T \leftarrow \emptyset$;
  while $\mathrm{CT} \neq 0$ **do**
    $\lambda \leftarrow$ Take the smallest element in CT w.r.t. $\succ$; $\mathrm{CT} \leftarrow \mathrm{CT} \setminus \{\lambda\}$;
    $h \leftarrow p^\lambda + \sum_{\tau \in S} c_\tau p^\tau$; $(\diamond)$ `/*` c_τ: indeterminate, $\lambda \succ \tau, \tau \notin T$ `*/`
    $c_\tau \leftarrow$ Solve the system $\psi_1(h) = 0, \ldots, \psi_r(h) = 0$ on $K[x]/\mathfrak{p}$; $(*1)$
      if "the solution c_τ exists" **then**
        $h' \leftarrow$Substitute the solution into c_τs of h;
        $T \leftarrow T \cup \{\lambda\}$; $B \leftarrow B \cup \{h'\}$;
      else
        $S \leftarrow S \cup \{\lambda\}$;
        $\mathrm{CT} \leftarrow$ **Headcandidate**$(\lambda, S, T) \cup \mathrm{CT}$;
      end-if
  end-while
return B;
END

Note that as we described in Lemma 5, an arbitrary element in $\left\{ \overline{f}^G | f \in Q \right\} \subset (K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$ does not have any non-zero constant term. Hence, $(0, \ldots, 0) \notin S$.

Theorem 7 *Algorithm 2 returns a basis of a primary ideal Q and terminates.*

Proof The output B satisfies the property "$\forall h \in B,\ \psi_1(h), \ldots, \psi_r(h) \in \mathfrak{p}$" because of $(*1)$. Thus, by Theorem 1, $B \subset Q \subset K[x]$.

Let us consider the case $g \notin \langle B \rangle \subset K[x]$. Let G be a Gröbner basis of the ideal $\langle p_1 - z_1, p_2 - z_2, \ldots, p_\ell - z_\ell \rangle$ w.r.t. $\succ_{x \cup z}$ (this term order is from Lemma 5) in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$, and $\overline{B}^G = \{\overline{f}^G | f \in B\} \subset (K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$. Then, by the division algorithm in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$, we can obtain $g' \in (K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$ such that $\overline{g}^G \equiv g' \mod \langle \overline{B}^G \rangle$, and $\exists f \in \overline{B}^G$ s.t. $\mathrm{ht}(f) \succ \mathrm{ht}(g')$. Since we consider all possible polynomials whose head terms are $\overline{p^\lambda}^G = z^\lambda$ and head coefficients are 1, at $(\diamond)$ of the sub-algorithm, in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$ w.r.t. $\succ$ and $g \notin \langle B \rangle \subset K[x]$, there exists $\psi_i \in \mathrm{NB}$ such that $\psi_i(g) \notin \mathfrak{p}$. That is $\forall g \notin \langle B \rangle$ implies $g \notin Q$. Therefore, since $\left\langle \left\{ \overline{f}^G | f \in Q \right\} \right\rangle = \left\langle \overline{B}^G \right\rangle$, we have $Q = \langle B \rangle$. Moreover, since $\mathrm{ht}\left(\overline{B}^G\right) = \{z^\lambda | \lambda \in T\}$ and $\mathrm{LL}(\overline{B}^G) \cap \{z^\lambda | \lambda \in T\} = \emptyset$, $\overline{B}^G$ is the reduced Gröbner basis of $\left\langle \left\{ \overline{f}^G | f \in Q \right\} \right\rangle$ w.r.t. $\succ$ in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$.

As $\left\langle \overline{B}^G \right\rangle$ is zero-dimensional in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$, the set of standard monomials associated to the Gröbner basis $\overline{B}^G$ in $(K[x]/\mathfrak{p})[z_1, \ldots, z_\ell]$ must be finite. Hence, the set S cannot be infinite. Therefore, the set CT becomes an empty set at some point. The algorithm terminates. $\square$

We illustrate Algorithm 2 with the following example.

Example 3 In Section 4, we obtained $\{\langle x^2 - \frac{3}{4}, y + \frac{1}{2} \rangle, \{1, \partial_x + 2x\partial_y\}\}$ as a Noetherian representation of the primary ideal Q_2. Let us compute the set of generators of the primary ideal Q_2 from the Noetherian representation. Set $\mathfrak{p} = \langle x^2 - \frac{3}{4}, y + \frac{1}{2} \rangle$, $p_1 = x^2 - \frac{3}{4}$ and $p_2 = y + \frac{1}{2}$. Let $\succ$ be the total degree lexicographic term order with $(1,0) \succ (0,1)$ on $\mathbb{N}^2$ where $(1,0), (0,1)$ correspond to the symbols p_1 and p_2.

(Initialization) $G = \emptyset$; $\mathrm{CT} = \{(0,1), (1,0)\}$; $S = \emptyset$; $T = \emptyset$;

(1) Take the smallest exponent $(0,1)$ in CT and update $\mathrm{CT} = \{(1,0)\}$. Set $h = p_1^0 p_2^1 = p_2$. Then,

$$1(h) = p_2, (\partial_x + 2x\partial_y)(h) = 2x,$$

thus, $p_2 \equiv 0 \mod \mathfrak{p}$ and $2x \not\equiv 0 \mod \mathfrak{p}$. Hence, $h \notin Q$. Renew $S = \{(0,1)\}$ and

$$\mathrm{CT} = \textbf{Headcandidate}((0,1), S, T) \cup \mathrm{CT} = \{(1,0), (0,2)\}.$$

(2) Take the smallest exponent $(1,0)$ in CT and update $\mathrm{CT} = \{(0,2)\}$. Set $h = p_1 + c_{(0,1)} p_2$ where $c_{(0,1)}$ is an indeterminate. Then,

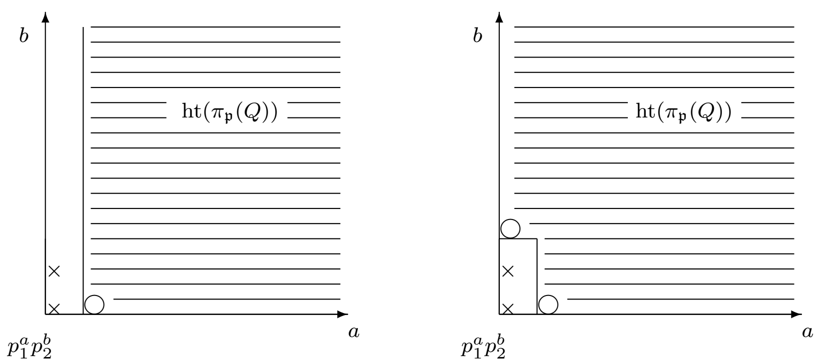
$$1(h) = p_1 + c_{(0,1)} p_2, (\partial_x + 2x\partial_y)(h) = (2c_{(0,1)} + 2)x,$$

thus, $p_1 + c_{(0,1)} p_2 \equiv 0 \mod \mathfrak{p}$ and $(2c_{(0,1)} + 2)x \equiv (2c_{(0,1)} + 2)x \mod \mathfrak{p}$.

Solve the linear equation $(2c_{(0,1)}+2)x = 0$ on $\mathbb{Q}[x,y]/\mathfrak{p}$. Then, $c_{(0,1)} = -1$. Thus, $h = p_1 - p_2 = x^2 - 2y - \frac{5}{4}$. We renew $T = \{(1,0)\}$ and $G = \{x^2 - 2y - \frac{5}{4}\}$.

In the left picture of **Fig.** 3, an element of the intermediate data S and $\{(0,0)\}$ is displayed as "×" and an element of the intermediate data T is displayed as "◯".

Fig. 3 Elements of $S \cup \{(0,0)\}$ and T



(3) Take the smallest exponent $(0,2)$ in CT and update CT $= \emptyset$. Set $h = p_2^2 + c_{(0,1)}p_2$ where $c_{(0,1)}$ is an indeterminate. Then,

$$1(h) = h, (\partial_x + 2x\partial_y)(h) = (4y + 2c_{(0,1)} + 2)x,$$

thus, $f \equiv 0 \mod \mathfrak{p}$ and $(4y + 2c_{(0,1)} + 2)x \equiv (4y + 2c_{(0,1)} + 2)x \mod \mathfrak{p}$. Solve the linear equation $(4y + 2c_{(0,1)} + 2)x = 0$ on $\mathbb{Q}[x,y]/\mathfrak{p}$. One can solve the system by utilizing linear algebra techniques or by computing a Gröbner basis. Let us compute a Gröbner basis in $\mathbb{Q}[c_{(0,1)}, x, y]$. The reduced Gröbner basis of $\langle (4y+2c_{(0,1)}+2)x, p_1, p_2\rangle$ w.r.t. the lexicographic term order with $c_{(0,1)} \succ x \succ y$ is

$$\left\{c_{(0,1)}, y + \tfrac{1}{2}, x^2 - \tfrac{3}{4}\right\}$$

where $p_1 = x^2 - \frac{3}{4}$ and $p_2 = y + \frac{1}{2}$. Hence, we obtain the solution $c_{(0,1)} = 0$, namely, $h = p_2^2 = y^2 + y + \frac{1}{4}$. We renew $T = \{(0,2),(1,0)\}$ and $G = \{y^2 + y + \frac{1}{4}, x^2 - 2y - \frac{5}{4}\}$.

Since CT $= \emptyset$, we stop the computation. Therefore, the basis of Q is

$$\left\{y^2 + y + \tfrac{1}{4}, x^2 - 2y - \tfrac{5}{4}\right\}.$$

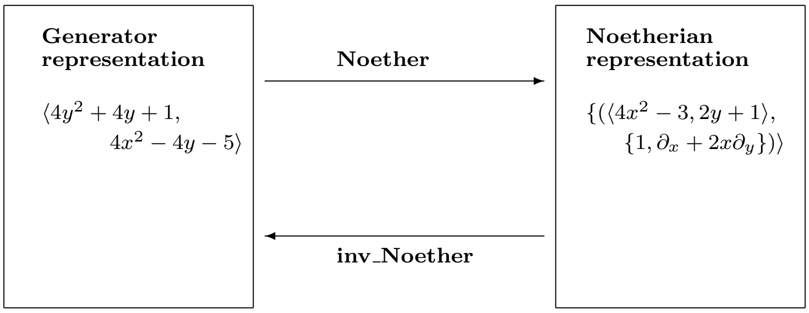
In the right picture of **Fig.** 3, an element of the set S and $\{(0,0)\}$ is displayed as "×" and an element of the set T is displayed as "◯".

7 Applications

Two kinds of representations of ideals are presented in this paper. One is using generators for ideals. Let us call it "generator representation". The other is Noetherian representation.

We have introduced two algorithms that connect the two representations. See **Fig.**4.

Fig. 4 Generator representation and Noetherian representation



Currently, many researchers are working and computing mathematical objects in "generator representation", but, certainly, we are able to work in Noetherian representation. In fact, if we have Noetherian representations of ideals, then we can easily compute sum of ideals, intersection of ideals and ideal quotients, like $7^{11}\times 7^5=7^{16}$, $\mathsf{gcd}(7^{11},7^5)=7^5$ and $7^{11}/7^6=7^5$ in integers. Noetherian representations are pretty useful to analyze and compute the ideals on the irreducible variety defined by the prime ideal $\mathfrak{p}$ when we study the localization.

Here we introduce new ideal computations as applications of Noetherian representations.

In this section, let $\mathfrak{p}$ be a zero-dimensional prime ideal, Q_1, Q_2 $\mathfrak{p}$-primary ideals and NB_{Q_i} the reduced basis of the vector space NT_{Q_i} w.r.t. $\succ$ where $i=1,2$.

Lemma 6 (Sum of ideals) *Let B be a basis of the vector space* $\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})\cap\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_2})$*. Then,*

$$\mathrm{Noether}(Q_1+Q_2)=\{(\mathfrak{p},B)\}.$$

Proof First we prove $\sqrt{Q_1+Q_2}=\mathfrak{p}$.
($\subset$) For all $f\in\sqrt{Q_1+Q_2}$, there exists $m\in\mathbb{N}$ s.t. $f^m\in Q_1+Q_2$. Since $Q_1,Q_2\subset\mathfrak{p}$, $f^m\in\mathfrak{p}$ and $f\in\mathfrak{p}$. Thus, $\sqrt{Q_1+Q_2}\subset\mathfrak{p}$.
($\supset$) For all $f\in\mathfrak{p}$, there exists $m'\in\mathbb{N}$ s.t. $f^{m'}\in Q_1$ and $f^{m'}\in Q_2$. Thus, $f^{m'}\in Q_1+Q_2$ and $f\in\sqrt{Q_1+Q_2}$. Thus, $\sqrt{Q_1+Q_2}=\mathfrak{p}$.

Therefore, $\sqrt{Q_1+Q_2}=\mathfrak{p}$.

Second, we show that a basis of $\mathrm{NT}_{Q_1+Q_2}$ is B. We prove

$$\mathrm{NT}_{Q_1+Q_2}=\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})\cap\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_2}).$$

($\subset$) For all $\psi\in\mathrm{NT}_{Q_1+Q_2}$ and $f\in Q_1+Q_2$, then we have $\psi(f)\in\mathfrak{p}$. Since $Q_1,Q_2\subset Q_1+Q_2$, for all $g\in Q_1$ and $h\in Q_2$, $\psi(g),\psi(h)\in\mathfrak{p}$, too. Thus, $\psi\in\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})$ and $\psi\in\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_2})$.
($\supset$) For all $\psi\in\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})\cap\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_2})$, $f\in Q_1$ and $g\in Q_2$, then we have $\psi(f),\psi(g)\in\mathfrak{p}$. When $h\in Q_1+Q_2$, there exist $h_1\in Q_1$ and $h_2\in Q_2$ such that $h=h_1+h_2$. Thus, $\psi(h)=\psi(h_1)+\psi(h_2)\in\mathfrak{p}$. Thus, $\psi\in\mathrm{NT}_{Q_1+Q_2}$.

Hence, $\mathrm{NT}_{Q_1+Q_2}=\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})\cap\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_2})$.

Therefore, $\mathrm{Noether}(Q_1+Q_2)=\{(\mathfrak{p},B)\}$. □

Example 4 Let us consider two Noetherian representations $\mathrm{Noether}(Q_1)=\{(\mathfrak{p},\{1,\partial_x,\partial_x^2-64x\partial_y\})\}$ and $\mathrm{Noether}(Q_2)=\{(\mathfrak{p},\{1,\partial_x,\partial_y,\partial_x\partial_y\})\}$ where $\mathfrak{p}=\langle 4x^2-3,2y+1\rangle\subset\mathbb{Q}[x,y]$ and $\partial_x=\frac{\partial}{\partial x},\partial_y=\frac{\partial}{\partial y}$. We compute a basis of the vector space $\mathrm{Span}_{K[x]/\mathfrak{p}}(\{1,\partial_x,\partial_x^2-64x\partial_y\})\cap\mathrm{Span}_{K[x]/\mathfrak{p}}(\{1,\partial_x,\partial_y,\partial_x\partial_y\})$.

The two base are written by

$$\begin{aligned}(\begin{array}{ccc}1 & \partial_x & \partial_x^2-64x\partial_y\end{array})&=(\begin{array}{ccccc}1 & \partial_x & \partial_y & \partial_x\partial_y & \partial_x^2\end{array})A,\\ (\begin{array}{cccc}1 & \partial_x & \partial_y & \partial_x\partial_y\end{array})&=(\begin{array}{ccccc}1 & \partial_x & \partial_y & \partial_x\partial_y & \partial_x^2\end{array})B\end{aligned}$$

where

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&-64x\\0&0&1\end{pmatrix},\quad B=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix},$$

and the reduced row echelon matrix of $(A|B)$ is

$$\left(\begin{array}{ccc|cccc}1&0&0&1&0&0&0\\0&1&0&0&1&0&0\\\hdashline 0&0&1&0&0&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{array}\right).$$

Hence, $\{1,\partial_x\}$ is a basis of the vector space. Therefore, $\mathrm{Noether}(Q_1+Q_2)=\{(\mathfrak{p},\{1,\partial_y\})\}$.

Lemma 7 (Intersection of ideals) *Let B be a basis of the vector space* $\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1}\cup\mathrm{NB}_{Q_2})$. *Then,*

$$\mathrm{Noether}(Q_1\cap Q_2)=\{(\mathfrak{p},B)\}.$$

Proof As $\sqrt{Q_1}=\sqrt{Q_2}=\mathfrak{p}$, it is obvious that $\sqrt{Q_1\cap Q_2}=\mathfrak{p}$. The meaning of $\mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1}\cup\mathrm{NB}_{Q_2})$ is "for all $\psi \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1} \cup \mathrm{NB}_{Q_2})$ and $f \in Q_1 \cap Q_2$, then $\psi(f) \in \mathfrak{p}$."

That is $\mathrm{NT}_{Q_1 \cap Q_2}$. (The definition of NT is in Proposition 1.) Hence, $\mathrm{NT}_{Q_1 \cap Q_2} = \mathrm{Span}_{K[x]/\mathfrak{p}}(B)$. Therefore, $\mathrm{Noether}(Q_1 \cap Q_2) = \{(\mathfrak{p}, B)\}$. □

Lemma 7 is easier than the standard method of computing an intersection of ideals in generator representation. The new method consists of linear algebra computations.

Example 5 Let us consider the same setting as Example 4. Then, by utilizing Gaussian elimination method over $\mathbb{Q}[x,y]/\mathfrak{p}$,

$$\mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}\left(\{1, \partial_x, \partial_x^2 - 64x\partial_y\} \cup \{1, \partial_x, \partial_y, \partial_x\partial_y\}\right) = \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}\left(\{1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2\}\right).$$

(See the reduced echelon matrix of $(A|B)$ of Example 4.)

Therefore, $\mathrm{Noether}(Q_1 \cap Q_2) = \{(\mathfrak{p}, \{1, \partial_y, \partial_x, \partial_x\partial_y, \partial_x^2\})\}$.

Let I and I' be zero-ideals in $K[x]$ and

$\mathrm{Noether}(I) = \{(\mathfrak{p}_1, B_1), (\mathfrak{p}_2, B_2), \ldots, (\mathfrak{p}_r, B_r), (\mathfrak{p}_{r+1}, B_{r+1}), \ldots, (\mathfrak{p}_s, B_s)\}$,
$\mathrm{Noether}(I') = \{(\mathfrak{p}'_1, B'_1), (\mathfrak{p}'_2, B'_2), \ldots, (\mathfrak{p}'_r, B'_r), (\mathfrak{p}'_{r+1}, B'_{r+1}), \ldots, (\mathfrak{p}_t, B'_t)\}$.

Assume that $\mathfrak{p}_i = \mathfrak{p}'_i$ for $i = 1, 2, \ldots, r$ and $\{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_s\} \cap \{\mathfrak{p}'_{r+1}, \ldots, \mathfrak{p}'_t\} = \emptyset$. Then

$$\mathrm{Noether}(I \cap I') = \bigcup_{i=1}^{r} \left\{(\mathfrak{p}_i, \mathrm{Span}_{K[x]/\mathfrak{p}_i}(B_i \cup B'_i))\right\} \cup U$$

where $U = \{(\mathfrak{p}_{r+1}, B_{r+1}), \ldots, (\mathfrak{p}_s, B_s), (\mathfrak{p}'_{r+1}, B'_{r+1}), \ldots, (\mathfrak{p}_t, B'_t)\}$. We emphasize that the Noetherian representation makes the intersection compute easy.

Lemma 8 (Ideal quotient) *Let $h \in K[x]$ and the ideal quotient $Q_1 : h$ be a $\mathfrak{p}$-primary. Let $N = \{\psi \bullet h \mid \psi \in \mathrm{NB}_{Q_1}\}$ and B be a basis of the vector space $\mathrm{Span}_{K[x]/\mathfrak{p}}(N)$. Then,*

$$\mathrm{Noether}(Q_1 : h) = \{(\mathfrak{p}, B)\}.$$

Proof We prove that $\mathrm{NT}_{Q_1:h} = \mathrm{Span}_{K[x]/\mathfrak{p}}(N)$.
($\subset$) For all $\psi \in \mathrm{NT}_{Q_1:h}$ and $f \in Q_1 : h$, then $hf \in Q_1$ and, for all $\varphi \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})$, $\varphi(hf) = \varphi(h(f)) \in \mathfrak{p}$. Since $\psi(f) \in \mathfrak{p}$ and $\varphi(h(f)) \in \mathfrak{p}$, there exists $\varphi' \in \mathrm{Span}_{K[x]/\mathfrak{p}}(\mathrm{NB}_{Q_1})$ such that $\psi(f) = \varphi'(h(f)) \in \mathfrak{p}$. Therefore, $\psi \in \mathrm{Span}_{K[x]/\mathfrak{p}}(N)$.
($\supset$) For all $\psi \in \mathrm{Span}_{K[x]/\mathfrak{p}}(N)$ and $f \in Q_1 : h$, then $hf \in Q_1$ and there exists $\varphi \in \mathrm{Span}_{K[x]/\mathfrak{p}}(Q_1)$ such that $\varphi(h \cdot f) = \psi(f) \in \mathfrak{p}$. Therefore, $\psi \in \mathrm{NT}_{Q_1:h}$. □

Example 6 Let us consider $h = 4x^2-3 \in \mathbb{Q}[x, y]$ and $\mathrm{Noether}(Q_1) = \{(\mathfrak{p}, \{1, \partial_x, \partial_y, \partial_x^2 + 4x\partial_x\partial_y\})\}$ where $\mathfrak{p} = \langle 4x^2 - 3, 2y + 1\rangle \subset \mathbb{Q}[x, y]$. Then,

$$\begin{aligned}
\partial_x \bullet h &= (4x^2 - 3)\partial_x + 8x \equiv 8x \mod \mathfrak{p},\\
\partial_y \bullet h &= (4x^2 - 3)\partial_y \equiv 0 \mod \mathfrak{p},\\
(\partial_x^2 + 4x\partial_x\partial_y) \bullet h &= 4x(4x^2 - 3))\partial_x\partial_y + (4x^2 - 3)\partial_x^2 + 32x^2\partial_y + 16x\partial_x + 8\\
&\equiv 16x(\partial_x + 2x\partial_y) + 8 \mod \mathfrak{p},
\end{aligned}$$

and thus we have $\mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(\{8x, 16x(\partial_x + 2x\partial_y) + 8\})$.

Since h is one of the generators of $\mathfrak{p}$ and the basis NB_{Q_1} has a first order partial differential operator ∂_x and a second order partial differential operator $\partial_x^2 + 4x\partial_x\partial_y$, thus it is clear that $Q_1 : h$ is $\mathfrak{p}$-primary. By utilizing Gaussian elimination method over $\mathbb{Q}[x, y]/\mathfrak{p}$, we obtain

$$\mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(\{8x, 16x(\partial_x + 2x\partial_y) + 8\}) = \mathrm{Span}_{\mathbb{Q}[x,y]/\mathfrak{p}}(\{1, \partial_x + 2x\partial_y\}).$$

Therefore, $\mathrm{Noether}(Q_1 : h) = \{(\mathfrak{p}, \{1, \partial_x + 2x\partial_y\})\}$.

As in the case of the intersection, the Noetherian representation makes an ideal quotient compute easy.

Our methods of this section are free from Gröbner basis computations.

References

1. Abbott, J., Bigatti, A., Kreuzer, M. and Robbiano, L.: Computing ideals of points, Journal of Symbolic Computation, **30**, 341–356, (2000)
2. Alonso, M. E., T. Mora and Raimondo, M.: Local decomposition algorithms Proc. AAECC 8, Lecture Notes in Computer Science, **508**, 208–221, Springer (1991)
3. Alonso, M.E., Marinari, M.G. and Mora, M.T.: The big Mother of all the dualities, II: Macaulay bases, Applicable Algebra in Engineering, Communication and Computing, **17**, 409–451, (2006)
4. Aoyama, T. and Noro, M.: Modular algorithms for computing minimal associated primes and radicals of polynomial ideals. Proc. ISSAC 2018, 31–38 (2018)
5. Becker, T. and Weispfenning, V.: Gröbner Bases, Graduate Texts in Math. **141**. Springer-Verlag, (1993)
6. Brumfiel, G.: Differential operators and primary ideals. Journal of Algebra, **51**, 375–398 (1978).
7. Caboara M., Conte, P. and Traverso C.: Yet another ideal decomposition algorithm, Lecture Notes in Computer Science, **1255**, 39–54, Springer, (1995)
8. Chen, J., Härkönen, M., Krone, R. and Leykin, A.: Noetherian operators and primary decomposition. Journal of Symbolic Computation, **110**, 1–23, (2022)
9. Chen, J., Cid-Ruiz, Y., Härkönen, M., Krone, R. and Leykin, A.: Noetherian operators in MACAULAY2. arXiv:2101.01002, (2021)
10. Cid-Ruiz, Y.: Noetherian operators, primary submodules and symbolic powers. Collect. Math., **72**, 175–202 (2021).
11. Cid-Ruiz, Y., Homs, R. and Sturmfels, B.: Primary ideals and their differential equations. Foundations of Computational Mathematics, **21**, 1363–1399 (2021)
12. Damiano, A., Sabadini, I. and Struppa, C, D.: Computational methods for the construction of a class of Noetherian operators. Experimental Mathematics, **16**, 41–53 (2007).

13. Decker, W., Greuel, G.-M. and Pfister, G.: Primary decomposition: algorithms and comparisons. Algorithmic algebra and number theory (Heidelberg, 1997), 187–220, Springer (1999).
14. Eisenbud, D., Huneke, C., Vasconcelos, W.: Direct methods for primary decomposition, Inventiones Math., **110**, 207–235 (1992).
15. Ehrenpreis, L.: A fundamental principle for system of linear differential equations with constant coefficients and some of its applications. Proc. Inter. Symp. on Linear Spaces, Jerusalem, 161–174, (1961)
16. Ehrenpreis, L.: Fourier Analysis in Several Complex Variables. Wily-Interscience Publishers, (1970)
17. Faugère, K., Gianni, P., Lazard, D. and Mora, T.: Efficient computation of zero-dimensional Gröbner bases by change of ordering. Journal of Symbolic Computation, **16**, 329–344, (1993)
18. Giusti M. and Heintz J.: Algorithmes – disons rapides – pour la décomposition d'une variété algébrique en composantes irréductibles. Progress in Mathematics, **94**, 169–194, (1990)
19. Gianni, P., Trager, B. and Zacharias, G.: Gröbner bases and primary decomposition of polynomial ideals. Journal of Symbolic Computation, **6**, 146–167, (1988)
20. Grayson, D. R. and Stillman, M. E.: Macaulay2: a software system for research in algebraic geometry. available at `http://www.math.uiuc.edu/Macaulay2`
21. Gröbner, W.: Über das Macaulaysche inverse System und dessen Bedeutung für die Theorie der linearen Differentialgleichungen mit konstanten Koeffizienten. Abhandlungen aus dem Mathematischen Seminaren der Universität Hamburg, **12**, 127–137 (1937) (English version: On the Macaulay inverse system and its importance for the theory of linear differential equations with constant coefficients. ACM Communications in Computer Algebra, **44**, 20–23 (2010))
22. Gröbner, W.: Über eine neue idealtheoretische Grundlegung der von linearen Differentiallgleichungen mit konstanten Koeffizienten. Monatshefte für Mathematik und Physik, **47**, 247–284 (1938)
23. Gröbner, W.: Algebraische Geometrie II. Hochschultaschenbücger, (1970)
24. Hörmander, L.: An Introduction to Complex Analysis in Several Variables. North-Holland, the third revised edition, (1990)
25. Ishihara, Y. and Yokoyama, K.: Computation of a primary component of an ideal from its associated prime by effective localization. Communications of Japan Society for Symbolic and Algebraic Computation, **4**, 1–31, (2020)
26. Kawazoe, T. and Noro, M.: Algorithms for computing a primary ideal decomposition without producing intermediate redundant components. Journal of Symbolic Computation, **46**, 1158–1172, (2011)
27. Krone, R.: Numerical algorithms for dual bases of positive-dimensional ideals. Journal of Algebra and Its Applications, **12**, No. 06, 1350018 (21 pages), (2013)
28. Krone, R. and Leykin, A.: Eliminating dual spaces. Journal of Symbolic Computation, **79**, 609–622, (2017)
29. Krone, R. and Leykin, A.: Numerical algorithms for detecting embedded components. Journal of Symbolic Computation, **82**, 1–18, (2017)
30. Macaulay, F. S.: The algebraic theory of modular systems. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1994). Revised reprint of the 1916 original, With an introduction by Paul Roberts.
31. Marinari, M. G., Möller, H. M. and Mora, T.: Groebner duality and multiplicities in polynomial system solving, Proc. ISSAC 1995, ACM (1995)
32. Marinari, M. G., Mora, T. and Möller, H. M.: Gröbner bases of ideals given by dual bases. Proc. ISSAC 1991, 55-63, ACM, (1991)
33. Marinari, M. G., Möller, H. M. and Mora, T.: On multiplicities in polynomial system solving, Trans. AMS, **348**, 3283–3321, (1996)
34. Marinari, M. G., Möller, H. M. and Mora, T.: Gröbner bases of ideals defined by functionals with an application to ideals of projective points. Applicable Algebra in Engineering, Communication and Computing **4**, 103–145, (1993)
35. Mora, T.: Solving Polynomial Equation Systems II. Macaulay's Paradigm and Gröbner Technology. Cambridge University Press, (2005)

36. Mourrain B.: A new criterion for normal form algorithms. In: Fossorier M., Imai H., Lin S., Poli A. (eds) Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. AAECC 1999. Lecture Notes in Computer Science, **1719**. Springer, (1999)
37. Mourrain B.: Bezoutian and quotient ring structure. Journal of Symbolic Computation, **39**, 397-415, (2005)
38. Nabeshima, K. and Tajima, S.: On the computation of algebraic local cohomology classes associated with semi-quasihomogeneous singularities. Advanced Studies in Pure Mathematics, **66**, 143–1159, (2015)
39. Nabeshima, K. and Tajima, S. : Algebraic local cohomology with parameters and parametric standard bases for zero-dimensional ideals. Journal of Symbolic Computation, **82**, 91–122, (2017)
40. Noro, M. and Takeshima, T.: Risa/Asir - A computer algebra system. Proc. ISSAC 1992, 387–396, ACM, (1992) available at `http://www.math.kobe-u.ac.jp/Asir/asir.html`
41. Noro, M. : New algorithms for computing primary decomposition of polynomial ideals . Proc. ICMS2010, Lecture Notes in Computer Science, **6327**, 233–244, Springer (2010)
42. Oberst, U.: The construction of Noetherian operators., Journal of Algebra, **222**, 595–620, (1999)
43. Ohara, K. and Tajima, S.: An algorithm for computing Grothendieck local residues I, – shape basis case –, Mathematics in Computer Sciences, **13**, 205–216, (2019)
44. Ohara, K. and Tajima, S.: An algorithm for computing Grothendieck local residues II – general case – Mathematics in Computer Sciences, **14**, 483–496, (2020)
45. Palamodov, V. P.: Linear differential operators with constant coefficients. Translated from the Russian by A. Brown. Die Gundlehren der mathematischen Wissenschaften, Band 168. Springer-Verlag, New York-Berlin, (1970)
46. Rouillier, F.: Solving zero-dimensional systems through the rational univariate representation. AAECC, **9**, 433–461, (1996)
47. Saito, M., Sturmfels, B. and Takayama, N.: Gröbner Deformations of Hypergeometric Differential Equations. Springer (1999)
48. Shimoyama, T. and Yokoyama, K.: Localization and primary decomposition of polynomial ideals. Journal of Symbolic Computation, **22**, 247–277, (1996)
49. Tajima, S.: An algorithm for computing the Noetherian operator representations and its applications to constant coefficients holonomic PDE's. Tools for Mathematical Modellings, St. Petersbourg, 154–160, (2001)
50. Tajima, S.: On Noether differential operators attached to a zero-dimensional primary ideal – shape basis case –. Proc. the 12th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications, 357–366, Kyushu University Press (2005)
51. Tajima, S.: Noetherian differential operators and Grothendieck local residues. RIMS Kôkyûroku, **1431**, 123-136 (2005) (in Japanese)
52. Tajima, S.: An algorithm for computing exponential polynomial solutions of constants holonomic PDE's –generic cases–. In ; Son, L.H., Tutschke, W., Jain, S. (eds) Methods of Complex and Clifford Analysis, 335–344. SAS international Publ., Delhi (2006)
53. Tajima, S. and Nakamura, Y.: Annihilating ideals for an algebraic local cohomology class. Journal of Symbolic Computation, **44**, 435–448, (2009)