Merging Multiple Algorithms for Computing Comprehensive Gröbner Systems Using Parallel Processing

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Abstract. We propose a new computational method for comprehensive Gröbner systems by merging algorithms developed by Kapur-Sun-Wang, Nabeshima, Suzuki-Sato-Nabeshima and Kalkbrener, and leveraging parallel processing. Furthermore, we have implemented this computational method in the computer algebra system Risa/Asir, we evaluate its effectiveness based on the results of computational experiments.

Keywords: Comprehensive Gröbner system · Gröbner basis · parallel computation.

1 Introduction

The concept of a comprehensive Gröbner system (and basis) was introduced, constructed, and studied by V. Weispfenning [24] as a special basis for a parametric polynomial ideal and has been regarded as one of the important tools for analyzing parametric ideals. After Weispfenning's paper was published, significant developments in comprehensive Gröbner systems were made by Kapur [5], Kapur-Sun-Wang [6,7], Kurata [8], Montes [10,11,12,13], Nabeshima [14,15], Suzuki-Sato [21,22], Sato-Suzuki-Nabeshima [18,19] and Weispfenning [24,25,26] in theory, software, and applications. However, the computational complexity of the comprehensive Gröbner systems remains quite high; therefore, efficient algorithms and implementations capable of faster computations are highly desired.

In this paper, we present a novel computational method for comprehensive Gröbner systems by integrating the algorithms developed by Kapur-Sun-Wang [6,7] and Nabeshima [15], along with the results of Sato-Suzuki-Nabeshima [19] and Kalkbrener [4], leveraging parallel processing techniques. In 2010, Kapur-Sun-Wang published an effective algorithm based on the results of Kalkbrener [4] for computing comprehensive Gröbner systems [6]. In 2024, Nabeshima introduced another different algorithm in [15], which utilizes Gröbner bases and ideal quotients for computing comprehensive Gröbner systems. It was reported in [15] that the speed of the two algorithms depends on the specific problem; namely, we cannot unconditionally determine which one is better. Notably, the two algorithms share something in common, namely, the structure. Since the structure of the two algorithms is essentially the same, we demonstrate that it is easy to integrate them using processing techniques. Moreover, we also integrate two more computational techniques that are for special types of comprehensive Gröbner systems, such as a comprehensive Gröbner basis on an affine variety defined by a prime ideal [4,19], and an alternative comprehensive Gröbner basis on a variety (ACGB-V) [19]. In some special situations, the algorithms for computing the two special types of comprehensive Gröbner systems seem fast; hence, the computational techniques may enhance the effectiveness of the new computational method.

A lot of current computers have multiple cores, and several computer algebra systems are equipped with the ability to perform parallel processing. By utilizing this capability, we have implemented the new computational method in the computer algebra system Risa/Asir. We assess the effectiveness of our implementations through computational experiments and analysis of the results.

This paper is organized as follows: In Section 2, we review the definition of comprehensive Gröbner systems. In Section 3, we recall two algorithms for computing comprehensive Gröbner systems: Kapur-Sun-Wang's algorithm and Nabeshima's algorithm. In Section 4, we also review how to compute a comprehensive Gröbner system on an affine variety defined by a zero dimensional ideal. Finally, in Section 5, we present a new computational method for comprehensive Gröbner systems along with the results of benchmark tests.

2 Preliminaries

Here we introduce some notation that will be utilized in this paper and recall comprehensive Gröbner systems.

Let $x = \{x_1, \dots, x_n\}$ and $t = \{t_1, \dots, t_m\}$ be sets of variables, K a field and $\overline{K}$ an any given algebraic closed extension of K. (We often regard t as parameters.) Symbols $Term(t)$, $Term(x)$ and $Term(t \cup x)$ mean the set of terms of t, the set of terms of x and the set of terms of $t \cup x$, respectively.

Fix a term order $\succ$ on $Term(x)$ and let $f \in K[t][x]$ where $K[t][x]$ is a polynomial ring over $K[t]$. Then $\mathrm{lt}(f), \mathrm{lm}(f)$ and $\mathrm{lc}(f)$ denote the leading term, leading monomial and leading coefficient of f i.e. $\mathrm{lm}(f) = \mathrm{lc}(f)\,\mathrm{lt}(f)$. For $F \subset K[t][x]$ and $f_1, \dots, f_\nu \in K[t][x]$, we define $\mathrm{lt}(F) = \{\mathrm{lt}(f) | f \in F\} \subset Term(x)$, $\mathrm{lc}(F) = \{\mathrm{lc}(f) | f \in F\}$ and $\langle f_1, \dots, f_\nu \rangle = \{\sum_{i=1}^{\nu} h_i f_i | h_1, \dots, h_\nu \in K[t][x]\}$.

The set of natural numbers $\mathbb{N}$ includes zero, $\mathbb{Q}$ is the field of rational numbers and $\mathbb{C}$ is the field of complex numbers.

Definition 1. *Let $\succ_1$ and $\succ_2$ be term orders on $Term(x)$ and $Term(t)$, respectively, and $x^{\alpha_1}, x^{\alpha_2} \in Term(x)$, $t^{\beta_1}, t^{\beta_2} \in Term(t)$,*

$$x^{\alpha_1} t^{\beta_1} \succ_{x,t} x^{\alpha_2} t^{\beta_2} \iff x^{\alpha_1} \succ_1 x^{\alpha_2} \text{ or } (x^{\alpha_1} = x^{\alpha_2}, \text{ and } t^{\beta_1} \succ_2 t^{\beta_2})$$

where $\alpha_1, \alpha_2 \in \mathbb{N}^n$ *and* $\beta_1, \beta_2 \in \mathbb{N}^m$. *This type of term order* $\succ_{x,t}$ *is called a block term order on* $Term(t \cup x)$. *The term order is written as* $(\succ_1, \succ_2)$.

Definition 2. *1. A basis* $\{x^{\alpha_1}, \ldots, x^{\alpha_\ell}\} \subset Term(x)$ *for a monomial ideal* I, *in* $K[x]$ *(or* $K[t][x]$, $\overline{K}[x]$*) is said to be minimal if no* x^{α_i} *in the basis divides other* x^j *for* $i \neq j$, *where* $\alpha_1, \ldots, \alpha_\ell \in \mathbb{N}^n$. *Then, the minimal basis of* I *is written as* $MB(I)$.

2. For $q \in K[t]$, *the squarefree part of* q *is written as* $\sqrt{q}$.

For $g_1, \ldots, g_\ell \in K[t]$, $\mathbf{V}(g_1, \ldots, g_\ell) \subset \overline{K}^m$ denotes the affine variety of $g_1, \ldots, g_\ell$, i.e. $\mathbf{V}(g_1, \ldots, g_\ell) = \{\bar{t} \in \overline{K}^m | g_1(\bar{t}) = \cdots = g_\ell(\bar{t}) = 0\}$, and $\mathbf{V}(0) = \overline{K}^m$. We call an algebraically constructible set of the form $\mathbf{V}(f_1, \ldots, f_\ell) \backslash \mathbf{V}(f'_1, \ldots, f'_{\ell'}) \subset \overline{K}^m$, a stratum where $f_1, \ldots, f_\ell, f'_1, \ldots, f'_{\ell'} \in \overline{K}[t]$. As it is clear that $\mathbf{V}(1) = \emptyset$, we have $\mathbf{V}(f_1, \ldots, f_\ell) \backslash \mathbf{V}(1) = \mathbf{V}(f_1, \ldots, f_\ell)$.

For $\bar{t} \in \overline{K}^m$, the canonical specialization homomorphism $\sigma_{\bar{t}} : K[t][x] \to \overline{K}[x]$ (or $K[t] \to \overline{K}$) is defined as the map that substitutes t by $\bar{t}$ in $f(t, x) \in K[t][x]$. The image $\sigma_{\bar{t}}$ of a set $F \subset K[t][x]$ is denoted by $\sigma_{\bar{t}}(F) = \{\sigma_{\bar{t}}(f) | f \in F\} \subset \overline{K}[x]$.

In this paper, we adopt the following definition of comprehensive Gröbner system.

Definition 3. *Fix a term order* $\succ$ *on* $Term(x)$. *Let* $F \subset K[t][x]$, $E_1, \ldots, E_s$, $N_1, \ldots, N_s \subset K[t]$, $G_1, \ldots, G_s \subset K[t][x]$. *If a finite set* $\mathcal{G} = \{(E_1, N_1, G_1), \ldots, (E_\ell, N_\ell, G_\ell)\}$ *of triples satisfies the properties such that*

(i) for each $i \in \{1, \ldots, \ell\}$, $\mathbf{V}(E_i) \backslash \mathbf{V}(N_i) \neq \emptyset$,

(ii) for $i \neq j$, $\big(\mathbf{V}(E_i) \backslash \mathbf{V}(N_i)\big) \cap \big(\mathbf{V}(E_j) \backslash \mathbf{V}(N_j)\big) = \emptyset$, *and*

(iii) for all $\bar{t} \in \mathbf{V}(E_i) \backslash \mathbf{V}(N_i)$, $\sigma_{\bar{t}}(G_i)$ *is a Gröbner basis of* $\langle \sigma_{\bar{t}}(F) \rangle$ *w.r.t.* $\succ$ *in* $\overline{K}^m[x]$,

then $\mathcal{G}$ *is called a comprehensive Gröbner system (CGS) of* $\langle F \rangle$ *w.r.t.* $\succ$ *on* $\bigcup_{i=1}^r (\mathbf{V}(E_i) \backslash \mathbf{V}(N_i))$. *We call a triple* (E_i, N_i, G_i) *segment of* $\mathcal{G}$. *We simply say that* $\mathcal{G}$ *is a comprehensive Gröbner system (CGS) of* $\langle F \rangle$ *w.r.t.* $\succ$ *if* $\bigcup_{i=1}^r (\mathbf{V}(E_i) \backslash \mathbf{V}(N_i)) = \overline{K}^m$.

We give an example of a comprehensive Gröbner system.

Example 1. Let $F = \{x^2y + ax^2 + y^2, x^2 + bxy + y, y^3 + bx^2y + xy\} \subset \mathbb{C}[a, b][x, y]$ where x, y are variable and a, b are parameters. Then, a comprehensive Gröbner system $\mathcal{G}$ of $\langle F \rangle$ w.r.t. the lexicographic term order $\succ$ such that $x \succ y$ is the following.

$\mathcal{G} = \{(\{a\}, \{a^3(b^4 + b) + 3a^2b^2 + 3ab^3 + b^4\}, \{by^2, xy + y^3, x^2 + y\}),$
$(\{a(b + 1) + b\}, \{a^3(b^2 - b + 1) + a^2(b^3 - 2b^2 + 3b) + a(-b^3 + 3b^2) + b^3\}, \{(-a - b)y^2 - a^2y, xy + y^3 - by^2 - a^2y, -x^2 + by^3 + (a^2 - 1)y\}),$
$(\{a, b\}, \{1\}, \{-y^5 - y^2, xy + y^3, x^2 + y\}),$
$(\{a^2(b^2 - b + 1) + a(-b^2 + 2b) + b^2\}, \{a + b\}, \{(-a - b)y^2 - a^2y, xy + y^3 - by^2 - a^2y, x^2 - by^3 + b^2y^2 + (a^2b + 1)y\}),$
$(\{0\}, \{a^4(b^3 + 1) + 3a^3b + 3a^2b^2 + ab^3\}, \{y, x^2 - by^3 + b^2y^2 + (a^2b + 1)y\})\}.$

The set $\mathcal{G}$ means the following.

- If (a,b) belongs to $\mathbf{V}(a)\backslash\mathbf{V}(a^3(b^4+b)+3a^2b^2+3ab^3+b^4)$ in $\mathbb{C}^2$, then $\{by^2, xy+y^3, x^2+y\}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.
- If (a,b) belongs to $\mathbf{V}(a(b+1)+b)\backslash\mathbf{V}(a^3(b^2-b+1)+a^2(b^3-2b^2+3b)+a(-b^3+3b^2)+b^3)$, then $\{(-a-b)y^2-a^2y, xy+y^3-by^2-a^2y, -x^2+by^3+(a^2-1)y\}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.
- If (a,b) belongs to $\mathbf{V}(a,b)$ (i.e. $a=b=0$), then $\{-y^5-y^2, xy+y^3, x^2+y\}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.
- If (a,b) belongs to $\mathbf{V}(a^2(b^2-b+1)+a(-b^2+2b)+b^2)\backslash\mathbf{V}(a+b)$, then $\{(-a-b)y^2-a^2y, xy+y^3-by^2-a^2y, x^2-by^3+b^2y^2+(a^2b+1)y\}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.
- If (a,b) belongs to $\mathbb{C}^2\backslash\mathbf{V}(a^4(b^3+1)+3a^3b+3a^2b^2+ab^3)$, then $\{y, x^2-by^3+b^2y^2+(a^2b+1)y\}$ is a Gröbner basis of $\langle F\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.

3 Two algorithms for computing comprehensive Gröbner systems

Here we review Kapur-Sun-Wang's algorithm and Nabeshima's algorithm, both known for their efficiency in computing comprehensive Gröbner systems. Further details can be found in [6,7,15].

In this section, we establish that $\succ_x$ denotes a term order on $Term(x)$, $\succ_t$ represents a term order on $Term(t)$, and $\succ_u$ signifies a term order on $Term(u)$ where $u \subset t$.

3.1 Kapur-Sun-Wang algorithm

In [6,7], Kapur-Sun-Wang introduced the following nice theorem that is based on the result of Kalkbrener [4].

Theorem 1 (Kapur-Sun-Wang [6,7]). *Let F be a finite set of polynomials in $K[t][x]$ and E a finite set of polynomials in $K[t]$ such that $\langle E\rangle$ is proper in $K[t]$. Regard $F\cup E$ as a set of polynomial in $K[t,x]$ and let G be a Gröbner basis of $\langle F\cup E\rangle$ w.r.t. a block term order $(\succ_x,\succ_t)$ in $K[t,x]$, and $G_b = G\backslash(G\cap\langle E\rangle)$. Assume that $G_b\neq\emptyset$ and let $MB(\langle \mathrm{lt}(G_b)\rangle)=\{w_1,\dots,w_\ell\}$ in $K[t][x]$. For each $i\in\{1,\dots,\ell\}$, let $G_{w_i}=\{g\in G|\,\mathrm{lt}(g)=w_i\}$ and $G'=\{g_1,\dots,g_\ell\}$ where $g_i\in G_{w_i}$ $(1\leq i\leq \ell)$.*

Then, for all $\bar{t}\in\mathbf{V}(E)\backslash\mathbf{V}\left(\sqrt{\prod_{j=1}^{\ell}\mathrm{lc}(g_j)}\right)$, $\sigma_{\bar{t}}(G')$ is a minimal Gröbner basis of $\langle\sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.

The following corollary is a direct consequence of Theorem 1.

Corollary 1. *Let F be a finite set of polynomials in $K[t][x]$. Regard F as a set of polynomial in $K[t,x]$ and let G be a Gröbner basis of $\langle F\rangle$ w.r.t. a block term order $(\succ_x,\succ_t)$ in $K[t,x]$. Let $MB(\langle\mathrm{lt}(G)\rangle)=\{w_1,\dots,w_\ell\}$ in $K[t][x]$. For* *each* $i \in \{1,\dots,\ell\}$*, let* $G_{w_i} = \{g \in G | \operatorname{lt}(g) = w_i\}$ *and* $G' = \{g_1,\dots,g_\ell\}$ *where* $g_i \in G_{w_i}$ $(1 \le i \le \ell)$.

Then, for all $\bar{t} \in \overline{K}^m \backslash \mathbf{V}\left(\sqrt{\prod_{j=1}^{\ell} \operatorname{lc}(g_j)}\right)$, $\sigma_{\bar{t}}(G')$ *is a minimal Gröbner basis of* $\langle \sigma_{\bar{t}}(F)\rangle$ *w.r.t.* $\succ_x$ *in* $\overline{K}[x]$.

We illustrate Corollary 1 with the following example.

Example 2. Let $F = \{t_1x^2 - xy + y^2, t_2xy + y, t_1x^2 - y, (t_2+1)xy^2 + t_1x\} \subset \mathbb{C}[t_1,t_2][x,y]$, $\succ_x$ the graded lexicographic term order with $x \succ y$ on $Term(\{x,y\})$ and $\succ_t$ the graded reverse lexicographic term order with $t_2 \succ t_1$ on $Term(\{t_1,t_2\})$. Then, the reduced Gröbner basis of $\langle F\rangle \subset \mathbb{C}[t_1,t_2,x,y]$ w.r.t. $(\succ_x,\succ_t)$ is

$$\begin{aligned} G = \{&(t_1+t_2^2+t_2)y, t_1(t_2-2)y^2+y, (t_1^2+6t_1-t_2-3)y, t_1x+t_2y,\\ &y^2-(t_1+2t_2+1)y, xy-(t_1+2t_2+2)y\}\end{aligned}$$

in $\mathbb{C}[t_1,t_2,x,y]$. Regard G as a set in $\mathbb{C}[t_1,t_2][x,y]$, then $MB(\langle \operatorname{lt}(G)\rangle) = \{x,y\}$. Set $G_x = \{t_1x + t_2y\}$, $G_y = \{(t_1+t_2^2+t_2)y, (t_1^2+6t_1-t_2-3)y\}$ and $G' = \{t_1x+t_2y, (t_1+t_2^2+t_2)y\}$.

Therefore, for all $\bar{t} \in \mathbb{C}^2 \backslash \mathbf{V}(t_1(t_1+t_2^2+t_2))$, $\sigma_{\bar{t}}(G')$ is a minimal Gröbner basis of $\langle\sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\mathbb{C}[x,y]$.

In this paper, we define $\mathbf{V}(\emptyset) = \overline{K}^m$, i.e. $\mathbf{V}(0) = \mathbf{V}(\emptyset) = \overline{K}^m$.

The contents of Theorem 1 are summarized in the following algorithm.

Algorithm 1 (Kapur-Sun-Wang [6,7])

Specification: $\mathsf{KSW}(E,p,F,\succ_x)$
Input: $E \subset K[t]$: finite set, $p \in K[t]$ s.t. $\mathbf{V}(E)\backslash\mathbf{V}(p) \neq \emptyset$,
$F \subset K[t][x]$ finite set, $\succ_x$: term order on $Term(x)$.
Output: (E',p',G',h') : If $E' \neq 1$ and $G' \neq \{0\}$, then $\mathbf{V}(E')\backslash\mathbf{V}(p') \neq \emptyset$ and, for all $\bar{t} \in \mathbf{V}(E')\backslash\mathbf{V}(p')$, $\sigma_{\bar{t}}(G')$ is a Gröbner basis of $\langle\sigma_{\bar{t}}(F)\rangle$ in $\overline{K}[x]$ and $\mathbf{V}(E')\backslash\mathbf{V}(p') \subset \mathbf{V}(E)\backslash\mathbf{V}(p)$. If $G' = \{0\}$, then for all $\bar{t} \in \mathbf{V}(E')\backslash\mathbf{V}(p')$, $\langle\sigma_{\bar{t}}(F)\rangle = \{0\}$. Moreover, $\mathbf{V}(p') = \mathbf{V}(p) \cup \mathbf{V}(h')$.
BEGIN
$\succ_t \leftarrow$ Set a term order on $Term(t)$;
$G \leftarrow$ Compute a reduced Gröbner basis of $\langle F \cup E\rangle$ w.r.t. $(\succ_x,\succ_t)$ in $K[t,x]$;
$G_b \leftarrow G\backslash(G \cap \langle E\rangle)$;
if $G_b = \emptyset$ **then**
  return $(E,p,\{0\},1)$;
end-if
$\{w_1,\dots,w_\ell\} \leftarrow MB(\langle\operatorname{lt}(G_b)\rangle)$;
$G' \leftarrow \emptyset$;
for each $i = 1$ **to** ℓ **do**
  $g \leftarrow$ Select one element $g \in G_b$ that satisfies $\operatorname{lt}(g) = w_i$;
  $G' \leftarrow G' \cup \{g\}$;
end-for

$h \leftarrow \sqrt{\prod_{g\in G'\subset K[t][x]} \mathrm{lc}(g)}$;
if $\mathbf{V}(E)\backslash\mathbf{V}(p\cdot h)=\emptyset$ **then**
  return $(1, p\cdot h, G', h)$;
end-if
return $(E, p\cdot h, G', h)$;
END

We remark that algorithms for checking whether $\mathbf{V}(E)\backslash\mathbf{V}(p\cdot h)=\emptyset$ or not are provided in [6,7,21]. Here, we omit the explanation of these algorithms.

In [6,7], Kapur-Sun-Wang developed an algorithm for computing a comprehensive Gröbner system by recursively utilizing Algorithm 1, as follows.

Algorithm 2 (CGS1) [6,7]

Specification: $\mathsf{CGS1}(E, p, F, \succ_x)$
Input: $E \subset K[t]$: finite set, $p \in K[t]$, $F \subset K[t][x]$ finite set,
  $\succ_x$: term order on $Term(x)$.
Output: $\mathcal{G}$: comprehensive Gröbner system of $\langle F\rangle$ on $\mathbf{V}(E)\backslash\mathbf{V}(p)$ w.r.t. $\succ_x$.
BEGIN
$\mathcal{G}\leftarrow\emptyset$;
if $\mathbf{V}(E)\backslash\mathbf{V}(p)=\emptyset$ **then** return $\mathcal{G}$; **end-if**
$\boxed{(E', p', G', h) \leftarrow \mathbf{KSW}(E, p, F, \succ);}$
if $E\neq 1$ **then**
  $\mathcal{G}\leftarrow\mathcal{G}\cup\{(E,\{p'\},G')\}$;
end-if
$q\leftarrow 1$;
$h_1h_2\cdots h_\ell \leftarrow \mathsf{factorization}(h)$; /*$h_i$ is an irreducible factor $(1\leq i\leq r)$. */
for each $i=1$ to ℓ **do**
  $\mathcal{G}\leftarrow\mathcal{G}\cup$ $\mathsf{CGS1}(E\cup\{h_i\}, p\cdot q, F, \succ)$;
  $q\leftarrow q\cdot h_i$;
end-for
return $\mathcal{G}$;
END

One of the optimizations is the use of factorization(h) that outputs the factorization of h in $K[t]$. The techniques described in [6,7,13,14,19], are applicable to obtain small and nice outputs of a comprehensive Gröbner system.

3.2 Nabeshima algorithm

In [15], the second author of this paper introduced an algorithm different from Kapur-Sun-Wang's for computing comprehensive Gröbner systems. Here, we review the algorithm.

Let u be a subset of t and $E\subset K[t]$ and $F\subset K[t][x]$. Then, $E\cup F$ can be regards as a subset of $K(t)[x]$ or $K(u)[x,\backslash u]$ where $K(t)$ and $K(u)$ are fields of rational functions with t and u, respectively. Let G be a (reduced) Gröbner basis of $\langle E\cup F\rangle$ in $K(t)[x]$ or $K(u)[x, t\backslash u]$.

In what follows, we assume that the Gröbner basis G satisfies either $G \subset K[t][x]$ or $G \subset K[u][x, t\backslash u]$, meaning that all coefficients are in $K[t]$ or $K[u]$.

Theorem 2 (Nabeshima [15]). *Let F be a finite set of polynomials in $K[t][x]$, G a reduced Gröbner basis of $\langle F\rangle$ w.r.t. $\succ_x$ in $K(t)[x]$. Let $h = \sqrt{\prod_{g\in G} \mathrm{lc}(g)}$ in $K[t][x]$. Consider F to be a subset of $K[t,x]$, and let S to be a reduced Gröbner basis of the ideal quotient $\langle F\rangle : \langle G\rangle$ w.r.t. a block term order $(\succ_x, \succ_t)$ in $K[t,x]$. Then, the following holds:*

(1) $S \cap K[t] \neq \emptyset$,
(2) for all $\bar{t} \in \overline{K}^m \setminus (\mathbf{V}(S\cap K[t]) \cup \mathbf{V}(h))$, $\sigma_{\bar{t}}(G)$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.

Corollary 2 (Nabeshima [15]). *Using the same notation as in Theorem 2, let $q \in S\cap K[t]$. Then, for all $\bar{t} \in \overline{K}^m \backslash \mathbf{V}(h\cdot q)$, $\sigma_{\bar{t}}(G)$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.*

We illustrate Theorem 2 and Corollary 2 with the following example.

Example 3. Let $F = \{t_1x^2 - xy + y^2, t_2xy + y, t_1x^2 - y, (t_2+1)xy^2 + t_1x\} \subset \mathbb{C}[t_1,t_2][x,y]$, $\succ_x$ the graded lexicographic term order with $x \succ y$ on *Term*$(\{x,y\})$ and $\succ_t$ the graded reverse lexicographic term order with $t_2 \succ t_1$ on *Term*$(\{t_1,t_2\})$. This setting is the same as Example 2.

The reduced Gröbner basis of $\langle F\rangle \subset \mathbb{C}(t_1,t_2)[x,y]$ w.r.t. $\succ_x$ is $\{x,y\}$. The reduced Gröbner basis S of the ideal quotient $\langle F\rangle : \langle x,y\rangle$ w.r.t. $(\succ_x, \succ_t)$ is

$$\begin{aligned} S = \{&(t_2-2)t_1^2 + t_1, t_1^2 + t_1t_2^2 + t_1t_2, -t_1^3 - 6t_1^2 + (t_2+3)t_1,\\ &-y + 2t_1^2 + (5t_2+3)t_1, -t_1x + t_1^2 + (2t_2+2)t_1\}.\end{aligned}$$

As $S\cap \mathbb{C}[t_1,t_2] = \{(t_2-2)t_1^2 + t_1, t_1^2 + t_1t_2^2 + t_1t_2, -t_1^3 - 6t_1^2 + (t_2+3)t_1\}$, for all $\bar{t} \in \mathbb{C}^2 \backslash \mathbf{V}((t_2-2)t_1^2 + t_1, t_1^2 + (t_2^2 + t_2)t_1, -t_1^3 - 6t_1^2 + (t_2+3)t_1)$, $\{x,y\}$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\mathbb{C}[x,y]$.

If we apply Corollary 2 and select $t_1^2 + t_1t_2^2 + t_1t_2$ from $S\cap\mathbb{C}[t_1,t_2]$, then it holds that for all $\bar{t} \in \mathbb{C}^2\backslash\mathbf{V}(t_1^2 + t_1t_2^2 + t_1t_2)$, $\{x,y\}$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ$ in $\mathbb{C}[x,y]$.

Definition 4. *Let I be a proper ideal of $K[x]$ and $u = \{u_1, \ldots, u_r\}$ a subset of x. Then, u is called independent modulo I if $I \cap K[u] = \{0\}$. Moreover, u is called a maximally independent set (MIS) modulo I if it is independent modulo I and the cardinality of u is equal to the dimension of I.*

By using a MIS modulo $\langle E\rangle$, Theorem 2 can be generalized as follows.

Theorem 3 (Nabeshima [15]). *Let F be a finite set of polynomials in $K[t][x]$, E a finite set of polynomials in $K[t]$ with $\langle E\rangle \neq \langle 1\rangle$, $u \subset t$ a MIS modulo $\langle E\rangle$ in $K[t]$. Regard $F\cup E$ as a set of $K(u)[x, t\backslash u]$, and let G be a Gröbner basis of $\langle F\cup E\rangle$ w.r.t. $(\succ_x, \succ_{t\backslash u})$ in $K(u)[x, t\backslash u]$, $G_b = G\backslash(G\cap\langle E\rangle)$ and $MB(\langle \mathrm{lt}(G_b)\rangle) = \{w_1, \ldots, w_\ell\}$ in $(K(u)[t\backslash u])[x]$. For each $i \in \{1, \ldots, \ell\}$, denote $G_{w_i} = \{f \in G_b \,|\, \mathrm{lt}(f) = w_i\}$ and take one polynomial g_i from G_{w_i}. Set $G' = \{g_1, \ldots, g_\ell\} \subset$*

$K[t][x]$. Let S be the reduced Gröbner basis of the ideal quotient $\langle F \cup E\rangle : \langle G'\rangle$ w.r.t. a block term order $(\succ_{t\backslash u,x}, \succ_u)$ in $K[t,x]$ where $\succ_{t\backslash u,x} = (\succ_x, \succ_{t\backslash u})$ is the block term order on Term$(x \cup \{t\backslash u\})$. Then, the following holds:

(1) $S \cap K[u] \neq \emptyset$,
(2) for all $\bar{t} \in \mathbf{V}(E)\backslash(\mathbf{V}(S \cap K[u]) \cup \mathbf{V}(h))$, $\sigma_{\bar{t}}(G')$ is a minimal Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$ where $h = \sqrt{\prod_{g \in G'} \mathrm{lc}(g)}$.

Corollary 3 (Nabeshima [15]). *Using the same notation as in Theorem 3, let $q \in S$ in $K[u]$. Then, for all $\bar{t} \in \mathbf{V}(E)\backslash\mathbf{V}(h \cdot q)$, $\sigma_{\bar{t}}(G')$ is a minimal Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.*

By utilizing Corollaries 2 and 3, we can construct the following algorithm.

Algorithm 3 (Nabeshima [15])

Specification: $\mathsf{NABESHIMA}(E, p, F, u, \succ_x)$
Input: $E \subset K[t]$: $\langle E\rangle$ is proper in $K[t]$, $p \in K[t]$ s.t. $\mathbf{V}(E)\backslash\mathbf{V}(p) \neq \emptyset$,
$F \subset K[t][x]$: finite set, $u \subset x$: MIS module $\langle E\rangle$,
$\succ_x$: term order on *Term*(x).
Output: (E', p', G', h') : If $E' \neq 1$ and $G' \neq \{0\}$, then $\mathbf{V}(E')\backslash\mathbf{V}(p') \neq \emptyset$ and, for all $\bar{t} \in \mathbf{V}(E')\backslash\mathbf{V}(p')$, $\sigma_{\bar{t}}(G')$ is a Gröbner basis of $\langle \sigma_{\bar{t}}(F)\rangle$ in $\overline{K}[x]$ and $\mathbf{V}(E')\backslash\mathbf{V}(p') \subset \mathbf{V}(E)\backslash\mathbf{V}(p)$. If $G' = \{0\}$, then for all $\bar{t} \in \mathbf{V}(E')\backslash\mathbf{V}(p')$, $\langle \sigma_{\bar{t}}(F)\rangle = \{0\}$. Moreover, $\mathbf{V}(p') = \mathbf{V}(p) \cup \mathbf{V}(h')$.
BEGIN
$G \leftarrow$ Compute a reduced Gröbner basis of $\langle F \cup E\rangle$ w.r.t. $(\succ_x, \succ_{t\backslash u})$ in $K(u)[t\backslash u][x]$
  where $\succ_{t\backslash u}$ is a term order on *Term*$(t\backslash u)$;
$G_b \leftarrow G\backslash(G \cap \langle E\rangle)$;
if $G_b = \emptyset$ **then**
  return $(E, p, \{0\}, 1)$;
end-if
$\{w_1, \ldots, w_\ell\} \leftarrow MB(\langle \mathrm{lt}(G_b)\rangle)$;
$G' \leftarrow \emptyset$;
for each $i = 1$ **to** ℓ **do**
  $g \leftarrow$ Select one element $g \in G_b$ that satisfies $\mathrm{lt}(g) = w_i$;
  $G' \leftarrow G' \cup \{g\}$;
end-for
$S \leftarrow$ Compute the reduced Gröbner basis of $\langle F \cup E\rangle : \langle G\rangle$ w.r.t. $(\succ_x, \succ_t)$
  in $K[x,t]$ where $\succ_t$ is a term order on *Term*(x);
$q \leftarrow$Take one element from $S \cap K[u]$;
$h \leftarrow \sqrt{q \times \prod_{g \in G' \subset K[t][x]} \mathrm{lc}(g)}$ in $K[t]$;
if $\mathbf{V}(E)\backslash\mathbf{V}(p \cdot h) = \emptyset$ **then**
  return $(1, p \cdot h, G', h)$;
end-if
return $(E, p \cdot h, G', h)$;
END

Note that if $u = t$, we regard $(\succ_x, \succ_{t\setminus u})$ as $\succ_x$. In fact, $\mathsf{NABESHIMA}$ (Algorithm 3) is essentially same as KSW (Algorithm 1) if $u = t$. Thus, we apply the algorithm $\mathsf{CGS1}$ to compute comprehensive Gröbner systems in the case.

In [15], the second author shows that by replacing KSW with $\mathsf{NABESHIMA}$ in Algorithm 2 we obtain the following algorithm for computing comprehensive Gröbner systems.

Algorithm 4 ($\mathsf{CGS2}$) [6,7]

Specification: $\mathsf{CGS2}(E, p, F, \succ_x)$
Computing a comprehensive Gröbner system.
Input: E: finite subsets of $K[t]$, $p \in K[t]$, F: finite subset of $K[t][x]$,
$\succ_x$: term order on $Term(x)$.
Output: $\mathcal{G}$: comprehensive Gröbner system of $\langle F\rangle$ on $\mathbf{V}(E)\backslash\mathbf{V}(p)$ w.r.t. $\succ_x$.
BEGIN
$\mathcal{G} \leftarrow \emptyset$;
$u \leftarrow$ MIS modulo $\langle E\rangle$ in $K[t]$;
if $\mathbf{V}(E)\backslash\mathbf{V}(N) = \emptyset$ **then** return $\mathcal{G}$; **end-if**
$\boxed{(E, p', G', h) \leftarrow \mathsf{NABESHIMA}(E, p, F, u, \succ_x);}$
if $E \neq 1$ **then**
  $\mathcal{G} \leftarrow \mathcal{G} \cup \{(E, \{p'\}, G')\}$;
end-if
$q \leftarrow 1$;
$h_1 h_2 \cdots h_\ell \leftarrow \mathsf{factorization}(h)$; /*$h_i$ is an irreducible factor $(1 \leq i \leq r)$. */
for each $i = 1$ to ℓ **do**
  $u' \leftarrow$ A MIS modulo $\langle E \cup \{h_i\}\rangle$ in $K[t]$;
  if $u' \neq \emptyset$ **then**
    $\mathcal{G} \leftarrow \mathcal{G}\cup$ $\mathsf{CGS2}(E \cup \{h_i\}, p \cdot q, u', F, \succ_x)$;
  else
    $\mathcal{G} \leftarrow \mathcal{G}\cup$ $\mathsf{CGS1}(E \cup \{h_i\}, p \cdot q, F, \succ_x)$;
  end-if
  $q \leftarrow q \cdot h_i$;
end-for
return $\mathcal{G}$;
END

We have reviewed two algorithms, namely $\mathsf{CGS1}$ and $\mathsf{CGS2}$, for computing comprehensive Gröbner systems. It was reported in [15] that we cannot unconditionally determine which one is superior.

In Section 5, we integrate the two algorithms using parallel processing techniques.

4 Zero-dimensional cases

Here we review special types of comprehensive Gröbner systems (or bases) on an affine variety defined by a zero dimensional ideal. As in Section 3, let $\succ_x$ be a term order on $Term(x)$ and $\succ_t$ a term order on $Term(t)$ in this section.

4.1 Gröbner basis in a polynomial ring over a commutative von Neumann regular ring

In 1987, V. Weispfenning studied, in [23], the theory of Gröbner bases in polynomial rings over a commutative von Neumann regular ring and gave an algorithm for computing them. After that A. Suzuki and Y. Sato connected the Weispfenning's theory of the Gröbner bases to comprehensive Gröbner bases. For the theory of Gröbner bases, we refer the reader to [17,18,21,23,26].

Definition 5. *A commutative ring R with identity 1 is called a commutative von Neumann regular ring if it has the following property:* $\forall a \in R,\ \exists b \in R$ *such that* $a^2 b = a$.

Lemma 1 ([18, Lemma 1]). *Let I be a zero dimensional radical ideal in $K[t]$. Then, $K[t]/I$ becomes a commutative von Neumann regular ring.*

The following theorem is borrowed from [18] and tells us how to compute a comprehensive Gröbner system (or basis) of an given ideal on $\mathbf{V}(E)$ where $\langle E \rangle$ is zero dimensional in $K[t]$.

Theorem 4 ([18, Theorem 2]). *Let F be a finite set of polynomials in $K[t][x]$ and E a finite set of polynomials in $K[t]$. Suppose that $\langle E \rangle$ is proper and a zero dimensional radical ideal in $K[t]$. Let G be a (stratified) reduced Gröbner basis of $\langle F \rangle$ w.r.t. $\succ_x$ in $(K[t]/\langle E \rangle)[x]$ where we regard $(K[t]/\langle E \rangle)[x]$ as a polynomial ring over a commutative von Neumann regular ring $K[t]/\langle E \rangle$. Then, for all $\bar{t} \in \mathbf{V}(E)$, $\sigma_{\bar{t}}(G)$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F) \rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.*

Remark 1. An algorithm for computing a Gröbner basis of an ideal in $R[x]$ is given in [23] where R is a commutative von Neumann regular ring. For the case $K[t]/\langle E \rangle$ above, Y. Kurata and M. Noro provided an effective algorithm, that utilizing modular dynamic evaluation method, for computing the Gröbner basis above in [9]. Moreover, they implemented the algorithm in the computer algebra system Risa/Asir [16]. One can obtain the Gröbner basis G, in Theorem 4, by utilizing their implementation.

Theorem 4 is summarized in the following algorithm.

Algorithm 5

Specification: $\mathsf{ZERO1}(E, F, \succ)$
Input: $E \subset K[t]$: finite set s.t. $\langle E \rangle$ is zero-dimensional in $K[t]$,
 $F \subset K[t][x]$ finite set, $\succ$: term order on $Term(x)$.
Output: (E', G) : G is a (stratified) reduced Gröbner basis of $\langle F \rangle$ w.r.t. $\succ$ in $(K[t]/\langle E' \rangle)[x]$ and $\langle E' \rangle = \sqrt{\langle E \rangle}$.
BEGIN
$E' \leftarrow$ Compute a basis of the radical of $\langle E \rangle$ in $K[t]$;
$G \leftarrow$ Computer a (stratified) reduced Gröbner basis of $\langle F \rangle$ with respect to $\succ$
 in $(K[t]/\langle E' \rangle)[x]$;
return (E', G);
END

Remark 2. The output (E', G) above satisfies that, for all $\bar{t} \in \mathbf{V}(E)$, $\sigma_{\bar{t}}(G)$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F) \rangle$ in $\overline{K}[x]$. However, for $\bar{a}, \bar{b} \in \mathbf{V}(E)$ $(\bar{a} \neq \bar{b})$, we may have $\mathrm{lt}(\sigma_{\bar{a}}(G) \neq \mathrm{lt}(\sigma_{\bar{b}}(G))$.

Remark 3. In [19], the output G of Algorithm 5 is referred to as an alternative comprehensive Gröbner basis on variety (ACGB-V).

4.2 Zero dimensional prime ideal

Let J be a zero dimensional prime ideal in $K[t]$ (i.e. J is maximal) and $\sqrt{J}$ a radical of J. Then, a comprehensive Gröbner system on $\mathbf{V}(J)$ can be easily obtained as follows.

Theorem 5 ([4, Theorem 3.3], [18]). *Let $F \subset K[t][x]$, $E \subset K[t]$ and $\succ_x$ a term order on $Term(t)$. Suppose that $\langle E \rangle$ is a zero dimensional prime ideal in $K[t]$. Let G be the reduced Gröbner basis of $\langle F \cup E \rangle$ w.r.t. $(\succ_x, \succ_t)$ in $K[t,x]$ and $G' = G \backslash (G \cap \langle E \rangle)$. Assume that $G \neq \emptyset$. Then, for all $\bar{t} \in \mathbf{V}(E)$, $\sigma_{\bar{t}}(G')$ is the reduced Gröbner basis of $\langle \sigma_{\bar{t}}(F) \rangle$ w.r.t. $\succ_x$ in $\overline{K}[x]$.*

Let $\sqrt{J} = P_1 \cap P_2 \cap \cdots \cap P_r$ be a minimal prime decomposition of $\sqrt{J}$ where $P_1, P_2, \ldots, P_r$ are prime ideals in $K[t]$. For each $i \in \{1, \ldots, r\}$, we can compute the redcued Gröbner basis of $\langle F \rangle \cup P_i$ w.r.t. $(\succ_x, \succ_t)$ in $K[t,x]$. Therefore, we can obtain a comprehensive Gröbner system of $\langle F \rangle$ on $\mathbf{V}(J)$ where $\mathbf{V}(J) = \bigcup_{i=1}^{r} \mathbf{V}(P_i)$.

Note that there exists several algorithm for computing the prime decomposition of the radical of an ideal. In general, the algorithm is faster than an algorithm for computing a primary decomposition of J. See [1,3,20].

Algorithm 6

Specification: $\mathsf{ZERO2}(E, p, F, \succ)$
Input: $E \subset K[t]$: finite set s.t. $\langle E \rangle$ is zero-dimensional in $K[t]$,
$p \in K[t]$ s.t. $\mathbf{V}(E) \backslash \mathbf{V}(p) \neq \emptyset$,
$F \subset K[t][x]$ finite set, $\succ$: term order on $Term(x)$.
Output: $\mathcal{L}$: comprehensive Gröbner system of $\langle F \rangle$ on $\mathbf{V}(E) \backslash \mathbf{V}(p)$ w.r.t. $\succ$.
BEGIN
$\mathcal{L} \leftarrow \emptyset$; $\succ_{t,x} \leftarrow$ A block term order with $x \gg t$;
$P_1, \ldots, P_r \leftarrow$ Compute a minimal prime decomposition of $\sqrt{\langle E \rangle}$ in $K[t]$;
for each $i = 0$ to r **do**
  if $\mathbf{V}(P_i) \backslash \mathbf{V}(p) \neq \emptyset$ **then**
    $G \leftarrow$ Compute reduced Gröbner basis of $\langle F \cup P_i \rangle$ w.r.t. $\succ_{t,x}$ in $K[t,x]$;
    $G' \leftarrow G \backslash (G \cap \langle P_i \rangle)$;
    if $G' \neq \emptyset$ **then**
      $\mathcal{L} \leftarrow \mathcal{L} \cup \{(P_i, \{p\}, G')\}$;
    else
      $\mathcal{L} \leftarrow \mathcal{L} \cup \{(P_i, \{p\}, \{0\})\}$;
    end-if
  end-if

end-for
end-if
return $\mathcal{L}$;
END

In Section 5, we will see how the utilization of ZERO2 significantly enhances computational efficiency in specific problems.

4.3 Standard way

As we describe in Section 3.2, CGS2 is essentially the same as CGS1 if the MIS modulo $\langle E \rangle$ is empty. Thus, we utilize CGS1 as the third method of the zero dimensional ideal $\langle E \rangle$ as follows.

Algorithm 7
Specification: ZERO3$(E, p, F, \succ)$
Input: $E \subset K[t]$: $\langle E \rangle$ is zero-dimensional in $K[t]$, $p \in K[t]$ s.t. $\mathbf{V}(E) \backslash \mathbf{V}(p) \neq \emptyset$,
$F \subset K[t][x]$ finite set, $\succ$: term order on $Term(x)$.
Output: $\mathcal{G}$: comprehensive Gröbner system of $\langle F \rangle$ w.r.t. $\succ$ on $\mathbf{V}(E) \backslash \mathbf{V}(p)$.
BEGIN
$\mathcal{G} \leftarrow$ **CGS1**$(E, p, F, \succ)$;
return $\mathcal{G}$;
END

5 Merging comprehensive Gröbner system algorithms using parallel processing

First, we present a new computational method for comprehensive Gröbner systems by integrating several algorithms introduced in Sections 3 and 4, utilizing parallel processing. Second, we provide the results of benchmark tests conducted after implementing the computational method in the computer algebra system Risa/Asir [16].

Our basic strategy for the new computational method of CGS is as follows: If the ideal $\langle E \rangle$ in $K[t]$ is not zero-dimensional, we execute two algorithms, KSW and NABESHIMA, in parallel and adopt the result that outputs the fastest among the two implementations. Otherwise, if $\langle E \rangle$ is zero-dimensional, we execute three algorithms, ZERO1, ZERO2, and ZERO3, in parallel and adopt the result that outputs the fastest among the three implementations

It is empirically known that the computational speed of comprehensive Gröbner systems depends on the result of the first recursive computation. Therefore, we also propose methods in which the initial selection, whether it is KSW or NABESHIMA, remains fixed. Hence, we have three strategies.

strategy 1: Execute KSW and NABESHIMA in parallel, and adopt the output of the fastest implementation. Terminate the ongoing process once the result is adopted. Repeat this strategy until $\langle E\rangle \subset K[t]$ is zero-dimensional.

strategy 2: Execute KSW, then proceed with *strategy 1*.

strategy 3: Execute NABESHIMA, then proceed with *strategy 1*.

Now, we are ready to present the following new computational method for comprehensive Gröbner systems.

Algorithm 8

Specification: MCGS($strategy, E, p, F, u, \succ$)

Input: strategy$\in \{1, 2, 3\}$, $E \subset K[t]$: finite set, $p \in K[t]$,
$F \subset K[t][x]$ finite set, $\succ$: term order on $Term(x)$.

Output: $\mathcal{G}$: comprehensive Gröbner system of $\langle F\rangle$ w.r.t. $\succ$ on $\mathbf{V}(E)\backslash\mathbf{V}(p)$.

BEGIN

$\mathcal{G} \leftarrow \emptyset$;

if $\mathbf{V}(E)\backslash\mathbf{V}(p) = \emptyset$ **then** return $\mathcal{G}$; **end-if**

if $strategy = 1$ **then**

$(E', p', G', h) \leftarrow$ [Execute KSW($E, p, F, \succ$) and NABESHIMA($E, p, F, u, \succ$) in parallel, and then adopt the result that is output the fastest among the two implementations. After adopting the result, terminate the process that is still executing.];

else if $strategy = 2$ **then**

$(E', p', G', h) \leftarrow$ KWS($E, p, F, \succ$);

else if $strategy = 3$ **then**

$(E', p', G', h) \leftarrow$ NABESHIMA($E, p, F, u, \succ$);

end-if

if $E' \neq 1$ **then**

$\mathcal{G} \leftarrow \mathcal{G} \cup \{(E, \{p'\}, G')\}$;

end-if

$h_1h_2\cdots h_\ell \leftarrow$ factorization(h); /*h_i is an irreducible factor ($1 \leq i \leq r$). */

$q \leftarrow 1$;

for each $i = 1$ to ℓ **do**

$u' \leftarrow$MIS modulo $\langle E \cup \{h_i\}\rangle$;

if $u' = \emptyset$ **then**

$\mathcal{L} \leftarrow$ [Execute ZERO1($E \cup h_i, F, \succ$), ZERO2($E \cup h_i, p \cdot q, F, \succ$), and ZERO3($E \cup h_i, p \cdot q, \succ$) in parallel, and then adopt the result that is output the fastest among the three implementations. After adopting the result, terminate the processes that are still executing.];

$\mathcal{G} \leftarrow \mathcal{G} \cup \mathcal{L}$;

else

$\mathcal{G} \leftarrow \mathcal{G}\cup$ MCGS($0, E \cup \{h_i\}, p \cdot q, F, u', \succ$);

end-if
$q \leftarrow q \cdot h_i$;
end-for
return $\mathcal{G}$;
END

Algorithm 8 shares the same structure as algorithms CGS1 and CGS2, ensuring correctness and termination in accordance with those algorithms.

We have implemented CGS1, CGS2 and MCGS in the computer algebra system Risa/Asir. One can download the source codes from the website

`https://www.rs.tus.ac.jp/~nabeshima/softwares.html`

Remark 4. As reported in [15], our experience indicates that the implementation of Risa/Asir is faster than that of SINGULAR in both Gröbner basis computation in $K(t)[x]$ and in the computation of ideal quotients. Therefore, the computer algebra system Risa/Asir is highly suitable for implementing the NABESHIMA algorithm. In fact, we were unable to replicate the same benchmark test results when using the computer algebra system Singular [2].

Here, we present the results of benchmark tests. Table 1 provides comparisons of the implementations. The implementations output a comprehensive Gröbner system w.r.t. the graded reverse lexicographic term order, such as (x, y) or (x, y, z). The examples in this paper were computed on a PC with the following specifications: OS: Windows 10, CPU: Intel(R) Core(TM) i9-C7900X CPU @ 3.30 GHz, RAM: 256GB. The time is measured in seconds. In Table 1, "> 24h" indicates that it took more than 24 hours, and **seg.** refers to the number of segments.

The columns labeled **methods** in Table 1 contain five-tuples $(n_1, n_2, n_3, n_4, n_5)$ where each component $(n_1, n_2, n_3, n_4, n_5) \in \mathbb{N}^5$. The value n_1 indicates how many times the KSW algorithm is utilized for CGS computation, while n_2 represents the frequency of the NABESHIMA algorithm's usage. Similarly, n_3, n_4, and n_5 correspond to the occurrences of ZERO1, ZERO2, and ZERO3, respectively.

Here, we present the results for the following 15 problems.

$S1 = \{y^4z + xy^2 + x^3, y^2zax^3 + 4bx^3y^2, xy^3 + y^3 + ayz\}$,
$S2 = \{y^2z^4 + xy + ax, y^3z + axyz + 2bx, xy^3z^4 + y^3 + axz\}$,
$S3 = \{x^5 + xy^2 + ay^3, x^3y^3 + x^2 + bx^2, x^2y + y^2 + xy\}$,
$S4 = \{x^6z^4 + ay^2, xy^5z^2 + bxy^2z + 2cx^2, x^2y^3z^4 + y^3 + dxz\}$,
$S5 = \{x^5y^5 + ax^2 + x^2y, x^5 + bxy^2 + y, x^2y + ax\}$,
$S6 = \{x^5y^4 + by^2 + x^2y, x^4y + axy^2 + y, x^2y + ax^2\}$,
$S7 = \{x^3y^6 + xy^3 + ay, x^2y + y^2 + xy, x^6y + by^2\}$,
$S8 = \{y^4z^4 + cy^2 + ax, y^5z + axy + 4bxy, x^2y^3z^4 + cy^3 + axz\}$,
$S9 = \{x^6 + xy^2 + ay^3, x^6 + y^5, x^3y^3 + x^2 + bx^3y, x^4y + x^3 + ax^2y\}$,
$S10 = \{x^5y^3 + az^4 + bxz + 5xy, x^4yz + ax^3z^2 + y^4, 4y^5z + xy^6 + xy^2 + by^3z + x\}$,
$S11 = \{y^3z^2 + xy + ax, y^5z + axyz + 2bx, xy^3z^4 + y^3 + axz\}$,

$S12 = \{y^3z^2 + ax^2y^2 + x^3, y^3z^2 + ax^2yz + 2bx, xy^3z^4 + y^3 + az\}$,
$S13 = \{x^3y + ax^4 + bxz + 5y, x^2y + ax^3z^2 + y^4, 4x^5z + xy^6 + y^2 + bx^5z^2\}$,
$S14 = \{x^4 + y^2x + y^3, y^3 + x^3 + bx^3y, x^3y^3 + y^4 + ax^2y\}$,
$S15 = \{x^3y^6 + xy^3 + ay, x^2y + y^2 + xy, x^6y + by^2\}$.

The main variables are x, y, z (or x, y) and the parameters are a, b, c, d. The graded reverse lexicographic term order with (x, y, z) (or (x, y)) is used for the benchmark tests.

Prob.	CGS1 (KSW)		CGS2 (NABESHIMA)		MCGS									
					strategy 1			strategy 2			strategy 3			
	seg.	time	seg.	time	methods	seg.	time	methods	seg.	time	methods	seg.	time
S1	4	0.01123	5	0.015625	(3,0,0,0,1)	4	0.011411	(3,0,0,0,1)	4	0.01855	(2,1,0,0,1)	4	0.03125
S2	9	0.09375	15	1.35938	(4,10,2,0)	9	0.046875	(3,1,0,2,0)	9	0.0625	(3,1,0,1,0)	9	0.04687
S3	15	12.1406	9	0.765625	(4,0,0,0,5)	15	1.67188	(4,0,0,0,5)	15	1.875	(2,1,0,0,2)	11	0.82813
S4	46	3.10938	42	4.78517	(35,1,0,0,2)	43	3.64062	(35,1,0,0,2)	43	3.39062	(35,1,0,0,2)	42	5.04688
S5	-	>24h	9	0.312	(1,1,0,0,1)	6	40.7031	(3,1,0,2,1)	15	9956.47	(2,1,0,1,0)	6	35.8125
S6	6	0.28125	6	3.10938	(2,1,0,0,1)	7	0.03125	(2,1,0,0,1)	7	0.078125	(2,1,0,0,1)	7	0.09375
S7	9	6118.27	12	161.344	(3,0,0,0,1)	10	267.172	(3,0,0,0,1)	10	224.562	(1,1,0,0,1)	9	318.938
S8	43	49.1094	24	2093.36	(30,1,0,0,1)	32	17.2656	(30,1,0,0,1)	32	16.25	(29,1,0,0,1)	32	190.703
S9	-	>24h	4	9.01562	(2,1,0,0,2)	5	7.76562	-	-	>24h	(1,1,0,0,2)	4	17.375
S10	17	16.9219	20	7.45312	(6,0,0,2,8)	16	1.71875	(6,0,0,0,10)	16	1.5625	(5,1,0,1,9)	16	10.6094
S11	-	>24h	-	>24h	(6,0,0,3,0)	14	12.1094	(6,0,0,3,0)	14	13.2344	(4,1,0,3,0)	13	888.859
S12	4	0.01563	5	74.1406	(3,0,0,0,1)	4	0.0625	(3,0,0,0,1)	4	0.140625	(2,1,0,0,1)	4	8309.31
S13	-	>24h	9	1.53125	(5,0,0,2,1)	10	1.59375	(5,0,0,1,2)	9	2.92188	(3,1,0,0,3)	7	2.73438
S14	-	>24h	4	2.67188	(4,0,0,2,1)	16	15422	(4,0,0,2,1)	16	17115.5	(2,1,0,1,0)	7	605.062
S15	9	1032.08	12	88.3125	(3,0,0,0,1)	10	267.172	(3,0,0,0,1)	10	269.938	(1,1,0,0,1)	9	318.938

Table 1. Comparison of comprehensive Gröbner systems

In problems S6 and S8, both *strategy 1* and *strategy 2* of MCGS outperform CGS1 (KSW) and CGS2 (NABESHIMA), likely due to the utilization of two methods, KSW and NABESHIMA. In problem S9, *strategy 1* exhibits the highest speed, possibly because it employs both KSW and NABESHIMA. This trend is also observed in problem S10, where *strategy 1* and *strategy 2* of MCGS surpass CGS1 and CGS2, likely attributed to the use of ZERO2.

For problem S11, MCGS significantly outperforms CGS1 and CGS2, again owing to the utilization of ZERO2. However, in problems S12, CGS1 demonstrates the fastest performance, while in problems S5, S13, S14, and S15, CGS2 takes the lead. Although it's challenging to determine the unequivocal superiority between the methods, Table 1 indicates that both *strategy 1* and *strategy 2* of MCGS consistently deliver results within 24 hours across all the mentioned problems.

In all problems, ZERO1 is not utilized, indicating that it is slower compared to the others. Therefore, ZERO1 may not be necessary.

The significant contribution of this paper is the introduction of a novel computational method for comprehensive Gröbner systems, achieved by integrating several existing methods using parallel processing. This paper presents a new option for computing comprehensive Gröbner systems, distinct from CGS1 and CGS2, which are known for their effectiveness.

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