SUT Journal of Mathematics
Vol. ??, No. ? (20??), 1–???

Algorithms for computing Kashiwara operators and s-parametric annihilators associated with isolated hypersurface singularities

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(Received ???? ??, 20??)

Abstract. The s-parametric annihilators and local b-functions associated with isolated hypersurface singularities are considered in the context of symbolic computation. A method is described for computing Kashiwara operators associated with an isolated hypersurface singularity. As an application, a new efficient method is proposed for computing generators of the s-parametric annihilators and local b-functions. The key tool of our approach is the Poincaré-Birkhoff-Witt algebra.

AMS 2020 *Mathematics Subject Classification.* 1310, 14H20.

Key words and phrases. Poincaré-Birkhoff-Witt algebra, b-function, Kashiwara operator, s-parametric annihilator, D-module.

§1. Introduction

In this paper, we consider local b-functions associated with isolated hypersurface singularities and the s-parametric annihilators in the context of symbolic computation. By utilizing Poincaré-Birkhoff-Witt algebra, we provide a method for computing "bon opérateur" introduced by M. Kashiwara. We show, as an application, that kashiwara operators introduced in [12], can be effectively used for computing local b-functions.

In [29, 31], M. Sato introduced the notion of b-functions in the study on prehomogeneous vector spaces and conjectured their existence and the rationality of their roots for arbitrary holomorphic functions. In [4, 5], I. N. Bernstein independently defined b-functions in his study on fundamental solutions of linear partial differential equations with constant coefficients. He proved that any non-zero polynomial has a non-zero b-function. Soon thereafter, in [6, 7], J. E. Björk extended the result of I. N. Bernstein to show the existence of b-functions associated with holomorphic functions. In [11], M. Kashiwara investigated b-functions in the context of algebraic analysis and provided a framework for studying b-functions. More precisely, let $\mathcal{D}_X$ denote the sheaf, on a complex manifold X, of linear partial differential operators with holomorphic coefficients. Let $\mathcal{D}_X[s] = \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$ where s is an indeterminate that commutes with all differential operators. Let f be a non-zero holomorphic function on X. M. Kashiwara introduced the s-parametric annihilator $Ann_{\mathcal{D}_X[s]}(f^s)$ and the cyclic $\mathcal{D}_X[s]$-module N_f as

$$N_f = \mathcal{D}_X[s]f^s = \mathcal{D}_X[s]/Ann_{\mathcal{D}_X[s]}(f^s).$$

M. Kashiwara proved in [12] the existence of b-functions and the rationality of the roots of b-functions. He also determined, by using the fact that a holomorphic function is integral over its Jacobian ideal, the characteristic variety of N_f. Using this result, M. Kashiwara proved the existence in $Ann_{\mathcal{D}_X[s]}(f^s)$ of partial differential operators, called "bon opérateur" in French in the literature [8], of the form

$$P = s^m + A_1(x, \tfrac{\partial}{\partial x})s^{m-1} + A_2(x, \tfrac{\partial}{\partial x})s^{m-2} + \cdots + A_m(x, \tfrac{\partial}{\partial x}),$$

where $A_j(x, \frac{\partial}{\partial x}) \in \mathcal{D}_X$ is a differential operator of order at most j.

Definition 1.1. We call an annihilator of the form P above, Kashiwara operator for b-functions.

Note that the Kashiwara operator named above is completely different from Kashiwara operators used in the theory of quantized enveloping algebra.

In [33], T. Yano developed the theory of b-functions. He studied the structure of s-parametric annihilators in an explicit manner and computed many interesting b-functions. In [8], J. Briançon, M. Granger et al considered Newton non-degenerate singularities and gave an algorithm for computing b-functions associated with Newton non-degenerate singularities. Notably, the Kashiwara operators play a definitive role both in [8, 33]. See also [1].

In [26, 27], T. Oaku gave a general algorithm for computing b-functions that can be applied to an arbitrary polynomial. The main tool in his approach is Gröbner basis computation in the Weyl algebra. In [9], J. Briançon and Ph. Maisonobe showed that s-parametric annihilators can be computed by utilizing Poincaré-Birkhoff-Witt algebras. See [3, 10, 15, 19] for the implementation of the algorithms. These two approaches due to T. Oaku, J. Briançon and Ph. Maisonobe are regarded as standard methods for computing b-functions. However, since the cost of Gröbner basis computation in non-commutative rings is quite high, it is difficult in general to compute b-functions even for the case of isolated hypersurface singularities. It is desirable to design algorithms which do not require non-commutative Gröbner basis computation.

The aim of this paper is to propose an alternative approach for computing local b-functions associated with isolated hypersurface singularities and the s-parametric annihilators. We provide, upon utilizing the Poincaré-Birkhoff-Witt algebra, an effective method for computing Kashiwara operators. We also show as an application, a method for computing local b-functions.

This paper is organized as follows. In Section 2, we briefly review our previous method and the notations that will be utilized in this paper. In Section 3, we discuss the Poincaré-Birkhoff-Witt algebra and s-parametric annihilators, and present algorithms for computing s-parametric annihilators. In Section 4, we introduce algorithms for computing Kashiwara operators and generators of s-parametric annihilators. In Section 5, we discuss local b-functions and s-parametric annihilators.

§2. Preliminaries

Let X be an open neighborhood of the origin O of the n-dimensional complex space $\mathbb{C}^n$ with coordinate $x = x_1, \ldots, x_n$ and let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions, $\mathcal{O}_{X,O}$ the stalk at the origin of $\mathcal{O}_X$.

We assume that ℓ polynomials $f_1, \ldots, f_\ell$ in $K[x]$ satisfy the condition $\{x \in X \mid f_1(x) = \cdots = f_\ell(x) = 0\} = \{O\}$, where K is a field of characteristic zero. Let $\mathcal{I}_O$ be the ideal generated by $f_1, \ldots, f_\ell$ in $\mathcal{O}_{X,O}$. Let I be the ideal generated by $f_1, \ldots, f_\ell$ in the polynomial ring $K[x]$.

2.1. Solving extended ideal membership problem

Here we briefly review our previous method for solving extended ideal membership problems for the local ring [20, 22].

Let fix a term order $\succ$ on x. Let

$$\epsilon_i = \begin{pmatrix} f_i, & 0, & \ldots, & 0, & \overset{(i+1)\text{th}}{1}, & 0, & \ldots, & 0 \end{pmatrix} \in (K[x])^{\ell+1}$$

where $1 \leq i \leq \ell$.

Consider a module M generated by $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$ in $(K[x])^{\ell+1}$ and its reduced Gröbner basis G_M w.r.t. a POT (position over term) module order in $(K[x])^{\ell+1}$. Set

$$G = \{g_j \in K[x] \,|\, (g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in G_M, g_j \neq 0\,\},$$
$$R_G = \{(g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in (K[x])^{\ell+1} \mid (g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in G_M, g_j \neq 0\},$$
$$Syz = \left\{(d_1, \ldots, d_\ell) \in (K[x])^\ell \,\middle|\, (0, d_1, \ldots, d_\ell) \in G_M\,\right\}.$$

Then, it is known that the set G is the reduced Gröbner basis of I w.r.t. $\succ$ in $K[x]$, $(g_j, a_{j1}, a_{j2}, \ldots, a_{js}) \in R_G$ satisfies $g_j = \sum_{i=1}^{\ell} a_{ji} f_i$, and the set Syz is the reduced Gröbner basis of the module of syzygies of $f_1, f_2, \ldots, f_\ell$.

Procedure 1 **Extgb_syz**

Specification Extgb_syz$([f_1, \ldots, f_\ell], \succ)$
Input: $f_1, \ldots, f_\ell \in K[x]$ with $\{x \in X | f_1(x) = \cdots = f_\ell(x) = 0\} = \{O\}$, $[f_1, \ldots, f_\ell]$ is a list of ordered polynomials. $\succ$: a term order.
Output: (G, R_G, Syz).
BEGIN
$G_M \leftarrow$ Compute the reduced Gröbner basis of a module generated by $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$ in $(K[x])^{\ell+1}$;
$G \leftarrow \{g_j \in K[x] \,|\, (g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in G_M, g_j \neq 0\}$;
$R_G \leftarrow \{(g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in (K[x])^{\ell+1} \mid (g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in G_M, g_i \neq 0\}$;
$Syz \leftarrow \left\{(d_1, d_2, \ldots, d_\ell) \in (K[x])^\ell \,\middle|\, (0, d_1, d_2, \ldots, d_\ell) \in G_M\right\}$;
return (G, R_G, Syz);
END

Lemma 2.1 ([20])**.** *Let h be a polynomial in $K[x]$. Then, $h \in \mathcal{I}_O$ if and only if there exists $u \in I : h$ such that $u \notin \mathfrak{m}$, where $I : h = \{u \in K[x] | uh \in I\}$ is the ideal quotient in $K[x]$ and $\mathfrak{m}$ is an ideal generated by $x_1, \ldots, x_n$.*

Suppose that $h \in K[x]$ and $h \in \mathcal{I}_O$. Then, there exits $u \in I : \langle h \rangle$ such that $u(O) \neq 0$ in $K[x]$. As $uh \in I$, there exits $q_1, q_2, \ldots, q_\ell \in K[x]$ such that

$$uh = q_1 f_1 + q_2 f_2 + \cdots + q_\ell f_\ell.$$

Let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis of I w.r.t. $\succ$ in $K[x]$. Then, Procedure 1 returns $(g_j, a_{j1}, a_{j2}, \ldots, a_{j\ell}) \in R_G \subset (K[x])^{\ell+1}$ that satisfies

$$g_j = a_{j1} f_1 + a_{j2} f_2 + \cdots + a_{j\ell} f_\ell$$

where $1 \leq j \leq \ell$. As uh can be reduced to 0 by the Gröbner basis $\{g_1, \ldots, g_r\}$, that satisfy

$$uh = b_1 g_1 + b_2 g_2 + \cdots + b_r g_r$$

can be obtained by the division algorithm. Therefore,

$$uh = \left(\sum_{j=1}^{r} b_j a_{j1}\right) f_1 + \left(\sum_{j=1}^{r} b_j a_{j2}\right) f_2 + \cdots + \left(\sum_{j=1}^{r} b_j a_{j\ell}\right) f_\ell,$$

namely, $q_i = \left(\sum_{j=1}^{r} b_j a_{ji}\right)$, for $1 \leq i \leq \ell$.

This method will be utilized in Procedure 2.

2.2. Integral dependence relation

Definition 2.2 (Integral dependence relation)**.** Let J be an ideal in a ring R. An element $h \in R$ is said to be integral over J if there exists an integer λ and $a_i \in J^i$, $i = 1, 2, \ldots, \lambda$, such that

$$h^\lambda + a_1 h^{\lambda-1} + a_2 h^{\lambda-2} + \cdots + a_{\lambda-1} h + a_\lambda = 0.$$

The smallest number λ that satisfies the equation above, is called integral number of h w.r.t. J. The equation above is called an integral dependence relation of h over J.

The generalization of the integral dependence relation is the following.

Definition 2.3 (Generalized integral dependence relation)**.** Let h be integral over J, λ the integral number of h w.r.t. J and k a non-zero natural number with $k < \lambda$. If there exists $b \in R$ and $a_i \in J^i$, $i = 1, 2, \ldots, k$, such that

$$bh^k + a_1 h^{k-1} + a_2 h^{k-2} + \cdots + a_{k-1} h + a_k = 0,$$

then, we call the equation above a generalized integral dependence relation of h over J.

We consider integral number and generalized integral dependence relations in $\mathcal{O}_{X,O}$. The computation methods of integral numbers and generalized integral dependence relations are described in [20, 22, 23].

§3. Poincaré-Birkhoff-Witt algebra and s-parametric annihilator

In this section, we first recall some basics on the Poincaré-Birkhoff-Witt algebra and s-parametric annihilators. Next, we give a basic procedure and two algorithms.

Let f be a holomorphic function defined on X. We assume throughout this paper that f has an isolated singularity at the origin O. More precisely, let $Sing(f)$ be the singular set in X of f:

$$Sing(f) = \left\{ x \in X \;\middle|\; f(x) = \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0 \right\}.$$

Then the singular set in X is the origin O i.e. $Sing(f) = \{O\}$. Let $\mathcal{D}_X$ denote the sheaf on X of linear partial differential operators with holomorphic coefficients.

3.1. Poincaré-Birkhoff-Witt algebra and s-parametric annihilator

Let $\mathcal{D}_X[s, \frac{\partial}{\partial t}]$ be the Poincaré-Birkhoff-Witt algebra with

$$sP\left(x, \frac{\partial}{\partial x}\right) = P\left(x, \frac{\partial}{\partial x}\right)s, \ \frac{\partial}{\partial t}P\left(x, \frac{\partial}{\partial x}\right) = P\left(x, \frac{\partial}{\partial x}\right)\frac{\partial}{\partial t},$$

for $P(x, \frac{\partial}{\partial x}) \in \mathcal{D}_X$, and

$$s\frac{\partial}{\partial t} - \frac{\partial}{\partial t}s = \frac{\partial}{\partial t}.$$

Note that, s corresponds to $(-\frac{\partial}{\partial t})t$.

Let

$$T_0 = s + f\frac{\partial}{\partial t}, \ T_i = \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t}, \ i = 1, 2, \ldots, n.$$

Let $\mathcal{I}_{PBW}$ be the left ideal in $\mathcal{D}_X[s, \frac{\partial}{\partial t}]$ generated by $T_0, T_1, \ldots, T_n$, and let $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ be the annihilator in $\mathcal{D}_X[s]$ of f^s:

$$\mathcal{I}_{PBW} = (T_0, T_1, T_2, \ldots, T_n),$$

$$\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) = \{P(s) \in \mathcal{D}_X[s] \mid P(s)f^s = 0\}.$$

J. Briançon and Ph. Maisonobe [9] obtained the following.

Theorem 3.1. *The following holds.*

$$\mathcal{I}_{PBW} \cap \mathcal{D}_X[s] = \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s).$$

We refer the reader to [2, 3] for the proof. See also [27, 28] for a classical approach for computing s-parametric annihilators.

Note that, for $1 \leq i, j, \leq n$,

$$\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial f}{\partial x_j}T_i - \frac{\partial f}{\partial x_i}T_j$$

are in $\mathcal{I}_{PBW} \cap \mathcal{D}_X[s] = \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$. Note that T. Yano [33] showed the following.

Proposition 3.2. *Assume that f has an isolated singularity at the origin. Then, the following holds.*

$$\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) \cap \mathcal{D}_X = \sum \mathcal{D}_X\left(\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}\right).$$

We give some examples for illustration.

Example 3.3. Let $a_0(x), a_1(x), \ldots, a_n(x)$ be holomorphic functions such that

$$a_0(x)f(x) + a_1(x)\frac{\partial f}{\partial x_1} + a_2(x)\frac{\partial f}{\partial x_2} + \cdots + a_n(x)\frac{\partial f}{\partial x_n} = 0 \tag{3.1}$$

Then,

$$a_0 T_0 + a_1 T_1 + \cdots + a_n T_n = a_0 s + a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n} \in \mathcal{D}_X[s]$$

holds. Since $a_0 T_o + a_1 T_1 + \cdots + a_n T_n \in \mathcal{I}_{PBW}$, we have

$$a_0 s + a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n} \in \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s).$$

Note that T. Yano [33, p. 135, Th. 2.21] proved the following

Proposition 3.4. *Assume that f has an isolated singularity at the origin, Then, $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) \cap (\mathcal{D}_X s + \mathcal{D}_X)$ is generated by the first order annihilators given in Proposition 3.2 and Example 3.3.*

It is easy to see that, in the Poincaré-Birkhoff-Witt algebra $\mathcal{D}_X[s, \frac{\partial}{\partial t}]$, the following relations holds.

$$\begin{aligned}
&f^2 \left(\frac{\partial}{\partial t}\right)^2 = f\frac{\partial}{\partial t}T_0 - (s-1)T_0 + s(s-1),\\
&f\frac{\partial f}{\partial x_i}\left(\frac{\partial}{\partial t}\right)^2 = \left(f\frac{\partial}{\partial t} + 1\right)T_i - \frac{\partial}{\partial x_i}T_0 + (s-1)\frac{\partial}{\partial x_i},\\
&\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\left(\frac{\partial}{\partial t}\right)^2 = \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t}T_j - \frac{\partial}{\partial x_j}T_i + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial}{\partial t},
\end{aligned}$$

where $1 \le i, j \le n$.

Now we define $L_0^{(2)}, L_i^{(2)}, L_{i,j}^{(2)}$ as

$$\begin{aligned}
L_0^{(2)} &= s(s-1) - f^2\left(\frac{\partial}{\partial t}\right)^2,\\
L_i^{(2)} &= (s-1)\frac{\partial}{\partial x_i} - f\frac{\partial f}{\partial x_i}\left(\frac{\partial}{\partial t}\right)^2,\\
L_{i,j}^{(2)} &= \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2 f}{\partial x_i \partial x_j}\left(\frac{\partial}{\partial t}\right) - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\left(\frac{\partial}{\partial t}\right)^2.
\end{aligned}$$

Notice that these operators $L_0^{(2)}, L_i^{(2)}, L_{i,j}^{(2)}$ are in the ideal $\mathcal{I}_{PBW}$.

Example 3.5. Let $a_0(x),\ a_i(x), i = 1, 2, \ldots, n,\ a_{i,j}(x), 1 \leq i \leq j \leq n$ be holomorphic functions. Let

$$r_1(x) = \sum_i a_i(x) \frac{\partial f}{\partial x_i},\ r_2(x) = \sum_{i,j} a_{i,j}(x) \left(\frac{\partial f}{\partial x_i} \right) \left(\frac{\partial f}{\partial x_j} \right).$$

Assume that $a_0(x), r_1(x), r_2(x)$ satisfy the following

$$a_0(x) f(x)^2 + r_1(x) f(x) + r_2(x) = 0. \tag{3.2}$$

Let $R^{(2)} = a_0(x) L_0^{(2)} + \sum_i a_i(x) L_i^{(2)} + \sum_{i,j} a_{i,j}(x) L_{i,j}^{(2)}$.

Then, $R^{(2)} \in \mathcal{I}_{PBW}$ and

$$\begin{aligned} R^{(2)} = a_0(x)(s^2 - s) + (s-1) \sum_i a_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} \\ + \sum_{i,j} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} \left(\frac{\partial}{\partial t} \right). \end{aligned}$$

Now we can utilize $T_0 = s + f \frac{\partial}{\partial t},\ T_i = \frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i} \frac{\partial}{\partial t},\ i = 1, 2, \ldots, n$ to eliminate $\frac{\partial}{\partial t}$. Let $u_1(x), b_0(x), b_i(x), i = 1, 2, \ldots, n$ be holomorphic function such that

$$u_1(x) \left(\sum_{i,j} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} \right) + b_0(x) f(x) + \sum_i b_i(x) \frac{\partial f}{\partial x_i} = 0. \tag{3.3}$$

Then,

$$\begin{aligned} u_1(x) \left\{ a_0(x)(s^2 - s) + (s-1) \sum_i a_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} \right\} \\ + b_0(x) s + \sum_i b_i(x) \frac{\partial}{\partial x_i} \end{aligned}$$

belongs to $\mathcal{I}_{PBW} \cap \mathcal{D}_X[s] = \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$.

Notably, for the case where $a_0(x) = u_1(x) = 1$, the annihilating operator constructed in the example above is a Kashiwara operator. Note also that the relations (3.1), (3.2) can be regarded as a generalization of integral dependence relation, in the local ring, of f with respect to the Jacobian ideal $(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$.

Recall that T. Yano [33, p.150-p. 151, Prop 2.31] proved the following.

Proposition 3.6. *Assume that f has an isolated singularity at the origin and the integral number of f w.r.t. the Jacobian ideal is equal to 2. Assume that*

(i) $\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}$ *in Example 3.5 is not in the ideal* $(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})$,

(ii) there exist a Kashiwara operator of order 3.

Then, $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) \cap (\mathcal{D}_X s^2 + \mathcal{D}_X s + \mathcal{D}_X)$ *is generated by the first order annihilators given in Proposition 3.2 and Example 3.3, and the second order annihilators constructed in Example 3.5.*

An algorithm for solving system of equations of the forms (3.3) is in [20].

3.2. Algorithms for computing first and second order annihilators

First and second order annihilators were discussed in Section 3.1. Example 3.3 and 3.5 give computation methods of the first and second order annihilators. Here, we summarize the methods as algorithms.

Procedure 2 that computes standard bases of the ideal quotient $(f_1, \dots, f_\ell) : f$ and syzygies of $[f, f_1, \dots, f_\ell]$, is the main part of Algorithm 1 and Algorithm 2 for computing first and second order annihilators.

Procedure 2 **Ann_syz**

Specification Ann_syz$(f, [f_1, \dots, f_\ell], \succ)$
Input: $f, f_1, \dots, f_\ell \in K[x]$. $\succ$: a term order on x. ($\{x \in X | f(x) = f_1(x) = \cdots = f_\ell(x) = 0\} = \{O\}$, $[f_1, \dots, f_\ell]$ is a list of ordered polynomials.)
Output: $P \subset K[x]^{\ell+1}$: $\forall (p_0, p_1, \dots, p_\ell) \in P$, $p_0 f + p_1 f_1 + \cdots + p_\ell f_\ell = 0$.
BEGIN
$P \leftarrow \emptyset$; $I \leftarrow$ Make an ideal generated by $f_1, \dots, f_\ell$ in $K[x]$;
$(G, R_G, Syz) \leftarrow$ **Extgb_syz**$([f_1, \dots, f_\ell], \succ)$;
$Q \leftarrow$ Compute a basis of the ideal quotient $I : f$ in $K[x]$;
$S_B \leftarrow$ Compute the reduced standard basis of (Q) in $\mathcal{O}_{X,O}$;
while $S_B \neq \emptyset$ **do**
  Select s_b from S_B; $S_B \leftarrow S_B \backslash \{s_b\}$;
  $U \leftarrow$ Compute a basis of the ideal quotient $(Q) : s_b$ in $K[x]$;
  $u \leftarrow$ Select u from U such that $u(O) \neq 0$;
  $(b_1, \dots, b_r) \leftarrow$ Compute $(b_1, \dots, b_r)$ by dividing $us_b f$ by $G = \{g_1, \dots, g_r\}$;
  /* $us_b f = \sum_{i=1}^r b_i g_i$ */
  $(p_1, \dots, p_n) \leftarrow (\sum_{i=1}^r b_i a_{i1}, \sum_{i=1}^r b_i a_{i2}, \dots, \sum_{i=1}^r b_i a_{in})$
  where $g_j = \sum_{i=1}^\ell a_{ji} f_j$, $(g_j, a_{j1}, \dots, a_{jn}) \in R_G$;
  $(p'_1, \dots, p'_n) \leftarrow$ Reduce $(p_1, \dots, p_n)$ by Syz;
  $P \leftarrow P \cup \{((-1)us_b, p'_1, \dots, p'_n)\}$;

end-while
return P;
END

Note that the standard bases can be computed via local cohomology classes associated with the ideal. See [21].

We introduce an algorithm for computing first order annihilators as Algorithm 1.

Algorithm 1 (First order partial differential operators)

Input: $f \in K[x]$. $\succ$: a term order. ($Sing(f) = \{O\}$ in $\mathbb{C}^n$.)
Output: $P \subset \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$: a set of first order partial differential operators whose highest power of s is 1 (i.e. $a_0 s + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ where $a_0, a_1, \ldots, a_n \in K[x]$).
BEGIN
$P \leftarrow \emptyset$;
$G \leftarrow$ **Ann_syz**$(f, [\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}], \succ)$;
  while $G \neq \emptyset$ **do**
    Select $(a_0, a_1, \ldots, a_n)$ from G; $G \leftarrow G \backslash \{(a_0, a_1, \ldots, a_n)\}$;
    $P \leftarrow P \cup \{a_0 s + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\}$;
  end-while
return P;
END

Let $a_0 s + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ be a first order partial differential operator in $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ where $a_0 \neq 0, a_1, \ldots, a_n \in K[x]$. As **Ann_syz** computes the reduced standard basis S_B of the ideal quotient $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) : f$ in Algorithm 1, thus a_0 belongs to the ideal (S_B). Therefore, the output of Algorithm 1 and $\{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i, j \leq n\}$ generate first order partial differential operators whose highest power of s, is less than or equal to 1.

Example 3.7. Let us consider $f = x^4 + y^{14} + xy^{11}$ in $\mathbb{C}[x, y]$ and let $\mathcal{I}_{PBW} = (s + f\frac{\partial}{\partial t}, \frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial t}, \frac{\partial}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial}{\partial t})$ in $\mathcal{D}_X[s, \frac{\partial}{\partial t}]$. Then, we obtain the following relations that are from the output of **Ann_syz**$(f, [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}])$.

$$
\begin{aligned}
&(x + \tfrac{14}{11}y^3)f + (-\tfrac{1}{4}x^2 - \tfrac{7}{22}xy^3)\tfrac{\partial f}{\partial x} + (-\tfrac{3}{44}xy - \tfrac{1}{11}y^4)\tfrac{\partial f}{\partial y} = 0,\\
&(-1331y^2 + 10976)y^5 f + (\tfrac{1331}{4}xy^7 - 2744xy^5 + \tfrac{121}{2}y^{10})\tfrac{\partial f}{\partial x}\\
&\qquad\qquad + (-22x^2 + 28xy^3 + \tfrac{363}{4}y^8 - 784y^6)\tfrac{\partial f}{\partial y} = 0.
\end{aligned}
$$

Hence, we gets the following two first order partial differential operators that belong to $\mathcal{I}_{PBW} \cap \mathcal{D}_X[s]$.

$(x+\frac{14}{11}y^3)s+(-\frac{1}{4}x^2-\frac{7}{22}xy^3)\frac{\partial}{\partial x}+(-\frac{3}{44}xy-\frac{1}{11}y^4)\frac{\partial}{\partial y}$,
$(-1331y^2+10976)y^5s+(\frac{1331}{4}xy^7-2744xy^5+\frac{121}{2}y^{10})\frac{\partial}{\partial x}$
$+(-22x^2+28xy^3+\frac{363}{4}y^8-784y^6)\frac{\partial}{\partial y}$.

Notice that $\{x+\frac{14}{11}y^3, y^5\}$ is a standard basis of the ideal quotient $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) : f$.

Next, we introduce an algorithm for computing second order annihilators as Algorithm 2.

Algorithm 2 (Second order partial differential operators)

Input: $f \in K[x]$. $\succ$: a term order. ($Sing(f) = \{O\}$ in $\mathbb{C}^n$.)
Output: $P \subset \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$: a set of second order partial differential operators whose highest power of s, is 2.
BEGIN
$P \leftarrow \emptyset$; $J_1 \leftarrow \{f\cdot\frac{\partial f}{\partial x_1},\ldots,f\cdot\frac{\partial f}{\partial x_n}\}$;
$J_2 \leftarrow \{(\frac{\partial f}{\partial x_i})(\frac{\partial f}{\partial x_j}) | 1\le i\le j\le n\}$;
$G \leftarrow$ **Ann_syz**$(f^2, [J_1\cup J_2], \succ)$;
while $G \neq \emptyset$ **do**
Select $(a_0, a_1,\ldots,a_n, a_{1,1}, a_{1,2},\ldots,a_{i,j},\ldots,a_{n,n})$ from G;
$G \leftarrow G\backslash\{(a_0, a_1,\ldots,a_n, a_{1,1}, a_{1,2},\ldots,a_{i,j},\ldots,a_{n,n})\}$;
$p \leftarrow a_0(s^2-s)+(s-1)\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}+\sum_{i\le j}a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}$;
$h \leftarrow \sum_{i\le j}a_{i,j}\frac{\partial^2 f}{\partial x_i\partial x_j}$;
$H \leftarrow$ **Ann_syz**$(h, [f, \frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}], \succ)$;
  while $H \neq \emptyset$ **do**
  Select $(b', b_0, b_1,\ldots,b_n)$ from H; $H \leftarrow H\backslash\{(b', b_0, b_1,\ldots,b_n)\}$;
  $P \leftarrow P\cup\{b'p+b_0s+\sum_{i=1}^n b_i\frac{\partial}{\partial x_i}\}$;
  end-while
end-while
return P;
END

In Algorithm 2, the procedure **Ann_syz** computes the reduced standard bases of the ideal quotient $(f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, (\frac{\partial f}{\partial x})^2, (\frac{\partial f}{\partial x})(\frac{\partial f}{\partial y}), (\frac{\partial f}{\partial y})^2) : f^2$ and $(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) : \sum_{i,j}a_{i,j}(x)\frac{\partial^2 f}{\partial x_i\partial x_j}$. (See Example 3.5.) Hence, Algorithm 2 returns generators of second order partial differential operators whose highest power of s, is 2.

Example 3.8. Let us consider $f = x^4+y^{14}+xy^{11}$ in Example 1, again. Then, Algorithm 2 outputs the following three second order partial differential operators that belong to $\mathcal{I}_{PBW}\cap\mathcal{D}_X[s]$.

$p_1 = (1331y^2 - 10976)x(s^2 - s) + (s-1)(392x^2\frac{\partial}{\partial x}(\frac{54571}{392}x^2 + 1568xy)\frac{\partial}{\partial y}) + (588x^3 - \frac{1331}{16}x^3y^2 - \frac{91}{2}y^{11})(\frac{\partial}{\partial x})^2 + (-\frac{54571}{1568}x^3 - \frac{363}{8}x^2y^3 - 28x^2y - \frac{11}{4}xy^6 + \frac{7}{2}y^9)\frac{\partial^2}{\partial x\partial y} + (-\frac{14883}{1568}x^2y - \frac{2651}{392}xy^4 - 56xy^2 - \frac{1}{56}y^7)(\frac{\partial}{\partial y})^2 + (\frac{21417}{7}xy^2 + 21560x)s + (-\frac{57717}{112}x^2y^2 + 3626x^2)\frac{\partial}{\partial x} + (\frac{203401}{1568}x^2 - \frac{99055}{784}xy^3 + 2268xy - \frac{41}{56}y^6)\frac{\partial}{\partial y}$,

$p_2 = -x((1331y^2 - 10976)y(s^2 - s) + (\frac{190333}{392}x^2 + 5488xy)\frac{\partial}{\partial x} + (\frac{3993}{28}xy + \frac{17640}{11}y^2)\frac{\partial}{\partial y} + (-\frac{190333}{1568}x^3 - \frac{1331}{16}x^2y^3 - 686x^2y)(\frac{\partial}{\partial x})^2 + (-\frac{107811}{1568}x^2y - \frac{37147}{784}xy^4 - \frac{4410}{11}xy^2 - \frac{11}{56}y^7)\frac{\partial^2}{\partial x\partial y} + (-\frac{1089}{112}xy^2 + \frac{1}{11}x - \frac{759}{112}y^5 - \frac{644}{11}y^3)(\frac{\partial}{\partial y})^2) + (\frac{185009}{196}x^2 + \frac{124264}{11}xy)s + (-\frac{134431}{224}x^3 - \frac{3993}{16}x^2y^3 - \frac{53704}{11}x^2y + \frac{5}{2}y^9)\frac{\partial}{\partial x} + (-\frac{77319}{392}x^2y - \frac{8635}{98}xy^4 - 1568xy^2 - \frac{5}{28}y^7)\frac{\partial}{\partial y}$,

$p_3 = -y((1331y^2 - 10976)y(s^2 - s) + (\frac{190333}{392}x^2 + 5488xy)\frac{\partial}{\partial x} + (\frac{3993}{28}xy + \frac{17640}{11}y^2)\frac{\partial}{\partial y} + (-\frac{190333}{1568}x^3 - \frac{1331}{16}x^2y^3 - 686x^2y)(\frac{\partial}{\partial x})^2 + (-\frac{107811}{1568}x^2y - \frac{37147}{784}xy^4 - \frac{4410}{11}xy^2 - \frac{11}{56}y^7)\frac{\partial^2}{\partial x\partial y} + (-\frac{1089}{112}xy^2 + \frac{1}{11}x - \frac{759}{112}y^5 - \frac{644}{11}y^3)(\frac{\partial}{\partial y})^2) + (\frac{47916}{49}xy + \frac{128184}{11}y^2)s - (\frac{954327}{1568}x^2y + \frac{3993}{16}xy^4 + \frac{54684}{11}xy^2)\frac{\partial}{\partial x} - (\frac{156453}{784}\ xy^2 - \frac{10}{11}x + \frac{34595}{392}y^5 + \frac{17528}{11}y^3)\frac{\partial}{\partial y}$.

The reduced standard basis of $(f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, (\frac{\partial f}{\partial x})^2, (\frac{\partial f}{\partial x})(\frac{\partial f}{\partial y}), (\frac{\partial f}{\partial y})^2) : f^2$ is $\{x, y\}$.

Note that the leading coefficients of p_1, p_2, p_3 are

$$(1331y^2 - 10976)x, -x(1331y^2 - 10976)y, -y(1331y^2 - 10976)y,$$

respectively. Thus, the highest power of s in $p_2 - (-x)p_1$, is 1. Therefore, p_2 is a redundant element. In order to keep the algorithm simple and understandability, we omit the optimization technique. One can easily delete the redundant element to check the coefficients of s^2.

§4. s-parametric annihilators

The existence of the Kashiwara operator in $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ was proved by M. Kashiwara. First, we introduce an algorithm for computing the Kashiwara operator. Next, we present an algorithm for computing a basis of $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ where the integral number of f w.r.t. the Jacobian ideal is less than equal to 2, and the order of a Kashiwara operator is less than equal to 3.

4.1. Computing Kashiwara operators

As we describe in Theorem 3.1, a basis of $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ can be obtained by computing a Gröbner basis of $\mathcal{I}_{PBW}$ w.r.t. a elimination term order

$\frac{\partial}{\partial t} \gg s, x, \frac{\partial}{\partial x}$ in $K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$. We are able to follow the same way to compute a Kashiwara operator, namely, if we get a Kashiwara operator during the Gröbner basis computation of $\mathcal{I}_{PBW}$, we break the computation. Algorithm 3 for computing a Kashiwara operator is essentially the same as the Buchberger algorithm. We omit several optimization techniques for computing non-commutative Gröbner bases in the algorithm. In Algorithm 3, $S(p, q)$ means an S-polynomial of the pair (p, q) and $\overline{S(p,q)}^G$ means the reduction of $S(p, q)$ by G where $p, q \in K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$, $G \subset K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$ and main symbols are $s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}$. Likewise, $\overline{g}^h$ is the reduction of g by h where $g, h \in K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$.

Let P be a Kashiwara operator in $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$, and let $u \in K[x]$ such that $u(O) \neq 0$, Then, we also call uP a Kashiwara operator.

Algorithm 3 (Kashiwara operator)

Input: $f \in K[x]$. ($Sing(f) = \{O\}$ in $\mathbb{C}^n$.) $\succ$: a block term order such that $\frac{\partial}{\partial t} \gg \{s, \frac{\partial}{\partial x}\} \gg x$, and a graded degree term order is imposed on $\{s, \frac{\partial}{\partial x}\}$ with $s > \frac{\partial}{\partial x}$.
Output: a Kashiwara operator.
BEGIN
$G \leftarrow \{s + f\frac{\partial}{\partial t}\} \cup \{\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t} | i = 1, 2, \ldots, n\}$;
$P \leftarrow \{(p, q) | p, q \in G, p \neq q\}$;
while $P \neq \emptyset$ **do**
  Select (p, q) from P; $P \leftarrow P \backslash \{(p, q)\}$;
  $h \leftarrow \overline{S(p,q)}^G$ in $K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$;
  if h is a Kashiwara operator **then**
   return h;
  end-if
  if $h \neq 0$ **then**
    $G' \leftarrow \emptyset$;
    while $G \neq \emptyset$ **do**
      Select g from G;
        if $\overline{g}^h$ is a Kashiwara operator **then**
          return $\overline{g}^h$;
        end-if
      $G' \leftarrow G' \cup \{\overline{g}^h\}$;
    end-while
    $G \leftarrow G'$; $P \leftarrow P \cup \{(h, g) | g \in G\}$; $G \leftarrow G \cup \{h\}$;
  end-if
end-while
END

Theorem 4.1. *Algorithm 3 terminates in a finite number of steps and cor-*

rectly computes a Kashiwara operator.

Proof. As describe earlier in the algorithm, Algorithm 3 is essentially the same as the Buchberger algorithm for computing a Gröbner basis. Thus, Algorithm 3 terminates in a finite number of steps.

Let

$$P = s^m + A_1(x, \frac{\partial}{\partial x})s^{m-1} + A_2(x, \frac{\partial}{\partial x})s^{m-2} + \cdots + A_m(x, \frac{\partial}{\partial x}),$$

be a Kashiwara operator in $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ where $A_j(x, \frac{\partial}{\partial x}) \in \mathcal{D}_X$ is a differential operator of order at most j. Then, there exists $u \in K[x]$ such that $u(O) \neq 0$ and $uP \in K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$.

Let G be the reduced Gröbner basis of the ideal I_{PBW} generated by $\{s + f\frac{\partial}{\partial t}\} \cup \{\frac{\partial}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial}{\partial t} | i = 1, 2, \ldots, n\}$ w.r.t. the term order described in Algorithm 3 in $K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$. Then, since $P \in \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$, we have $uP \in I_{PBW}$ in $K[x][s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}]$. Hence, there exist $g_1, \ldots, g_d \in G$ such that the leading terms of $g_1, \ldots, g_d$, in $\{s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\}$, divide s^m, and there exist $a_1, \ldots, a_d \in K[x]$ such that

$$a_1 \mathrm{lc}(g_1) + a_2 \mathrm{lc}(g_2) + \cdots + a_d \mathrm{lc}(g_d) = u$$

where $\mathrm{lc}(g_1), \mathrm{lc}(g_2), \ldots, \mathrm{lc}(g_d) \in K[x]$ are the leading coefficients of $g_1, \ldots, g_d$, respectively. Since u has a non-zero constant term, at least one of $\mathrm{lc}(g_1), \mathrm{lc}(g_2), \ldots, \mathrm{lc}(g_d)$ must have a non-zero constant term as well. Without loss of generality, suppose that $\mathrm{lc}(g_1)$ has a non-zero constant term. Since the graded degree term order is used on $\{s, \frac{\partial}{\partial x}\}$ and $\mathrm{lc}(g_1)(O) \neq 0$, we conclude that g_1 is a Kashiwara operator. Hence, G includes a Kashiwara operator that is obtained by the Buchberger algorithm. Therefore, Algorithm 3 correctly computes a Kashiwara operator in $\mathcal{D}_X[s]$. □

As we know Proposition 3.6, we can replace "h is a Kashiwara operator" and "$\overline{g}^h$ is a Kashiwara operator" with "h is a Kashiwara operator with $\deg_s(h) \leq 2$" and "$\overline{g}^h$ is a Kashiwara operator with $\deg_s(h) \leq 2$" in Algorithm 3, respectively, if the integral number of f w.r.t. the Jacobian ideal is ≤ 2 where $\deg_s(h)$ is the highest power of the variable s in h.

Algorithm 3 has been implemented by the third author in the computer algebra system Risa/Asir [25].

Example 4.2. Let us consider $f = x^4 + y^{14} + xy^{11}$ in Example 1, again. Then, Algorithm 3 outputs the following Kashiwara operator that belongs to $\mathcal{I}_{PBW} \cap \mathcal{D}_X[s]$.

$(8348032y^2 - 68841472)s^3 + (2458624x\frac{\partial}{\partial x} + 1341648x\frac{\partial}{\partial y} + 14751744y\frac{\partial}{\partial y} + 10245312y^2 - 56548352)s^2 + (526848x^2(\frac{\partial}{\partial x})^2 - 15972x^2(\frac{\partial}{\partial x})(\frac{\partial}{\partial y}) -$

$351232xy(\frac{\partial}{\partial x})(\frac{\partial}{\partial y}) + 1443288xy^2\frac{\partial}{\partial x} - 87120xy(\frac{\partial}{\partial y})^2 - 1053696y^2(\frac{\partial}{\partial y})^2 - 6409984x\frac{\partial}{\partial x} + 1026564x\frac{\partial}{\partial y} + 7024640y\frac{\partial}{\partial y} + 5617304y^2 - 24235008)s + (-99568y^{11} - 130438x^3y^2 + 790272x^3)(\frac{\partial}{\partial x})^3 + (8232y^9 - 6468xy^6 - 106722x^2y^3 - 79860x^3 - 37632x^2y)(\frac{\partial}{\partial x})^2(\frac{\partial}{\partial y}) + (-84y^7 - 31812xy^4 - 22869x^2y + 12544xy^2)(\frac{\partial}{\partial x})(\frac{\partial}{\partial y})^2 + (-3042y^5 - 297xy^2 + 25088y^3)(\frac{\partial}{\partial y})^3 + (-1392468x^2y^2 + 7639296x^2)(\frac{\partial}{\partial x})^2 + (-11844y^6 - 594330xy^3 - 381150x^2 + 470400xy)(\frac{\partial}{\partial x})(\frac{\partial}{\partial y}) + (-61278y^4 - 76824xy - 213248y^2)(\frac{\partial}{\partial y})^2 + (-2535918xy^2 + 12117504x)(\frac{\partial}{\partial x}) + (-448200y^3 - 76230x + 1467648y)(\frac{\partial}{\partial y})$.

Note that the highest power of s in the Kashiwara operator, is 3.

4.2. s-parametric annihilators

As we describe in Proposition 3.4 and Proposition 3.6, one can construct an algorithm for computing a basis of $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ by utilizing Algorithm 1, 2 and 3, if the integral number of f w.r.t. the Jacobian ideal is ≤ 2 and the order of a Kashiwara operator is ≤ 3.

Here, we summarize the fact as follows.

Algorithm 4 (Annihilators of f^s)

Input: $f \in K[x]$.($Sing(f) = \{O\}$ in $\mathbb{C}^n$.) The integral number of f w.r.t. the Jacobian ideal is ≤ 2.
Output: $P \subset \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$: if the order of a Kashiwara operator is ≤ 3, $P \cup \{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i < j \leq n\}$ is a basis of $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$.
BEGIN
$p \leftarrow$ Compute a Kashiwara operator by **Algorithm** 3;
$k \leftarrow \deg_s(p)$; $P \leftarrow \{p\}$;
if $k \leq 3$ **then**
    $j \leftarrow k$;
else
    $j \leftarrow 3$;
end-if
for $i = 1$ **to** $j - 1$ **do**
    $Q \leftarrow$ Execute **Algorithm** i;
    $P \leftarrow P \cup Q$;
end-for
return P;
END

Theorem 4.3. *Algorithm 4 terminates in a finite number of steps and correctly computes a Kashiwara operator.*

Proof. The termination of the algorithm follows from Algorithm 1, 2, 3.

If the order of a Kashiwara operator p is 1 i.e. $p = s + A(x, \frac{\partial}{\partial x})$, then any s-parametric annihilator can be reduced by p to a partial differential operator that does not contain the variables s. Thus, by Proposition 3.2, all s-parametric annihilators can be generated by p and $\{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i < j \leq n\}$.

If the order of a Kashiwara operator p is 2, then any s-parametric annihilator can be reduced by p to a partial differential operator whose highest power of s is less that equal to 1. By Proposition 3.4 and 3.2, $P \cup \{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i < j \leq n\}$ generate $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ where P is the output of Algorithm 4.

For the same reason, if the order of a Kashiwara operator p is 3, then by Proposition 3.6 and 3.2 $P \cup \{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i < j \leq n\}$ generate $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$. Otherwise, there is no guarantee that $P \cup \{\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} | 1 \leq i < j \leq n\}$ generate $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$. □

Note that only Algorithm 3 requires the non-commutative multiplication rules. Algorithm 1 and 2 work in commutative polynomial ring $K[x]$. This is an advantage of the algorithm.

All algorithms introduced in this paper have been implemented in the computer algebra system Risa/Asir [25][1]. Here, we give results of comparisons between a Gröbner basis computation of I_{PBW} in $D[s, \frac{\partial}{\partial t}]$ ("GB for $I_{PBW} \cap D[s]$" in Table 1) and Algorithm 4 where $D = K[x, \frac{\partial}{\partial x}]$. (An algorithm for computing a Gröbner basis of I_{PBW} in $D[s, \frac{\partial}{\partial t}]$ has been also implemented in Risa/Asir, like Algorithm 3. Notice that an element of $I_{PBW} \cap D[s]$ is an s-parametric annihilator.)

All tests presented in Table 1 have been performed on a machine [OS: Windows 10 (64 bit), CPU: Intel Core i9-7900 @ 3.30 GHz, RAM 128 GB]. "#" means the number of elements and "size" means the sum of all bit lengths of each output. The time is given in second (CPU time) and "$> 5h$" means that it takes more than five hours. Note that each time of Algorithm 4 includes the time of computing the integral number. The block term order $\succ$ with $\frac{\partial}{\partial t} \gg s \gg \frac{\partial}{\partial x} \succ \frac{\partial}{\partial y} \succ \frac{\partial}{\partial z} \gg x \succ y \succ z$, and the graded degree lexicographic term order in each block is used.

The following polynomials define an isolated singularity at the origin in $\mathbb{C}^2$ or $\mathbb{C}^3$, the integral numbers are ≤ 2 and the orders of Kashiwara operators are ≤ 3.

[1]The implementations are available at the following URL: `https://www.rs.tus.ac.jp/~nabeshima/softwares.html`.

$$
\begin{aligned}
f_1 &= x^4 + x^2y^3 + y^7 + y^8,\\
f_2 &= x^4 + xz^4 + y^3 + yz^2 + y^2z^2,\\
f_3 &= x^5 + x^2z^4 + y^5 + yz^4 + y^2z^4,\\
f_4 &= x^6 + x^4z^4 + y^7z + z^8,\\
f_5 &= x^5 + xy^4 + xz^5 + y^3z^4,\\
f_6 &= x^4 + yz^3 + xy^7 + y^{11} + y^{12},\\
f_7 &= x^2z + yz^2 + xy^4 + y^6 + z^3,\\
f_8 &= x^6z + yz^2 + xy^4 + y^6 + z^4,\\
f_9 &= x^2y + z^4 + y^5 + y^4z + y^3z^2 + y^4z^2,\\
f_{10} &= x^3 + yz^2 + y^{10} + xy^7 + xz^2.
\end{aligned}
$$

The big advantage of Algorithm 4 is to make a smaller number of essential s-parametric annihilators that decide local b-function. The keys are the Kashiwara operators, Algorithm 1 and 2. In particular, the Kashiwara operator makes the number possible to small. Therefore, Algorithm 4 is faster than "GB for $I_{PBW} \cap D[s]$" in speed.

In f_1, f_7, f_8, f_9, the size of Algorithm 4 is bigger than the other, because the output is not reduced by any partial differential operators. However, in f_2, f_3, f_4, f_5, f_6, the size is smaller, because of the number of partial differential operators.

Table 1: Comparison

f	GB for $I_{PBW} \cap D[s]$			Algorithm 4		
	size	#	time	size	#	time
f_1	1269	7	0.8125	670	3	0.125
f_2	1164	11	2.031	88	1	0.04688
f_3	2243	13	11.55	1962	4	0.1875
f_4	415	8	14.5	387	4	0.01563
f_5	4129	20	3.047	2722	9	1.125
f_6	32991	10	12.69	10962	4	1.219
f_7	14359	11	73.95	41526	4	1.578
f_8	8096	16	118.1	8389	4	19.92
f_9	21388	13	719.5	679713	7	9.578
f_{10}	–	–	>5h	13809	4	161

§5. Local b-functions

Let $\mathcal{D}_X[s](f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n})$ be a left ideal in $\mathcal{D}_X[s]$ generated by $f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}$. Then, according to [26, 27], the reduced local b-function $\tilde{b}_{f,O}$ of f at the origin O can be defined as a monic generator of the ideal

$$\left(\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) + \mathcal{D}_X[s]\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)\right) \cap \mathbb{C}[s].$$

M. Kashiwara [12] proved in general that the roots of b-function are negative rational numbers. For a rational number α, we consider a left ideal $\mathcal{I}_\alpha$ defined as

$$\mathcal{I}_\alpha = \mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) + \mathcal{D}_X[s]\left(f, \tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\right) + \mathcal{D}_X[s](s-\alpha),$$

and the $\mathcal{D}_X$-module $\mathcal{M}_\alpha = \mathcal{D}_X[s]/\mathcal{I}_\alpha$.

Then, it is well-known that α is a root of the reduced b-function $\tilde{b}_{f,O}$ if and only if $\mathcal{M}_\alpha \neq \{0\}$, or equivalently,

$$\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s) + \mathcal{D}_X[s]\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right) + \mathcal{D}_X[s](s-\alpha) \neq \mathcal{D}_X.$$

Let $\mathcal{H}^n_{\{O\}}(\mathcal{O}_X)$ be the sheaf of the highest local cohomology supported at the origin O. We have the following.

Proposition 5.1 ([11, 33])**.** *Let α is a rational number. Then α is a root of the reduced b-function $\tilde{b}_{f,O}$ if and only if*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{H}^n_{\{O\}}(\mathcal{O}_X)) \neq \{0\}.$$

Let

$$H_{T_f} = \left\{\psi \in \mathcal{H}^n_{\{O\}}(\mathcal{O}_X) \,\middle|\, f\psi = \frac{\partial f}{\partial x_1}\psi = \cdots = \frac{\partial f}{\partial x_n} = 0\right\}.$$

Since f has an isolated singularity at the origin O, H_{T_f} is a finite dimensional vector space. The dimension is equal to the Tjurina number τ defined to be $\tau = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}))$. Since the ideal $\mathcal{I}_\alpha$ contains $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ as generators, any solution in $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{H}^n_{\{O\}}(\mathcal{O}_X))$ belongs to the finite dimensional vector space H_{T_f}.

Proposition 5.2. *Let $P_1(s), P_2(s), \ldots, P_q(s) \in \mathcal{D}_X[s]$ be a set of generators of the s-parametric annihilator $\mathcal{A}nn_{\mathcal{D}_X[s]}(f^s)$ of f. Then, the following holds.*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\alpha, \mathcal{H}^n_{\{O\}}(\mathcal{O}_X)) = \left\{\psi \in H_{T_f} \,\middle|\, P_1(\alpha)\psi = \cdots = P_q(\alpha)\psi = 0\right\}.$$

Notably, in this approach, a Gröbner basis of the s-parametric annihilator is not required. Generally, the computational complexity of computing a Gröbner basis of $\mathcal{A}nn_{\mathcal{D}_X[s]}\ (f^s) + (f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is quite high. In order to overcome the difficulty, we adopt the idea introduced by Levandovskyy and Martín-Morales [16], that is "to check roots of the b-function".

In several singularities, possible candidates of roots of local b-functions can be obtained from the properties of the singularities, for instance, semi-weighted homogeneous and Newton non-degenerate singularities. Here, we present the outline of the computation method for semi-weighted homogeneous (isolated) hypersurface singularity. The details are in [24].

Method (Semi-weighted homogeneous case)

Step 1: Compute a set P of s-parametric annihilators by Algorithm 4;

Step 2: Compute a set E of possible candidates of roots of the local b-functions;

Step 3: For each $\gamma \in E$, set

$$\mathcal{I}_\gamma = P + \mathcal{D}_X[s]\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) + \mathcal{D}_X[s](s-\gamma)$$

and $\mathcal{M}_\gamma = \mathcal{D}_X[s]/\mathcal{I}_\gamma$. Compute

$$\kappa_\gamma = \dim_{\mathbb{C}}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_\gamma, \mathcal{H}^n_{\{O\}}(\mathcal{O}_X))).$$

If $\kappa_\gamma \neq 0$, then γ is a root of the local b-function.

Step 4: If the sum of all κ_γ is the Milnor number of the singularity, then we get all roots of $\tilde{b}_{f,O}$.

An algorithm for computing κ_γ is introduced in [19], and the details of the method for semi-weighted singularities are described in [24]. Notice that, in Step 2,3 and 4, we do not need any non-commutative Gröbner basis.

Example 5.3. Let us consider a semi-weighted homogeneous polynomial $f = x^4y + y^6 + xy^5 + x^2y^5$ that defines an isolated singularity at the origin, the Milnor number at the singularity is 19 and the integral number w.r.t. the Jacobian ideal is 2. It takes 2.297 seconds for Algorithm 4 to return a set P of s-parametric annihilators. From the weighted vector $(5,4)$ and the Poincaré polynomial, we obtain, as possible candidates of roots of the local b-functions, the following

$$E \cup \{\gamma+1 | \gamma \in E\} \cup \{\gamma + 2 | \gamma \in E\}$$

where $E = \{-\frac{i}{24} | i \in \{9, 13, 14, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31, \underline{34}, \underline{35}, \underline{39}\}\}$. (See [12, 24].) By computing each dimension of the vector space $\mathcal{M}_\gamma$, we obtain the following local b-function

$$\tilde{b}_{f,O} = \prod_{i \in E'} (s - i)$$

where $E' = \{-\frac{i}{24} | i \in \{9, \underline{10}, \underline{11}, 13, 14, \underline{15}, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31\}\}$. It takes 4.334 second for our implementation to get $\tilde{b}_{f,O}$. If we use the Risa/Asir function "bfct" that computes a b-function, then it takes 52.03 seconds because the function "bfct" computes a costly non-commutative Gröbner basis.

Acknowledgments

This work has been partly supported by JSPS Grant-in-Aid for Scientific Research (C) (Nos. 22K03334 and 23K03076).

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